ON THE RANK OF THE SUM OF TWO RECTANGULAR MATRICES

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The purpose of this note is to present a short proof for the following theorem.

THEOREM. *Let* A *and* B *be two complex* $m \times n$ *matrices. If* $B^*A = 0$ *and* $AB^* = 0$ *then* $\operatorname{rank}(A + B) = \operatorname{rank}(A) + \operatorname{rank}(B)$.

Proof. Let $A^\dagger$ and $B^\dagger$ be the generalized inverses of A and B, respectively, in the sense of Penrose [1]. Now,

$$\begin{aligned}
B^*A &= 0 \Rightarrow (B^*B)^\dagger B^*A = 0 \Rightarrow B^\dagger A = 0 \\
B^*A &= 0 \Rightarrow A^*B = 0 \\
AB^* &= 0 \Rightarrow AB^*(BB^*)^\dagger = 0 \Rightarrow AB^\dagger = 0 \\
AB^* &= 0 \Rightarrow BA^* = 0 \Rightarrow BA^*(AA^*)^\dagger = 0 \Rightarrow BA^\dagger = 0 .
\end{aligned}$$

Using these together with the fact that $A^*AA^\dagger = A^*$, we may write

$$\begin{bmatrix} A^* \\ BB^\dagger \end{bmatrix} (A + B) \; [A^\dagger \quad B^\dagger B] = \begin{bmatrix} A^* & 0 \\ 0 & B \end{bmatrix} .$$

Therefore,

$$\operatorname{rank}(A) + \operatorname{rank}(B) = \operatorname{rank}(A^*) + \operatorname{rank}(B) = \operatorname{rank} \begin{bmatrix} A^* & 0 \\ 0 & B \end{bmatrix} \leq \operatorname{rank}(A + B).$$

Since it is always true that $\operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$, we have $\operatorname{rank}(A + B) = \operatorname{rank}(A) + \operatorname{rank}(B)$.

REFERENCE

1. R. Penrose, A generalized inverse for matrices. Proc. Cambridge Philos. Soc. 51 (1955) 406-413.

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