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IP-SETS ON THE CIRCLE

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ABSTRACT. Let P be an *IP-set of integers* namely $P = \{\sum_{n\in S} a_n : S \subseteq \mathbf{N} \text{ finite}\}$ for a certain sequence $(a_n)_{n=1}^{\infty}$ in $\mathbf{Z}$. The main questions studied here are : (1) Under what conditions on (a_n) is $P\alpha$ dense modulo 1 for every irrational α? (2) Under what conditions on (a_n) is $P\alpha$ (considered as a sequence ordered in a way to be subsequently defined) uniformly distributed modulo 1 for every irrational α?

1. Introduction. A classical result in the theory of distribution module 1 states that, if α is an irrational, then the sequence $(n\alpha)_{n=1}^{\infty}$ is dense, and even uniformly distributed, modulo 1. There are also some rather "thin" subsequences of $(n\alpha)_{n=1}^{\infty}$ enjoying the same property, for instance $(P(n)\alpha)_{n=1}^{\infty}$, P being a non-constant polynomial taking integer values at the integer points (cf. [5, p 27, Th. 3.2]). Let us now introduce the following general definitions.

Definition 1.1. A set $S \subseteq \mathbf{Z}$ is a *DD set* if $S\alpha$ is dense modulo 1 for every irrational α. (DD-Dense Dilations).

Definition 1.2. A sequence $(n_k)_{k=1}^{\infty}$ of integers is a *UDD sequence* if $(n_k\alpha)_{k=1}^{\infty}$ is uniformly distributed modulo 1 for every irrational α. (UDD-Uniformly Distributed Dilations).

To characterize all DD sets (or all UDD sequences) is probably a hopeless task. A humbler goal is doing this for certain classes of sets of a special type. For example, in [**3**, Ch. IV] a full characterization of DD multiplicative semigroups was given. In this paper we consider this problem for the class of IP-sets, that is "infinite-dimensional parallelepipeds" of the form

$$(1.1) \qquad P = \{a_{n_1} + a_{n_2} + \cdots + a_{n_r} : r \in \mathbf{N},\ n_1 < n_2 < \ldots < n_r\},$$

$(a_n)_{n=1}^{\infty}$ being a sequence of integers [**4**, Def. 2.3]. The elements of P can be ordered in a natural way to form a sequence as follows. If $l = \sum_{k=1}^{d} \epsilon_k 2^{k-1}$ is the binary representation of l, then the l-th term of our sequence is $\sum_{k=1}^{d} \epsilon_k a_k$ (note that the sequence may assume some values more than once). The set P defined in (1.1), as well as the sequence formed from it, are denoted by *IP*-$(a_n)_{n=1}^{\infty}$ (or simply *IP*-(a_n)). We shall say that (a_n) *generates a DD set* (resp. *generates a UDD sequence*) if *IP*-(a_n) is a DD set (resp. a UDD sequence). In this terminology, the main questions to be dealt with in this paper may be put as:

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(1) Which sequences of integers generate DD sets?

(2) Which sequences of integers generate UDD sequences?

In [1] the analogue of the first of these questions was studied for multiplicative IP-sets. It is worthwhile mentioning that the techniques used there, as well as the type of results obtained, are very different from those we have here for additive IP-sets.

Given an IP-set P in $\mathbf{Z}$ and a real number α, the set $P\alpha$ modulo 1 forms an IP-set in the circle group $\mathbf{T} = \mathbf{R}/\mathbf{Z}$. It is clear therefore that a study of general IP-sets in $\mathbf{T}$ may be helpful for our problem. This is accomplished in Section 2. A by-product of our results is finding a close relation between our problem and some questions concerning the convergence of series of the form $\sum_{k=1}^{\infty} \sin(n_k x + \mu_k)$ tackled by Erdös and Taylor [2].

Obviously, the faster (a_n) grows, the slimmer is the "chance" of IP-(a_n) being a DD set. Section 3 deals with the effect of the rate of growth of (a_n) on its generating a DD set (or a UDD sequence). However, we note that the rate of growth of (a_n) by itself is in general not enough for deciding our question. For example, on the one hand in Section 4 we obtain a criterion for generating DD sets for sequences in which each term is divisible by its predecessor, showing in particular that $(n!)$ generates a non-DD set (see Example 4.2). On the other hand, it follows from example 6.4 that $(n! - 1)$, say, does generate a DD set.

Section 5 is concerned with sequences satisfying a linear recurrence with constant coefficients. Our results provide a somewhat surprising example. Denoting by (F_n) the Fibonacci sequence, we show that $(F_n)_{n=2}^{\infty}$ generates a DD set, whereas $(F_n)_{n=3}^{\infty}$ does not. We mention that in view of Corollary 2.2 such a phenomenon cannot occur for UDD sequences.

In section 6 we deal with a class of sequences satisfying some sort of recurrence (see Theorem 6.1). We show, for instance, that $([p/q)^n])$ generates a UDD sequence for any rational $p/q > 1$ (see Example 6.3).

2. General IP-Sets on the Circle. Let (α_n) be a sequence in $\mathbf{T}$. The sequence IP-(α_n) is constructed analogously to the sequence IP-(a_n), namely its terms are $\alpha_1,\ \alpha_2,\ \alpha_1+\alpha_2,\ \alpha_3,\ \alpha_1+\alpha_3,\ \alpha_2+\alpha_3,\ \alpha_1+\alpha_2+\alpha_3,\ \ldots$. In this section we shall try to find out for which sequences (α_n) is IP-(α_n) dense, or uniformly distributed (henceforward – u.d.), in $\mathbf{T}$.

We shall not distinguish between a point in $\mathbf{T}$ and real numbers lying above it. As a set, $\mathbf{T}$ will be regarded as any half open half closed interval of length 1 in $\mathbf{R}$, usually $[-1/2,\ 1/2)$ or $[0,\ 1)$. For a real number x, denote by $\|x\|$ its distance from the nearest integer.

