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# NEF VECTOR BUNDLES ON A QUADRIC THREEFOLD WITH FIRST CHERN CLASS TWO

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*Abstract* We classify nef vector bundles on a smooth hyperquadric of dimension three with first Chern class two over an algebraically closed field of characteristic zero. In particular, we see that they are globally generated.

*Keywords:* nef vector bundles; Fano bundles; full strong exceptional collections

*2020 Mathematics subject classification:* Primary 14J60;
Secondary 14J45; 14F08

## 1. Introduction

In [17, §2 Theorem 2], Peternell–Szurek–Wiśniewski classified nef vector bundles on a smooth hyperquadric $\mathbb{Q}^n$ of dimension $n \geq 3$ with first Chern class $\leq 1$ over an algebraically closed field $K$ of characteristic zero. In [12, Theorem 9.3], we provided a different proof of this classification, which was based on an analysis with a full strong exceptional collection of vector bundles on $\mathbb{Q}^n$.

In this paper, we classify nef vector bundles on a smooth quadric threefold $\mathbb{Q}^3$ with first Chern class two. (In the subsequent paper [14], we classify those on a smooth hyperquadric $\mathbb{Q}^n$ of dimension $n \geq 4$.) The precise statement is as follows.

**Theorem 1.1.** *Let $\mathcal{E}$ be a nef vector bundle of rank $r$ on a smooth hyperquadric $\mathbb{Q}^3$ of dimension 3 over an algebraically closed field $K$ of characteristic zero, and let $\mathcal{S}$ be the spinor bundle on $\mathbb{Q}^3$. Suppose that $\det \mathcal{E} \cong \mathcal{O}(2)$. Then $\mathcal{E}$ is isomorphic to one of the following vector bundles or fits in one of the following exact sequences:*

(1) $\mathcal{O}(2) \oplus \mathcal{O}^{\oplus r-1}$;
(2) $\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-2}$;
(3) $\mathcal{O}(1) \oplus \mathcal{S} \oplus \mathcal{O}^{\oplus r-3}$;
(4) $0 \to \mathcal{O}(-1) \to \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r} \to \mathcal{E} \to 0$;
(5) $0 \to \mathcal{O}^{\oplus a} \to \mathcal{S}^{\oplus 2} \oplus \mathcal{O}^{\oplus r-4+a} \to \mathcal{E} \to 0$, *where* $a = 0$ *or* 1, *and the composite of the injection* $\mathcal{O}^{\oplus a} \to \mathcal{S}^{\oplus 2} \oplus \mathcal{O}^{\oplus r-4+a}$ *and the projection* $\mathcal{S}^{\oplus 2} \oplus \mathcal{O}^{\oplus r-4+a} \to \mathcal{O}^{\oplus r-4+a}$ *is zero;*
(6) $0 \to \mathcal{S}(-1) \oplus \mathcal{O}(-1) \to \mathcal{O}^{\oplus r+3} \to \mathcal{E} \to 0$;
(7) $0 \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}^{\oplus r+2} \to \mathcal{E} \to 0$;
(8) $0 \to \mathcal{O}(-2) \to \mathcal{O}^{\oplus r+1} \to \mathcal{E} \to 0$;
(9) $0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^{\oplus 4} \to \mathcal{O}^{\oplus r+3} \to \mathcal{E} \to 0$.

Note that this list is effective: in each case exists an example. For example, if we denote by $\mathcal{N}$ a null correlation bundle on $\mathbb{P}^3$, then $\pi_p^*(\mathcal{N}(1))$ belongs to Case (9) of Theorem 1.1, where $\pi_p : \mathbb{Q}^3 \to \mathbb{P}^3$ is the projection from a point $p \in \mathbb{P}^4 \setminus \mathbb{Q}^3$. (Similarly, $\pi_p^*(\Omega_{\mathbb{P}^3}(2))$ belongs to Case (9) of Theorem 1.1.) Under the stronger assumption that $\mathcal{E}$ is globally generated, Ballico–Huh–Malaspina provided a classification of $\mathcal{E}$ on $\mathbb{Q}^3$ with $c_1 = 2$ in [3] and [2].

Note also that the projectivization $\mathbb{P}(\mathcal{E})$ of the bundle $\mathcal{E}$ in Theorem 1.1 is a Fano manifold of dimension $r+2$, i.e. the bundle $\mathcal{E}$ in Theorem 1.1 is a Fano bundle on $\mathbb{Q}^3$ of rank $r$. As a related result, Langer classified smooth Fano 4-folds with adjunction theoretic scroll structure over $\mathbb{Q}^3$ in [10, Theorem 7.2].

Our basic strategy and framework for describing $\mathcal{E}$ in Theorem 1.1 is to give a minimal locally free resolution of $\mathcal{E}$ in terms of some twists of the full strong exceptional collection

$$(\mathcal{O}, \mathcal{S}, \mathcal{O}(1), \mathcal{O}(2))$$

of vector bundles (see [12] for more details).

The content of this paper is as follows. In § 2, we briefly recall Bondal's theorem [1, Theorem 6.2] and its related notions and results required in the proof of Theorem 1.1. In particular, we recall some finite-dimensional algebra $A$ and fix some symbols, e.g. $G$, $P_i$ and $S_i$, related to $A$ and to finitely generated right $A$-modules. We also recall the classification [13, Theorem 1.1] of nef vector bundles on a smooth quadric surface $\mathbb{Q}^2$ with Chern class $(2, 2)$ in Theorem 2.3. In § 3, we recall some basic properties of the spinor bundle $\mathcal{S}$ on $\mathbb{Q}^3$. In § 4, we state Hirzebruch–Riemann–Roch formulas for vector bundles $\mathcal{E}$ on $\mathbb{Q}^3$ with $c_1 = 2$ and for $\mathcal{S}^\vee \otimes \mathcal{E}$. In § 5, we show some key lemmas required later in the proof of Theorem 1.1. In § 6, we provide a lower bound for the third Chern class of a nef vector bundle $\mathcal{E}$, if $h^0(\mathcal{E}(-D)) \neq 0$ for some effective divisor $D$. In § 7, we provide the set-up for the proof of Theorem 1.1. The proof of Theorem 1.1 is carried out in § 8–19, according to which case of Theorem 2.3 $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to.

