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Parabolic Note: Co-Normal Points.

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1. If the coordinates of a point on a parabola,

$$y^2 - 4ax = 0,$$

be $(am^2, 2am)$, in which I call m the parameter, then the equations to the tangent and normal at the point are

$$x - my + am^2 = 0 \quad \ldots \quad \ldots \quad \text{(i.)}$$

and $$mx + y - a(m^3 + 2m) = 0 \quad \ldots \quad \ldots \quad \text{(ii.)},$$

and to the chord through (m), (m') is

$$y(m + m') - 2x - 2amm' = 0 \quad \ldots \quad \ldots \quad \text{(iii.)}.$$

If we write (ii.) in the form

$$am^3 + (2a - x)m - y = 0 \quad \ldots \quad \ldots \quad \text{(iv.)},$$

we see that from a given point (x, y) we can draw three normals to the curve with the condition

$$\Sigma m = 0.$$

Let O be the point (x, y), and $P(m_1)$, $Q(m_2)$, $R(m_3)$ the corresponding points on the parabola: then I call these latter *co-normal* points, and the circle through them a *co-normal* circle.

2. We have
$$S_1 \equiv \Sigma m = 0,$$
$$S_2 \equiv \Sigma m^2 = -2\Sigma m_1 m_2,$$
$$S_3 = 3m_1 m_2 m_3 = 3\mu,$$
$$S_4 = S_2^2/2\,;$$
also $$m_1^2 - m_2 m_3 = m_1^2 + m_1 m_2 + m_2^2 = \cdots = S_2/2.$$

3. In the case when P, Q, R are *any* three points on the curve the circle PQR is

$$x^2 + y^2 - ax[S_2 + \Sigma m_1 m_2 + 4] + ay[S_1 \,.\, \Sigma m_1 m_2 - \mu]/2 - a^2\mu S_1 = 0 \quad \ldots \quad \text{(i.)}$$

and the tangent-circle pqr is

$$x^2 + y^2 - ax[1 + \Sigma m_1 m_2] - ay[S_1 - \mu] + a^2\Sigma m_1 m_2 = 0, \quad \ldots \quad \text{(ii.)}$$

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If the points are co-normal points, then these equations take the form

$$x^2+y^2-ax(S_2+8)/2-ay\mu/2=0, \quad \ldots \quad \ldots \text{(iii)}$$

$$x^2+y^2+ax(S_2-2)/2+ay\mu-a^2S_2/2=0 \ldots \quad \ldots \text{(iv.)}.$$

4. The co-ordinates of p (§ 3) for co-normal points, are

$$(am_2m_3, \ -am_1).^*$$

5. Through P, Q, R draw parallels to

(i.) the tangents at Q, R, P;

(ii.) ,, R, P, Q;

and let P_q, Q_r, R_p; P_r, Q_p, R_q be the points where the sets (i.), (ii.) respectively meet the parabola (C.P. § 19). If PP_q, QQ_p meet in R_r, and in like manner for the other pairs, then R_r is given by $a(m_1^2+m_2^2-m_1m_2)$, $-am_3$. (C.P. § 24.)

Then the area of $P_pQ_qR_r$

$$=\pm\frac{a^2}{2}\begin{vmatrix} 1 & 1 & 1 \\ S_2/_2-2m_2m_3 & S_2/_2-2m_3m_1 & S_2/_2-2m_1m_2 \\ m_1 & m_2 & m_3 \end{vmatrix} = \text{area of } \triangle PQR;$$

and the equation to the circle is

$$x^2+y^2+(1-4S_2)ax/2+4\mu ay+3(S_2^2-S_2)a^2/4=0.$$

Again, since the coordinates of midpoint of Q_qR_r are

$$a(S_2/_2+m_1^2),\ am_1/2,$$

the N.P. circle of the triangle is given by

$$x^2+y^2-(6S_2+1)ax/4-2\mu ay+(4S_2^2+S_2)a^2/8=0.$$

6. If we draw Pr_1, Qr_1 parallel to the normals at Q, P, since Pr_1 is given by

$$y+m_2x=2am_1+am_1^2m_2,$$

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* The first four articles in the text are taken from my paper, entitled *Some Properties of Co-normal Points on a Parabola* (*Proceedings of London Mathematical Society*, vol. xxi., pp. 442-451. Subsequent references are to the sections of this paper, *C.P.*

we see that r_1 is given by

$$-a(2+m_1m_2), \quad -a(2m_3+\mu),$$

hence $\triangle p_1q_1r_1 = \pm\frac{1}{2}a^2 \begin{vmatrix} 1 & 1 & 1 \\ 2+m_2m_3 & 2+m_3m_1 & 2+m_1m_2 \\ \mu+2m_1 & \mu+2m_2 & \mu+2m_3 \end{vmatrix} = \triangle PQR;$

and since $p_1q_1 = PQ$, the triangles are congruent.