PROPOSITION 2.1. (1) *If* $\sum_{n=1}^{\infty} \|h\alpha_n\| = \infty$ *for every positive integer* h, *then* IP-(α_n) *is dense in* $\mathbf{T}$.

(2) *If* IP-(α_n) *is dense in* $\mathbf{T}$, *then* $\sum_{n=1}^{\infty} \|h\alpha_n\| \geqq 1$ *for every positive integer* h.

PROPOSITION 2.2. IP-(α_n) *is u.d. in* $\mathbf{T}$ iff *for every positive integer* h *either (1)* $\|h\alpha_n\| = 1/2$ *for some* $n \in \mathbf{N}$, *or (ii)* $\sum_{n=1}^{\infty} \|h\alpha_n\|^2 = \infty$.

Before turning to prove these propositions, we shall show that neither of the conditions regarding $\sum_{n=1}^{\infty}\|h\alpha_n\|$ in Proposition 2.1 is actually both necessary and sufficient for IP-(α_n) to be dense in $\mathbf{T}$.

Example 2.1. Let $\alpha_n = 2^{-n}$, $n \geqq 1$. Then IP-(α_n) is clearly dense in $\mathbf{T}$ (and even u.d.), but $\sum_{n=1}^{\infty}\|h\alpha_n\| < \infty$ for every $h \in \mathbf{N}$.

Example 2.2. Let $\alpha_n = 2/5$ for $n \leqq 3$ and $\alpha_n = 2^{3-n}/5$ for $n \geqq 4$. It is easy to see that IP-$(\alpha_n) \cap (3/5,\ 4/5) = \phi$. We claim that $\sum_{n=1}^{\infty}\|h\alpha_n\| \geqq 1$ for every $h \in \mathbf{N}$. In fact, this is readily verified for $h \leqq 4$, while for $h \geqq 5$ even the set IP-$(\alpha_n)_{n=4}^{\infty}$ is dense in $\mathbf{T}$ so that the conclusion follows from Proposition 2.1. (2).

Proof of Proposition 2.1. The closure of IP-(α_n) clearly contains the closed group generated by all limit points of (α_n). Since the only closed proper subgroups of $\mathbf{T}$ are finite cyclic groups we may assume that all the limit points of (α_n) belong to the set $\{0,\ 1/d,\ 2/d, \dots,\ (d-1)/d\}$ for a certain $d \in \mathbf{N}$. Split (α_n) into $2d$ subsequences $(\alpha_n^{(j)})_{n=1}^{\infty}$, $1 \leqq j \leqq 2d$, by putting α_n in $(\alpha_n^{(j)})$ if $\alpha_n \in [(j-1)/2d,\ j/2d)$. For some $1 \leqq j \leqq 2d$ we have $\sum_{n=1}^{\infty}\|h\alpha_n^{(j)}\| = \infty$ for every $h \in \mathbf{N}$. In fact, if this is not the case, then for every $1 \leqq j \leqq 2d$ there exists an $h^{(j)}$ with $\sum_{n=1}^{\infty}\|h^{(j)}\alpha_n^{(j)}\| < \infty$. Setting $h = \prod_{j=1}^{2d} h^{(j)}$ we find that

$$\sum_{n=1}^{\infty}\|h\alpha_n^{(j)}\| \leqq \prod_{i\neq j} h^{(i)} \cdot \sum_{n=1}^{\infty}\|h^{(j)}\alpha_n^{(j)}\| < \infty, \quad 1 \leqq j \leqq 2d,$$

and therefore

$$\sum_{n=1}^{\infty}\|h\alpha_n\| = \sum_{j=1}^{2d}\sum_{n=1}^{\infty}\|h\alpha_n^{(j)}\| < \infty,$$

contradicting the assumptions. Hence we indeed have some $1 \leqq j \leqq 2d$ with $\sum_{n=1}^{\infty}\|h\alpha_n^{(j)}\| = \infty$ for every $h \in \mathbf{N}$. As it certainly suffices to show that IP-$(\alpha_n^{(j)})$ is dense in $\mathbf{T}$, we may assume to begin with that $(j-1)/2d \leqq \alpha_n < j/2d$ for each n, and $\alpha_n \underset{n\to\infty}{\longrightarrow} (j-1)/2d$ or $\alpha_n \underset{n\to\infty}{\longrightarrow} j/2d$ depending on whether j is odd or even, respectively. Assume that, say, $\alpha_n \underset{n\to\infty}{\longrightarrow} j/2d$. Define a sequence (β_n) by $\beta_n = \sum_{i=1}^{d} \alpha_{(d-1)n+i}$, $n \in \mathbf{N}$. From our assumptions it follows that $-1/2 \leqq \beta_n < 0$ for each n, $\beta_n \underset{n\to\infty}{\longrightarrow} 0$ and $\sum_{n=1}^{\infty}\beta_n = -\infty$. It is therefore evident that IP-(β_n) is dense in $\mathbf{T}$, whence the same is true for IP-(α_n).

(2): Obviously, IP-$(\alpha_n) \subseteq [\sum_{-1/2<\alpha_n<0}\alpha_n,\ \sum_{0\leqq\alpha_n\leqq 1/2}\alpha_n]$. The density of IP-(α_n) in $\mathbf{T}$ yields

$$\sum_{n=1}^{\infty}\|\alpha_n\| = \sum_{0\leqq\alpha_n\leqq 1/2}\alpha_n - \sum_{-1/2<\alpha_n<0}\alpha_n \geqq 1$$

which concludes the case $h = 1$. For general h we now just have to observe that IP-$(h\alpha_n)$ is also dense in $\mathbf{T}$. This completes the proof.

