Proof: The implication $\Leftarrow$ is evident (Figures 2 and 4). In order to show the implication $\Rightarrow$ we can argue as follows: Since the P-Lemoine axis is always orthogonal to the straight line PP' (Figure 4) passing through the circumcentre O, to satisfy the property on the left-hand side of the Lemma P must lie on the straight line $OX(15)$. Since P and P' are *inverse* points with respect to the circumcircle it is easy to see that the midpoint of PP' (i.e. Q' in Figure 4) is the nearer to the circumcircle the nearer P lies to it (in other words: on the line $OX(15)$ the distance of Q' to the circumcircle increases *monotonically* with the distance of P from the circumcircle, in both directions). Therefore, there are exactly two possible positions for P, namely $X(15)$ and $X(16)$.

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108.18 A one-line proof of the Finsler-Hadwiger inequality

Every proof is a one-line proof if you start sufficiently far to the left, [1].

The *Finsler-Hadwiger inequality* asserts that, in the triangle ABC with side-lengths a, b, c and area Δ,

$$\sum a^2 \geqslant \sum (b - c)^2 + 4\sqrt{3}\Delta, \tag{1}$$

with equality if, and only if, triangle ABC is equilateral.

In this short Note, we adapt an idea from [2] to give a very quick proof.

Let r and R denote the inradius and circumradius of triangle ABC, let r_A be the radius of the excircle opposite A which touches BC (with r_B, r_C similarly defined) and, as usual, let $s = \frac{1}{2}(a + b + c)$.

Then $\sum r_A = r + 4R$, $\sum r_A r_B = s^2$, $r_A r_B r_C = rs^2$: these are reasonably standard triangle identities but, for the convenience of readers, we outline their proofs.

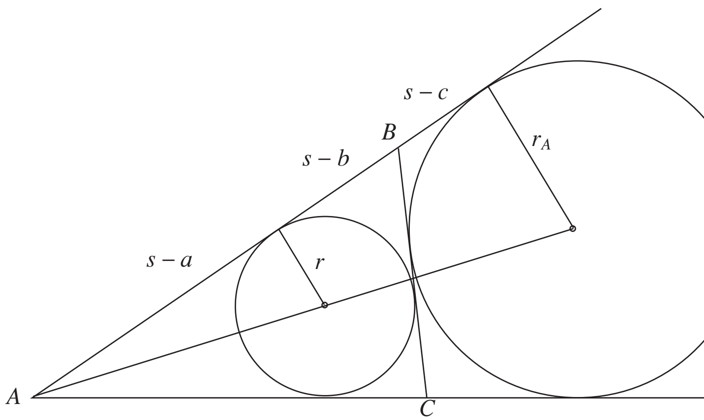

FIGURE: Similar triangles give $\frac{r}{s - a} = \frac{r_A}{s}$

From the Figure, $r_A = \dfrac{rs}{s - a}$. This gives

- $r_A r_B r_C = \dfrac{r^3 s^3}{(s-a)(s-b)(s-c)} = rs^2$, since $rs = \Delta = \sqrt{s(s-a)(s-b)(s-c)}$;
- $\sum r_A r_B = \sum \dfrac{r^2 s^2}{(s-a)(s-b)} = \dfrac{r^2 s^2}{(s-a)(s-b)(s-c)} \cdot \sum (s-a) = s.s = s^2$;
- $\sum r_A = \dfrac{rs}{(s-a)(s-b)(s-c)} \cdot \sum (s-a)(s-b) = \dfrac{1}{r}. \sum (s-a)(s-b)$.

Now, on the one hand,

$$\sum (s - a)(s - b) = \sum \left[s^2 - s(a + b) + ab\right] = \sum ab - s^2, \quad (2)$$

but we also have

$$r^2 s = \frac{\Delta^2}{s} = (s - a)(s - b)(s - c) = s^3 - (\Sigma a) s^2 + (\Sigma ab) s - abc$$

$$= -s^3 + (\Sigma ab) s - 4rRs,$$

and hence

$$\sum ab - s^2 = r^2 + 4rR. \quad (3)$$

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From (2) and (3), $\sum r_A = \frac{1}{r} \cdot \left(r^2 + 4rR\right) = r + 4R$.

It follows that r_A, r_B, r_C are the roots of the cubic

$$f(X) = X^3 - (r + 4R)X^2 + s^2X - rs^2 = 0.$$

In [2, p. 206], the discriminant of this cubic is used to prove the triangle inequalities of Rouché and Blundon. Here we simply observe that, because the graph $y = f(X)$ always has stationary points, the discriminant of the quadratic $f'(X) = 3X^2 - 2(r + 4R)X + s^2$ is non-negative. It follows that $(r + 4R)^2 \geqslant 3s^2$ or $r + 4R \geqslant \sqrt{3}s$. This completes the proof because the latter is a known equivalent of (1). Indeed, $\Delta = rs$ and, using (2) and (3),

$$\sum a^2 - \sum (b - c)^2 = \sum (a - b + c)(a + b - c)$$

$$= 4\sum \left(s - b\right)\left(s - c\right) = 4\left(r^2 + 4rR\right).$$

An alternative to using the discriminant is to employ the standard inequality for three variables, $\sum r_A^2 \geqslant \sum r_A r_B$. This gives

$$\left(\sum r_A\right)^2 = \sum r_A^2 + 2\sum r_A r_B \geqslant 3\sum r_A r_B,$$

from which $(r + 4R)^2 \geqslant 3s^2$ again follows.

We also note that the inequality of arithmetic and harmonic means gives $\frac{1}{3}\sum r_A \geqslant \frac{3r_A r_B r_C}{\sum r_A r_B}$ or $\frac{1}{3}(r + 4R) \geqslant \frac{3rs^2}{s^2}$ which rearranges to Euler's inequality, $R \geqslant 2r$.

Finally, it is a remarkable fact that, although the Finsler-Hadwiger inequality seems clearly stronger than the famous Weitzenböck inequality $\sum a^2 \geqslant 4\sqrt{3}\Delta$, it is actually equivalent to it, [2, 3].

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