The equation to Ar_1 is $\quad y = m_3x,$
hence Ar_1 and the normal at R make equal angles with the axis.

The circle $p_1q_1r_1$ is given by

$$x^2+y^2-(8-S_2/2)ax+5a\mu y/2+3a^2(\mu^2-2S_2+8)/2=0;$$

and the N.P. circle is given by

$$x^2+y^2+(2-S_2/4)ax+7a\mu y/4-a^2(2S_2-3\mu^2)/4=0.$$

7. The equations to PP_p, QQ_q (§ 6) are

$$m_1x-m_2m_3y=am_1(m_1^2-2m_2m_3),$$
$$m_2x-m_3m_1y=am_2(m_2^2-2m_3m_1),$$

hence they intersect in r_2, on the ordinate of R, given by

$$am_3^2, \quad -am_1m_2/m_3.$$

Similarly for the analogous points p_2, q_2.

Hence $\qquad \triangle p_2q_2r_2 = \frac{1}{2}\triangle PQR.$

The circle $p_2q_2r_2$ is given by

$$x^2+y^2-(S_2+1)ax+(\mu+S_2^2/4\mu)ay+a^2(4S_2+S_2^2)/4=0. \quad \ldots \quad \text{(i.)}$$

The points $p_2q_2r_2$, lie on the rectangular hyperbola

$$xy=-\mu a^2 \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii.)}$$

which cuts (i.) again in $\quad (a, -a\mu)$.

The equation to the perpendicular from r_2 on p_2q_2 is

$$m_3y+m_1m_2x=am_1m_2(m_3^2-1),$$

hence the orthocentre of $p_2q_2r_2$ is $(-a, a\mu)$.

This point is on (ii) and *coincides with the orthocentre* of pqr (C.P. §13).

The N.P. circle of $p_2q_2r_2$ is the co-normal circle

$$x^2+y^2+ax(1-S_2)/2+ay(S_2^2-4\mu^2)/8\mu=0.$$

The radical axis of this circle and of PQR is

$$36\mu^2x+S_2^2y=0.$$

The tangent from the focus to (i.) is $aS_2/2$.

The equation to pp_2 is

$$S_2x-6\mu y=2a(m_1^4-m_1\mu+3m_2^2m_3^2),$$

hence pp_2, qq_2, rr_2 are parallel.

Also the equation to p_2q_2 is

$$m_1m_2y-m_3x=-a(m_1^2+m_2^2)m_3,$$

i.e., the line is parallel to Rr.

The points p, q, r lie on the hyperbola (ii.): hence we see otherwise that the orthocentres of pqr, $p_2q_2r_2$ coincide, and that the two circles cut the Latus Rectum in the same point $(a, -a\mu)$, the join of which with the common orthocentre is a diameter of the hyperbola.

8. The orthocentre of $p_1q_1r_1$ (§6) being $(2a, -a\mu/2)$ is on the hyperbola (§7, ii.). See C.P. §12.

From §11 of C.P. we see that the centre of perspective of the triangles PQR, pqr, viz. $(aS_2/6, -6a\mu/S_2)$ is also on the same curve.

9. If the sides QR, RP, PQ produced cut the diameters through P, Q, R in L, M, N these points are given by

$$L, [a(m_1^2+m_2m_3), 2am_1], \text{ etc.};$$

hence $$\triangle LMN=2\triangle PQR.$$

The circle LMN has for its equation

$$x^2+y^2-2ax-2a\mu y-a^2(S_2^2+4S_2)/4=0.$$

The orthocentre of LMN is

$$a(S_2-4)/2, \ -2a\mu.$$

If n is the midpoint of LM, it is given by

$$(-am_1m_2, \ -am_3)$$

and therefore it and the analogous points l, m, lie on the rectangular hyperbola $xy=\mu a^2$. ... ... (i.)

From the above we see that pl, qm, rn are diameters of the parabola, and lm, pq; mn, qr; nl, rp; intersect on the tangent at the vertex and are isoclinals to it.

The equation to the circle lmn is

$$x^2+y^2+ax(2-S_2)/2+a\mu y-a^2S_2/2=0;$$

and it is therefore § 3 (iv.) equal to the circle pqr.

The perpendiculars from l, m, n on QR, RP, PQ respectively meet in $(2a, a\mu/2)$, which is on (i.); and the perpendiculars from P, Q, R on mn, nl, lm meet in $[a(S_2-4)/2, -a\mu]$, *i.e.* O′ of C.P. § 15.