Proof of Proposition 2.2. Let (β_j) denote the sequence IP-(α_n) preceded by a zero. Evidently, (β_j) is u.d. iff IP-(α_n) is. Assume first that IP-(α_n) is u.d. By Weyl's criterion for equidistribution [9] we have

$$\frac{1}{N}\sum_{j=1}^{n} e^{2\pi i h\beta_j} \underset{n\to\infty}{\longrightarrow} 0, \qquad h \in \mathbf{N}.$$

In particular

$$\frac{1}{2^k}\sum_{j=1}^{2^k} e^{2\pi i h\beta_j} \underset{n\to\infty}{\longrightarrow} 0, \qquad h \in \mathbf{N}.$$

It is easy to check that

$$\left|\frac{1}{2^k}\sum_{j=1}^{2^k} e^{2\pi i h\beta_j}\right|^2 = \left|\frac{1}{2^k}\sum_{n=1}^{k} (1 + e^{2\pi i h\alpha_n})\right|^2$$
$$= \sum_{n=1}^{k} \cos^2 \pi h\alpha_n.$$

The last expression approaches 0 as $k \to \infty$ iff either (a) $\cos \pi h\alpha_n = 0$ for some n, or (b) $\sum_{n=1}^{\infty} (1 - \cos^2 \pi h\alpha_n) = \infty$. The first of these possibilities is equivalent to there existing some n with $\|h\alpha_n\| = 1/2$. Now

$$\sum_{n=1}^{\infty} (1 - \cos^2 \pi h\alpha_n) = \sum_{n=1}^{\infty} \sin^2 \pi h\alpha_n = \sum_{n=1}^{\infty} \sin^2 \pi\|h\alpha_n\|.$$

The series $\sum_{n=1}^{\infty} \sin^2 \pi\|h\alpha_n\|$ and $\sum_{n=1}^{\infty} \|h\alpha_n\|^2$ converge and diverge together, whence (b) above amounts to $\sum_{n=1}^{\infty} \|h\alpha_n\|^2 = \infty$. This concludes the proof of the "only if" part and also shows that, if the condition in the proposition is satisfied, then

$$\frac{1}{2^k}\sum_{j=1}^{2^k} e^{2\pi i h\beta_j} \underset{k\to\infty}{\longrightarrow} 0, \ h \in \mathbf{N}.$$

It remains to prove that

$$\frac{1}{N}\sum_{j=1}^{N} e^{2\pi i h\beta_j} \underset{N\to\infty}{\longrightarrow} 0$$

for every $h \in \mathbf{N}$. Let $\epsilon > 0$. Choose k with

$$\left| \frac{1}{2^k} \sum_{j=1}^{2^k} e^{2\pi i h \beta_j} \right| < \epsilon/2.$$

Let $N > 2^k(1 + 2/\epsilon)$. Write $N = 2^k L + r$, where $0 \leqq r < 2^k$. Then, since

$$\left| \sum_{j=2^k(l-1)+1}^{2^k l} e^{2\pi i h \beta_j} \right| = \left| \sum_{j=1}^{2^k} e^{2\pi i h \beta_j} \right|$$

for $1 \leqq l \leqq L$, we get

$$\left| \frac{1}{N} \sum_{j=1}^{N} e^{2\pi i h \beta_j} \right| \leqq \frac{1}{2^k L} \sum_{l=1}^{L} \left| \sum_{j=2^k(l-1)+1}^{2^k l} e^{2\pi i h \beta_j} \right| + \frac{1}{2^k L} \left| \sum_{j=2^k L+1}^{2^k L+r} e^{2\pi i h \beta_j} \right| \leqq \frac{1}{2^k} \left| \sum_{j=1}^{2^k} e^{2\pi i h \beta_j} \right| + \frac{1}{L} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This proves the proposition.

COROLLARY 2.1. *Let (a_n) be a sequence of integers. If (a_n) generates a non-*DD *set, then there exists an irrational α with $\sum_{n=1}^{\infty} \|a_n\alpha\| < \infty$.*

In [2] Erdös and Taylor studied, among other questions, the absolute convergence of series of the form $\sum_{k=1}^{\infty}\{n_k x\}$ (where $\{t\}$ denotes the fractional part of a real number t). Corollary 2.1 makes it clear that our problem on density of IP-sets is closely related to theirs. Even though the results obtained there do not seem to have direct implications for our problem, their approach, and in particular the kind of examples they considered, had a significant influence on the present paper.

COROLLARY 2.2. *A sequence of integers (a_n) generates a* UDD *sequence iff $\sum_{n=1}^{\infty} \|a_n\alpha\|^2 = \infty$ for every irrational α. In particular, if (a_n) generates a non-*UDD *sequence, then there exists an irrational α with $\|a_n\alpha\| \underset{n\to\infty}{\longrightarrow} 0$.*

Obviously, if a subsequence (a_{n_k}) of (a_n) generates a DD set, then so does (a_n) itself. Corollary 2.2 shows that the same is true with respect to generating UDD sequences. Similarly, the properties of generating DD sets and UDD sequences are invariant under arbitrary permutations of the underlying sequence. On the other hand, whereas it follows from Corollary 2.2 that omitting finitely many terms from a sequence which generates a UDD sequence leaves us with a sequence having the same property, the analogous statement for sequences generating DD sets is false (see Example 5.2 *infra*).

3. Growth Rate Conditions. The sequences (a_n) we deal with will be usually implicitly assumed to consist of positive numbers (Reformulating our results for general sequences is straightforward). In view of the remark following Corollary 2.2, we may also assume (a_n) to be non-decreasing. In this section we shall see that in some cases we can determine whether or not (a_n) generates a DD set (or a UDD sequence) just by the rate of growth of (a_n).

The *upper density* $D^*(S)$ and the *upper Banach density* $\mathrm{BD}^*(S)$ of a set $S \subseteq \mathbf{N}$ are defined by (compare with [4, Def. 3.7]).

$$D^*(S) = \varlimsup_{N \to \infty} \#(S \cap [1, N])/N,$$

$$BD^*(S) = \varlimsup_{N-M \to \infty} \#(S \cap [M, N-1])/(N-M),$$

where $\#(F)$ denotes the cardinality of a finite set F. A sequence (α_n) in $\mathbf{T}$ is *well distributed* if for any interval $[a, b) \subseteq [0, 1)$ we have

$$\#\left(\{n \in [M, N-1] : \alpha_n \in [a, b)\}\right)/(N-M) \underset{N_M \to \infty}{\longrightarrow} b - a$$

(cf. [5, p. 40, Def. 5.1]).

THEOREM 3.1. *If the set $\{a_n : n\epsilon\mathbf{N}\}$ is of positive upper Banach density, then (a_n) generates a UDD sequence.*

Proof. Let α be an irrational. Put $S = \{a_n : n \in \mathbf{N}\}$ and $A = \overline{S\alpha} \subseteq \mathbf{T}$. Since the sequence $(n\alpha)_{n=1}^{\infty}$ is well-distributed [5, p. 42, Ex. 5.2] it is easy to see that $\mu(A) \geqq BD^*(S)$, where μ denotes the Haar measure on $\mathbf{T}$. Consequently, A is a set of positive measure, and in particular $\|a_n\alpha\| \underset{n \to \infty}{\not\to} 0$. The theorem now follows from Corollary 2.2.

An improvement of the theorem will be presented later (Corollary 6.1). In this form, however, the theorem amends itself to the following generalization. Call a sequence of integers (n_k) a *WDD sequence* if $(n_k\alpha)$ is well-distributed for every irrational α. Given a sequence (b_n) and a subsequence (b_{n_k}) thereof, we define the *upper density* (resp. *upper Banach density*) of (b_{n_k}), regarded as a subsequence of (b_n), as $D^*(\{n_k : k \in \mathbf{N}\})$ (resp. $BD^*(\{n_k : k \in \mathbf{N}\})$).

THEOREM 3.1$'$. *Suppose that (a_n) is either (i) a subsequence of positive upper Banach density of some WDD sequence, or (ii) a subsequence of positive upper density of some UDD sequence. Then (a_n) generates a UDD sequence.*

The proof is the same as that of Theorem 3.1.

THEOREM 3.2. (1) *If $\sum_{n=1}^{\infty} a_n/a_{n+1} < \infty$, then for any $\epsilon > 0$ there exist uncountably many α's for which IP-$(a_n\alpha) \subseteq [0, \epsilon]$. In particular, (a_n) generates a non-DD set.*

(2) *If $\sum_{n=1}^{\infty} (a_n/a_{n+1})^2 < \infty$, then there exist uncountably many α's for which IP-$(a_n\alpha)$ is not u.d. in $\mathbf{T}$. In particular, (a_n) generates a non-UDD sequence.*

The first part of the theorem should be compared with [2, Th. 5].

Proof. (1) Let $\epsilon > 0$. Our assumption regarding (a_n) implies that $\sum_{j=1}^{n-1} a_j/a_n \underset{n\to\infty}{\longrightarrow} 0$. Select k with $\sum_{j=1}^{k-1} a_j/a_k < 1/2$ and $\sum_{n=k}^{\infty} a_n/a_{n+1} < \epsilon/6$. For any $\alpha \in [0, \epsilon/a_k]$ we have $\sum_{j=1}^{k-1} a_j\alpha < \epsilon/2$. To conclude the proof it suffices to construct a sequence $(\epsilon_n)_{n=k}^{\infty}$ of positive numbers with $\sum_{n=k}^{\infty} \epsilon_n < \epsilon/2$ such that the set of solutions $\alpha \in [0, \epsilon/a_k]$ of the system

$$a_n\alpha \in [0, \epsilon_n], \qquad n = k, k+1, \ldots$$

contains a Cantor-type set. In fact, $\epsilon_n = 3a_n/a_{n+1}$ for $n \geqq k$. The condition $a_k\alpha \in [0, \epsilon_k]$ is satisfied (at least) on the interval $[0, \epsilon_k/a_k]$. For any $n \geqq k$ we have $a_{n+1}\cdot(\epsilon_n/a_n) = 3 > 2+\epsilon_{n+1}$. Hence any interval of length ϵ_n/a_n contains at least two subintervals of length ϵ_{n+1}/a_{n+1} which are mapped upon multiplication (mod. 1) by a_{n+1} to the interval $[0, \epsilon_{n+1}]$. This proves part (1).

The proof of part (2) is similar.

In the next section we shall see that for any positive integer a the sequence (a^n) generates a UDD sequence. One might have expected that, at least sequences growing much slower than exponentially (say polynomially), always have this property. That this is not the case follows from the following proposition, which is of a somewhat different flavour than our other results.

PROPOSITION 3.1. *Let α be any irrational. Given a sequence (b_n) of positive integers satisfying $\sum_{n=1}^{\infty} 1/b_n \leqq 1$ (resp. $\sum_{n=1}^{\infty} 1/b_n^2 < \infty$), there exists a sequence (a_n) with $a_n < b_n$ for each n such that IP-$(a_n\alpha)$ is not dense (resp. not u.d.) in* $\mathbf{T}$.

In fact, by Dirichlet's theorem for each n we can find a positive integer $a_n < b_n$ with $\|a_n\alpha\| < 1/b_n$. According to Propositions 2.1 and 2.2, this proves our proposition.

In the same spirit we have

PROPOSITION 3.2. *Let α be an irrational and P a polynomial degree of d without free term assuming integer values at the integer points. Set $D = (2^{d+1} - 2)/3$ if d is even and $D = (2^{d+1} - 1)/3$ if d is odd. Then for any $\epsilon > 0$:*

(1) *There exists a sequence (e_n) of positive integers satisfying $e_n < n^{D+\epsilon}$ for all sufficiently large n such that IP-$(P(e_n)\alpha)$ is not dense in* $\mathbf{T}$.

(2) *There exists a sequence (e_n) of positive integers satisfying $e_n < n^{D/2+\epsilon}$ for all sufficiently large n such that IP-$(P(e_n)\alpha)$ is not u.d. in* $\mathbf{T}$.