10. If the join of P to the midpoint of QR cuts the parabola in p_3, the parameter of this point is $-S_2/3m_1$, hence the corresponding tangent circle of the triangle $p_3q_3r_3$ is given by

$$x^2+y^2-ax-ayS_2^2(9+2S_2)/54\mu=0.$$

The vertices of this tangent triangle are

$$(aS_2^2/9m_1m_2, \ am_3S_2/3m_1m_2)$$

so that its centroid is $(0, \ -S_2^2/18\mu)$;

and its orthocentre $(-a, \ 7aS_2^3/54\mu)$.

11. Through P, Q, R draw lines parallel to QR, RP, PQ respectively, these lines meet the parabola in the co-normal points, whose parameters are $-2m_1$, $-2m_2$, $-2m_3$; and the lines cut one another in

$$(-2am_2m_3, \ -4am_1), \ (-2am_3m_1, \ -4am_2), \ (-2am_2m_3, \ -4am_3).$$

Take the images of these points in the vertex, viz $(2am_1m_3, 4am_1)$, etc., and we find its circumcircle to be given by

$$x^2 + y^2 - (8 - S_2)ax - a\mu y - 8S_2a^2 = 0,$$

the centre of which is the orthocentre of PQR (C.P. § 13.)

12. The lines QR, AP cut in $p_a\,(-am_2m_3/2, -am_2m_3/m_1)$, RP, AQ in q_a, and PQ, AR in r_a; hence $p_aq_ar_a$, which is the *central* triangle of the quadrilateral APQR,

$$= \tfrac{1}{2}\triangle PQR.$$

The circle $p_aq_ar_a$ has for its equation

$$x^2 + y^2 + 2ax + ay(S_2^2/\mu^2)/4\mu + a^2S_2/2 = 0.$$

The equation to p_aq_a is

$$m_1m_2y + 2m_3x = +a\mu.$$

13. If (cf. C.P. §29) we draw lines from P, Q, R through the point, $x = ka$ on the axis to cut the curve in T_1', T_2', T_3', then as T_1' is given by $(-k/m_1)$ the equation to the tangent-circle for $T_1'T_2'T_3'$ will differ from that to the tangent-circle for $T_1T_2T_3$ only in the sign of k, *i.e.*, it will be

$$x^2 + y^2 - ax - ay(k\,.\,S_2 + 2k^3)/2\mu = 0.$$

14. If through p, q, r we draw the corresponding diameters, the vertices of these diameters are co-normal points, viz.,

$$(am_1^2/4, \; -am_1), \text{ etc.},$$

and the co-normal circle through the vertices is

$$x^2 + y^2 - ax(S_2 + 32)/8 + ay\mu/16 = 0.$$

15. Parallels through p to QR and through q parallel to RP intersect on the diameter through r.

16. Parallels through P to AQ, AR, meet the parabola in points whose parameters are

$$(m_2 - m_1), \; (m_3 - m_1),$$

hence we get two sets of co-normal points.

The equations to the co-normal circles are

$$x^2 + y^2 - ax(3S_2 + 8)/2 \mp ayk/2 = 0,$$

where $$k \equiv m_2 - m_1 \,.\, m_3 - m_1 \,.\, m_1 - m_2.$$

17. The median of PQR which passes through P cuts the parabola in the point whose parameter is $(-S_2/3m_1)$, hence the corresponding tangent-circle has for its equation

$$x^2 + y^2 - ax - ayS_2^2(9 + 2S_2)/54\mu = 0.$$

18. If in § 12 q_a, r_a are outside the curve, then the midpoints of QR, AP, $q_a r_a$ are given by

$a(m_2^2 + m_3^2)/2, \ -am_1$; $am_1^2/2, \ am_1$; $am_1^2/4, \ -am_1(m_2^2 + m_3^2)/2m_2m_3$;

hence the *central* axis of APQR is

$$-2m_2m_3y + 4m_1x = m_1S_2a.$$

19. The poles of the co-normal chords are

$-a(m_1^2 + 2), \ -2a/m_1$; $-a(m_2^2 + 2), \ -2a/m_2$; $-a(m_3^2 + 2), \ -2a/m_3$.

These poles lie upon the line

$$\mu y - 2x = a(S_2 + 4). \qquad \text{(cf. C.P. § 17.)}$$

The diameters through the poles meet the curve in

$$a/m_1^2, \ -2a/m_1 \ ; \text{ etc. };$$

hence the circle through the vertices of these diameters is

$$x^2 + y^2 - ax(S_2^2 + 4\mu^2)/\mu^2 + ay/2\mu + a^2S_2/2\mu^2 = 0 \ ;$$

and the corresponding tangent-circle is

$$x^2 + y^2 - ax - ay(S_2 + 2)/2\mu = 0.$$

The sides of this last triangle are

$$m_1y - 2m_2m_3x = 2a, \text{ etc.},$$

∴ the perpendiculars are

$$m_1^2x + 2\mu y = a(1 - 4m_2m_3), \text{ etc.},$$

whence the orthocentre is

$$[-4a, \ a(1 + 2S_2)/2\mu].$$