Proof. Let $\epsilon > 0$. According to [**8**, Th. 7B], given any $\delta > 0$, for all sufficiently large M there exists some $m \leqq M$ with $\|P(m)\alpha\| < 1/M^{1/D-\delta}$. Taking $M = n^{D+\epsilon/2}$ we find that for any sufficiently large n there exists a positive integer $b_n \leqq n^{D+\epsilon/2}$ such that

$$\|P(b_n)\alpha\| < 1/M^{1/D-\delta} \leqq 1/n^{(D+\epsilon/2)(1/D-\delta)} = 1/n^{1+\eta},$$

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where $\eta > 0$ if δ is chosen sufficiently small. Hence there exists an r with $\sum_{n=r+1}^{\infty} \|P(b_n)\alpha\| < 1$. Putting $e_n = b_{n+r}$ for each n, we get $e_n \leqq (n+r)^{D+\epsilon/2} < n^{D+\epsilon}$ for all sufficiently large n, and by Proposition 2.1 (2) IP-$(P(e_n)\alpha)$ is not dense in $\mathbf{T}$. this proves part (1). The second part is proved the same way.

4. Divisibility Sequences. Call a sequence (a_n) a *divisibility sequence* if $a_n | a_{n+1}$ for each n. It turns out to be possible to fully characterize those divisibility sequences generating *DD* sets and UDD sequences.

THEOREM 4.1. *Let (b_i) be a strictly increasing divisibility sequence and (r_i) an arbitrary sequence of positive integers. The divisibility sequence (a_n) defined by $a_n = b_i$ for $\sum_{j=1}^{i-1} r_j < n \leqq \sum_{j=1}^{i} r_j$ generates*

(1) a DD set – iff $\underline{\lim}_{i\to\infty} \sum_{j=1}^{i-1} r_j b_j / b_i > 0.$

(2) a UDD sequence – iff $\underline{\lim}_{i\to\infty} \sum_{j=1}^{i-1} r_j b_j^2 / b_i^2 > 0.$

A divisibility sequence (a_n) without repetitions satisfies $a_{n+1} \geqq 2a_n$ for each n. Hence, Theorem 4.1 easily yields

COROLLARY 4.1. *Let (a_n) be a strictly increasing divisibility sequence. Then the following conditions are equivalent:*
(1) *(a_n) generates a DD set.*
(2) *(a_n) generates a UDD sequence.*
(3) *The sequence (a_{n+1}/a_n) is bounded.*

Compare the equivalence (1) $\Leftrightarrow$ (3) with [**2**, *Th.* 3].

Proof of Theorem 4.1. (1) The "only if" part will follow from the following, somewhat stronger assertion: if

$$\lim_{i\to\infty} \sum_{j=1}^{i-1} r_j b_j / b_i = 0$$

then for any $\epsilon > 0$ there exist uncountably many α's for which IP-$(a_n\alpha) \subseteq [0, \epsilon]$. In fact, let $\epsilon > 0$. Select a sequence (i_k) such that

$$\sum_{k=1}^{\infty} \left(\sum_{j=1}^{i_k - 1} r_j b_j / b_{i_k} \right) < \epsilon.$$

Set

$$\alpha = \sum_{k=1}^{\infty} 1/b_{i_k}.$$

Then

$$\sum_{n=1}^{\infty} \|a_n\alpha\| = \sum_{j=1}^{\infty} r_j \|b_j\alpha\| \leqq \sum_{j=1}^{\infty} r_j \left(\sum_{k=1}^{\infty} \|b_j/b_{i_k}\| \right)$$
$$= \sum_{k=1}^{\infty} \left(\sum_{j=1}^{i_k - 1} r_j b_j / b_{i_k} \right) < \epsilon.$$

Since all the $a_n\alpha$'s lie on the "positive side" of 0 in $\mathbf{T}$, this implies that $IP\text{-}(a_n\alpha) \subseteq [0, \epsilon]$. Now replacing (i_k) by any subsequence thereof we get another α having the same property. This provides us with an uncountable number of such α's, thereby proving the required implication.

For the "if" part it suffices to show, according to Corollary 2.1, that our condition implies

$$\sum_{n=1}^{\infty} \|a_n\alpha\| = \infty$$

for any irrational α. Put

$$\eta = \varliminf_{i \to \infty} \sum_{j=1}^{i-1} r_j b_j / b_i$$

(with, say, $\eta = 1$ if the limit is infinite). This will be accomplished by proving that for any positive integer l we have

$$\text{(4.1)} \qquad \sum_{j=1}^{k} r_j \|b_j\alpha\| > l\eta/4$$

if k is sufficiently large. We first show it for $l = 1$. Choose s such that $\sum_{j=1}^{i-1} r_j b_j / b_i > \eta/2$ for every $i \geqq s$. Let t be the least positive integer for which $(b_t/b_s)\|b_s\alpha\| \geqq 1/2$. Then

$$\sum_{j=1}^{t-1} r_j \|b_j\alpha\| \geqq \sum_{j=1}^{s-1} r_j (b_j/b_s) \|b_s\alpha\| + \sum_{j=s}^{t-1} r_j \|b_j\alpha\|$$
$$= \sum_{j=1}^{s-1} r_j (b_j/b_s) \|b_s\alpha\| + \sum_{j=s}^{t-1} r_j (b_j/b_s \|b_s\alpha\|$$
$$= \sum_{j=1}^{t-1} r_j b_j / b_t \cdot (b_t/b_s) \|b_s\alpha\| > ((\eta/2) b_t/b_s) \|b_s\alpha\| > \eta/4.$$

This proves the case $l = 1$. We proceed by induction. Suppose that, for some k, (4.1) is satisfied. Applying the previous considerations to the sequences $(b_i)_{i=k+1}^{\infty}$

and $(r_i)_{i=k+1}^{\infty}$ instead of $(b_i)_{i=1}^{\infty}$ and $(r_i)_{i=1}^{\infty}$, we find an m with $\sum_{j=k+1}^{m} r_j\|b_j\alpha\| > \eta/4$, and so $\sum_{j=1}^{m} r_j\|b_j\alpha\| > (l+1)\eta/4$. This proves part (1).
The second part is analogously proved.

Example 4.1. For any positive integer a, the sequence (a^n) generates a UDD sequence. Note that for $a = 2$ the sequence IP-(a^n) is nothing but the sequence of all positive integers, but for $a \geqq 3$ it is a much thinner subsequence. More generally, if $a \geqq 2$ and $(n_k)_{k=1}^{\infty}$ is an increasing sequence of positive integers, then (a^{n_k}) generates a UDD sequence (or, equivalently, a *DD* set) iff the sequence $(n_{k+1} - n_k)_{k=2}^{\infty}$ is bounded.

Example 4.2. The sequence $(n!)$ generates a non-*DD* set.

5. Recurrence Sequences. In this section we shall consider sequences (a_n) satisfying a recurrence of the form

$$(5.1) \qquad a_{n+r} = d_1 a_{n+r-1} + d_2 a_{n+r-2} + \cdots + d_r a_n, \quad n = 1, 2, \ldots$$

for certain fixed integers $d_1, d_2, \ldots, d_r (r \geqq 1)$. The sequence (a_n) is said to satisfy an *order-r linear recurrence with constant coefficients*. It is well-known that, denoting by E the set of roots of the polynomial $P(x) = x^r - \sum_{j=1}^{r} d_j x^{r-j}$ corresponding to the recurrence (5.1), we have

$$(5.2) \qquad a_n = \sum_{\lambda \in E} P_\lambda(n)\lambda^n, \quad n = 1, 2, \ldots$$

for suitable polynomials $P_\lambda \in \mathbf{Q}(\lambda)[x], \lambda \in E$. Let K be the number field obtained from $\mathbf{Q}$ upon adjoining E to it. Denote by G the Galois group of the extension $K/\mathbf{Q}$. Put $E_{\geqq 1} = \{\lambda \in E : |\lambda| \geqq 1\}$, and let H be the subgroup of G generated by the set $\{\psi \in G : \psi(E_{\geqq 1}) \cap E_{\geqq 1} \neq \phi\}$.

THEOREM 5.1. *Suppose (a_n) satisfies the recurrence* (5. 1) *but no lower order recurrence of this type. Then, with the notations introduced above:*
(1) *(a_n) generates a UDD sequence iff $H = G$.*
(2) *If $H \subsetneq G$ there exists a positive integer s such that $(a_n)_{n=s}^{\infty}$ does not generate a DD set.*

The theorem follows straightforwardly from

PROPOSITION 5.1. *In the setup of Theorem* 5. 1, *and denoting by F the fixed field of H in K, we have*
(1) *IP-$(a_n\alpha)$ is* u.d. *in* $\mathbf{T}$ *iff $\alpha \notin F$.*
(2) *If $\alpha \in F$ then there exists a positive integer s such that IP-$(a_n\alpha)_{n=s}^{\infty}$ is not dense in* $\mathbf{T}$.

Proof. Assume that $IP\text{-}(a_n\alpha)$ is not u.d. in $\mathbf{T}$ for a certain α. According to Proposition 2.2 we have in particular $\|ha_n\alpha\| \underset{n\to\infty}{\longrightarrow} 0$ for some $h \in \mathbf{N}$. By **[7]** this implies $h\alpha \in \mathbf{Q}(\lambda)$ for every $\lambda \in E_{\geqq 1}$, and in particular $\alpha \in K$. Given a $\psi \in G$, extend the action of ψ to $K[x]$ by letting it act on polynomials coefficient-wise. An equality of the form $\psi(\lambda_1) = \lambda_2$ for some $\psi \in G$ and $\lambda_1, \lambda_2 \in E$ implies $\psi(P_{\lambda_1}) = P_{\lambda_2}$, where the P_λ's are defined by (5.2). The convergence $\|ha_n\lambda\| \underset{n\to\infty}{\longrightarrow} 0$ also implies by [7] that, if $\psi(\lambda_1) = \lambda - 2$ for some $\psi \in G$ and $\lambda_1, \lambda_2 \in E_{\geqq 1}$, then $\psi(h\alpha P_{\lambda_1}) = h\alpha P_{\lambda_2}$, which, combined with the equality $\psi(P_{\lambda_1}) = P_{\lambda_2}$, yields $\psi(\alpha) = \alpha$. According to our assumptions no P_λ is the zero polynomial. Hence, if for a given $\psi \in G$ we have $\psi(E_{\geqq 1}) \cap E_{\geqq 1} \neq 0$, then $\psi(\alpha) = \alpha$. Thus, α is fixed under any automorphism in H, that is $\alpha \in F$.

We now have to prove that, given an $\alpha \in F$, $IP\text{-}(a_n\alpha)$ is not u.d. in $\mathbf{T}$ and there exists a positive integer s such that $IP\text{-}(a_n\alpha)_{n=s}^{\infty}$ is not dense in $\mathbf{T}$. Take a positive integer l such that all coefficients of the polynomials lP_λ, $\lambda \in E$, are algebraic integers. Since it clearly suffices to prove that $IP\text{-}(la_n\alpha)_{n=s}^{\infty}$ is not dense in $\mathbf{T}$. For any $\lambda \in E$ define α_λ as follows. Take a $\lambda_0 \in E_{\geqq 1}$ and a $\psi \in G$ with $\psi(\lambda_0) = \lambda$, and set $\alpha_\lambda = \psi(\alpha)$. Note that α_λ is well-defined. In fact, if $\psi_1(\lambda_1) = \psi_2(\lambda_2) = \lambda$ for some $\lambda_1, \lambda_2 \in E_{\geqq 1}$, then $\psi_2^{-1}\psi_1(\lambda_1) = \lambda_2$, whence $\psi_2^{-1}\psi_1 \in H$. Since $\alpha \in F$ this implies $\psi_2^{-1}\psi_1(\lambda) = \lambda$, that is $\psi_1(\lambda) = \psi_2(\lambda)$. Evidently, $\alpha_\lambda = \alpha$ for $\lambda \in E_{\geqq 1}$. It is also routinely verified that, if $\psi(\lambda_1) = \lambda_2$ for some $\psi \in G$ and $\lambda_1, \lambda_2 \in E$, then $\psi_1(\alpha_{\lambda_1}) = \alpha_{\lambda_2}$. It follows that the sequence (t_n) defined by

$$t_n = \sum_{\lambda\in E} \alpha_\lambda P_\lambda(n)\lambda^n, \quad n = 1, 2, \ldots$$

is invariant under any $\psi \in G$, and therefore consists of rationals. As each t_n is obviously an algebraic integer, it is actually a rational integer. Consequently

$$\begin{aligned}\|a_n\alpha\| = \|a_n\alpha - t_n\| &= \left\| \sum_{\lambda\in E} (\alpha - \alpha_\lambda)P_\lambda(n)\lambda^n \right\| \\ &= \left\| \sum_{\lambda\in E_{\geqq 1}} (\alpha - \alpha_\lambda)P_\lambda(n)\lambda^n \right\| \leqq (|\lambda_0| + \epsilon)^n\end{aligned}$$

for all sufficiently large n, where λ_0 is an element of maximal norm in $E\backslash E_{\geqq 1}$ and $\epsilon > 0$ arbitrary. Thus $\sum_{n=s}^{\infty} \|a_n\lambda\| < 1$ if s is chosen sufficiently large, so that by Proposition 2.1 $IP\text{-}(a_n\lambda)_{n=s}^{\infty}$ is not dense in $\mathbf{T}$. This completes the proof.

Remark 5.1. In a sense, Theorem 5.1 indicates recurrence sequences usually generate UDD sequences. In fact, assume for simplicity that the polynomial corresponding to the recurrence is irreducible over $\mathbf{Q}$. The product of all roots of this polynomial is at least 1 in absolute value, and hence "on the average" at least half of these roots belong to $E_{\geqq 1}$. Now if more than half of the roots belong to $E_{\geqq 1}$, then clearly $H = G$, and therefore our sequence generates a UDD sequence. Actually, on "probabilistic" grounds one would expect the equality $H = G$ to hold as long as $\#(E_{\geqq 1})$ is not very small compared to $\#(E)$. In Examples 5.2–5.4 we present some recurrence sequences generating non-UDD sequences.

Example 5.1. For any positive integer a, the sequence (a^n) generates a UDD sequence (Compare with Example 4.1).

Example 5.2. Let $(F_n)_{n=1}^{\infty}$ be the Fibonacci sequence: $F_1 = F_2 = 1$ and $F_{n+2} = F_n + F_{n+1}$ for $n \geqq 1$. We claim that IP-$(F_n)_{n=2}^{\infty}$ is a DD set, but IP-$(F_n)_{n=3}^{\infty}$ is not. In fact, since $F_2 = 1$ and $F_{n+1} \leqq 2F_n$ for each n, the set IP-$(F_n)_{n=2}^{\infty}$ contains all positive integers and is therefore a DD set. Put $\lambda_1 = (1 + \sqrt{5})/2$ and $\lambda_2 = (1 - \sqrt{5})/2$. A routine calculation shows that $F_n\lambda_1 = F_{n+1} - \lambda_2^n$ for each n. Thus

$$\sum_{n=3}^{\infty} \|F_n\lambda_1\| = \sum_{n=3}^{\infty}((\sqrt{5} - 1)/2)^n = (\sqrt{5} - 1)/2 < 1,$$

and in particular IP-$(F_n)_{n=3}^{\infty}$ is not a DD set. It is worthwhile to note that $(F_n)_{n=2}^{\infty}$ manages to generate a DD set even though $\sum_{n=2}^{\infty} \|F_n\lambda_1\|$ is exactly 1.

Example 5.3. A general example of a recurrence sequence generating a non-UDD sequence, including as a special case the Fibonacci sequence, is as follows. Let λ be a *PV-number*, of degree $\geqq 2$ over $\mathbf{Q}$, that is an algebraic integer > 1 all of whose conjugates over $\mathbf{Q}$ lie inside the unit circle in the complex plane (cf.[**6**, p. 25, Def. 10]). Take an arbitrary algebraic integer θ in $\mathbf{Q}(\lambda)$ (Some non-integers will also do, as is the case in Example 5.2; see [**6**, p. 27, Prop. 4]). Let $\lambda_1 = \lambda, \lambda_2, \ldots, \lambda_r$ be the conjugates of λ over $\mathbf{Q}$, and $\theta_1 = \theta, \theta_2, \ldots, \theta_r$ the corresponding conjugates of θ. Then the sequence (a_n) defined by $a_n = \sum_{i=1}^{r} \theta_i\lambda_i^n$ for each n generates a non-UDD sequence.

Example 5.4. Let $\omega = e^{i\pi/4}$. Then $\mathbf{Q}(\omega) = \mathbf{Q}(i, \sqrt{2})$. The Galois group of $\mathbf{Q}(\omega)/\mathbf{Q}$ is (the non-cyclic group) of order 4, and its elements are defined by $\sigma_1(\omega) = \omega$, $\sigma_2(\omega) = \omega^3$, $\sigma_3(\omega) = \omega^5$ and $\sigma_4(\omega) = \omega^7$. Let $\lambda = 1 + \omega$ and $\lambda_i = \sigma_i(\lambda)$, $1 \leqq i \leqq 4$. Since λ_1 and λ_4 are outside the unit circle, while λ_2 and λ_3 are inside, we have, using the notations introduced at the beginning of the section, $H = \{\sigma_1, \sigma_4\}$ and $F = \mathbf{Q}(\sqrt{2})$. Thus, for instance, the sequence (a_n), given by $a_n = \sum_{i=1}^{4} \lambda_i^n$ for each n, generates a non-UDD sequence.

6. Approximate Recurrence Sequences. The following result, which is quite easy to prove, will provide us with several interesting examples of UDD sequences.

THEOREM 6.1. *Let r be a positive integer, $(n_{kj})_{k=1}^{\infty}$, $l \leqq j \leqq r$, sequences of positive integers, each consisting of distinct terms, $(d_{kj})_{k=1}^{\infty}$, $1 \leqq j \leqq r$, arbitrary* *sequences of integers and* $(c_k)_{k=1}^{\infty}$ *a bounded sequence of non-zero integers such that:*

$$d_{k1}a_{n_{k1}} + d_{k2}a_{n_{k2}} + \cdots + d_{kr}a_{n_{kr}} = c_k, \quad k = 1, 2, \ldots .$$

Set $D_k = \max_{1 \leqq j \leqq r} |d_{k_j}|$ *for each* k. *Then:*

(1) *If* $\sum_{k=1}^{\infty} 1/D_k = \infty$, (a_n) *generates a* DD *set.*
(2) *If* $\sum_{k=1}^{\infty} 1/D_k^2 = \infty$, (a_n) *generates a* UDD *sequence.*

Proof. Passing to a subsequence if necessary, we may assume that (c_k) is constant, say $c_k = c$ for each k. Given an irrational α we have

$$\|c\alpha\| \leqq |d_{k1}|\|a_{n_{k1}}\alpha\| + |d_{k2}|\|a_{n_{k2}}\alpha\| + \cdots + |d_{kr}|\|a_{n_{kr}}\alpha\| \leqq D_k \sum_{j=1}^{r} \|a_{n_{kj}}\alpha\|$$

for each k. To prove (1) we observe that

$$\sum_{n=1}^{\infty} \|a_n\alpha\| \geqq \frac{1}{r} \sum_{j=1}^{r} \sum_{k=1}^{\infty} \|a_{n_{kj}}\alpha\| \geqq (\|c\alpha\|/r) \sum_{k=1}^{\infty} 1/D_k$$

and employ Corollary 2.1. Part (2) is similarly proved using Corollary 2.2.

A straightforward consequence improving Theorem 3.1 is

COROLLARY 6.1. *Let* $\{a_n : n \in \mathbf{N}\} = \{b_n : n \in \mathbf{N}\}$, *with* (b_n) *strictly increasing. If* $\varliminf_{n \to \infty} (b_{n+1} - b_n) < \infty$, *then* (a_n) *generates a* UDD *sequence.*

Example 6.1. Suppose the sequence (a_n) satisfies

$$a_{n+r} = d_1 a_{n+r-1} + d_2 a_{n+r-2} + \cdots + d_r a_n + p_n, \quad n = 1, 2, \ldots,$$

where (p_n) is a bounded sequence, taking non-zero values for infinitely many n's. Then (a_n) generates a UDD sequence.

A sequence (y_n) is a *bounded perturbation* of a sequence (x_n) if $(y_n - x_n)$ is bounded, and a *non-zero perturbation* if $y_n \neq x_n$ for all sufficiently large n.

Example 6.2. Let (s_n) be a bounded sequence of positive integers with $s_n \geqq 2$ infinitely often. Set $S_n = \prod_{i=1}^{n} s_i$. Then any bounded perturbation of (S_n) generates a UDD sequence. In fact, suppose $a_n = S_n + p_n$, (p_n) being bounded. Then

$$a_{n+1} - s_{n+1}a_n = p_{n+1} - s_{n+1}p_n. \tag{6.1}$$

If the right hand side of (6.1) is non-zero infinitely often, then the required conclusion follows from Theorem 6.1. Otherwise, we get $p_{n+1} = s_{n+1}p_n$ for all sufficiently large n, whence the boundedness of (p_n) implies that $p_n = 0$ for all sufficiently large n, and by Theorem 4.1 we are done.

Example 6.3. Any bounded perturbation of $([\theta u^n])$, θ being an arbitrary non-zero real number and $u > 1$ rational, generates a UDD sequence. In fact, let (a_n) be such a sequence. Write $u = p/q$, $(p, q) = 1$. The sequence $(qa_{n+1} - pa_n)_{n=1}^{\infty}$ is obviously bounded. If $q > 1$ then this sequence is non-zero infinitely often, so that Theorem 6.1 proves our claim. We may assume therefore that $q = 1$ and $a_{n+1} = pa_n$ for all sufficiently large n. In this case the conclusion follows from Corollary 4.1.

Example 6.4. Any bounded non-zero perturbation of $(n!)$ generates a *DD* set (Note that, in view of Theorem 3.2, it does not generate a UDD sequence). In fact, let $a_n = n! + p_n$, where $|p_n| \leqq P$ for each n and $p_n \neq 0$ for all sufficiently large n. Define sets I_m by

$$I_m = \{n \in \mathbf{N} : p_n = p_{n+m}\}, \quad m = 2P+1,\ 2P+2,\ \ldots, (2P+1)^2.$$

Obviously, $\bigcup_{m=2P+1}^{(2P+1)^2} I_m$ is a set with bounded gaps (not exceeding $1 + (2P+1)^2$). Hence there exists an $l \geqq 2P + 1$ such that $\sum_{n \in I_l} 1/n = \infty$. Write $I_l = \{n_k : k \in \mathbf{N}\}$ with (n_k) strictly increasing. For each n we have

$$a_{n+1} - (n+1)a_n = p_{n+1} - (n+1)p_n,$$

so that for each k

$$\begin{aligned} a_{n_k+l+1} &- (n_k + l + 1)a_{n_k+l} - a_{n_k+1} + (n_k+1)a_{n_k} \\ &= p_{n_k+l+1} - (n_k + l + 1)p_{n_k+l} - p_{n_k+1} + (n_k + 1)p_{n_k} \\ &= p_{n_k+l+1} - p_{n_k+1} - lp_{n_k}. \end{aligned} \tag{6.2}$$

From our assumptions it follows that the right hand side of (6.2) is non-zero. Also

$$\sum_{k=1}^{\infty} 1/(n_k + l + 1) = \infty.$$

By Theorem 6.1, this implies our claim.

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