### 1.1. Notation and conventions

Throughout this paper, we work over an algebraically closed field $K$ of characteristic zero. Basically, we follow the standard notation and terminology in algebraic geometry.

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We denote by $\mathbb{Q}^3$ a smooth quadric threefold over $K$, by $\mathbb{Q}^2$ a smooth quadric surface over $K$ and by

- $\mathcal{S}$ the spinor bundle on $\mathbb{Q}^3$.

Note that we follow Kapranov's convention [9, p. 499]; our spinor bundle $\mathcal{S}$ is globally generated, and it is the dual of that of Ottaviani's [16]. For a coherent sheaf $\mathcal{F}$, we denote by $c_i(\mathcal{F})$ the $i$th Chern class of $\mathcal{F}$ and by $\mathcal{F}^\vee$ the dual of $\mathcal{F}$. In particular,

- $c_i$ stands for $c_i(\mathcal{E})$ of the nef vector bundle $\mathcal{E}$ we are dealing with.

For a vector bundle $\mathcal{E}$, $\mathbb{P}(\mathcal{E})$ denotes $\operatorname{Proj} S(\mathcal{E})$, where $S(\mathcal{E})$ denotes the symmetric algebra of $\mathcal{E}$. The tautological line bundle

- $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is also denoted by $H(\mathcal{E})$.

Let $A^*\mathbb{Q}^3$ be the Chow ring of $\mathbb{Q}^3$. We denote

- by $H$ a hyperplane section of $\mathbb{Q}^3$ and by $h$ its class in $A^1\mathbb{Q}^3$: $A^1\mathbb{Q}^3 = \mathbb{Z}h$;
- by $L$ a line in $\mathbb{Q}^3$ and by $l$ its class in $A^2\mathbb{Q}^3$: $A^2\mathbb{Q}^3 = \mathbb{Z}l$.

Note that $h^2 = 2l$. Via the map $\deg : A^3\mathbb{Q}^3 \cong \mathbb{Z}$, we identify elements $A^3\mathbb{Q}^3$ with its corresponding integer; thus, we have $h^3 = 2$ and $hl = 1$. For any closed subscheme $Z$ in $\mathbb{Q}^3$, $\mathcal{I}_Z$ denotes the ideal sheaf of $Z$ in $\mathbb{Q}^3$; for a point $p \in \mathbb{Q}^3$, $\mathcal{I}_p$ denotes the ideal sheaf of $p \in \mathbb{Q}^3$ and $k(p)$ denotes the residue field of $p \in \mathbb{Q}^3$. For coherent sheaves $\mathcal{F}$ and $\mathcal{G}$, we set

- $\operatorname{ext}^q(\mathcal{F}, \mathcal{G}) = \dim \operatorname{Ext}^q(\mathcal{F}, \mathcal{G})$;
- $\hom(\mathcal{F}, \mathcal{G}) = \dim \operatorname{Hom}(\mathcal{F}, \mathcal{G})$.

Finally we refer to [11] for the definition and basic properties of nef vector bundles.

## 2. Preliminaries

Throughout this paper, $G_0$, $G_1$, $G_2$, $G_3$ denote respectively $\mathcal{O}$, $\mathcal{S}$, $\mathcal{O}(1)$, $\mathcal{O}(2)$ on $\mathbb{Q}^3$. An important and well-known fact [9, Theorem 4.10] of the collection $(G_0, G_1, G_2, G_3)$ is that it is a full strong exceptional collection in $D^b(\mathbb{Q}^3)$, where $D^b(\mathbb{Q}^3)$ denotes the bounded derived category of (the abelian category of) coherent sheaves on $\mathbb{Q}^3$. Here we use the term 'collection' to mean 'family', not 'set'. Thus, an exceptional collection is also called an exceptional sequence. We refer to [7] for the definition of a full strong exceptional sequence.

Denote by $G$ the direct sum $\bigoplus_{i=0}^3 G_i$ of $G_0$, $G_1$, $G_2$ and $G_3$, and by $A$ the endomorphism ring $\operatorname{End}(G)$ of $G$. The ring $A$ is a finite-dimensional $K$-algebra, and $G$ is a left $A$-module. Note that $\operatorname{Ext}^q(G, \mathcal{F})$ is a finitely generated right $A$-module for a coherent sheaf $\mathcal{F}$ on $\mathbb{Q}^3$. We denote by $\operatorname{mod} A$ the category of finitely generated right $A$-modules and by $D^b(\operatorname{mod} A)$ the bounded derived category of $\operatorname{mod} A$. Let $p_i : G \to G_i$ be the projection, and $\iota_i : G_i \hookrightarrow G$ the inclusion. Set $e_i = \iota_i \circ p_i$. Then $e_i \in A$. Set

$$P_i = e_i A.$$

Then $A \cong \oplus_i P_i$ as right $A$-modules, and $P_i$'s are projective right $A$-modules. We see that $P_i \otimes_A G \cong G_i$. Any finitely generated right $A$-module $V$ has an ascending filtration

$$0 = V^{\leq -1} \subset V^{\leq 0} \subset V^{\leq 1} \subset V^{\leq 2} \subset V^{\leq 3} = V$$

by right $A$-submodules, where $V^{\leq i}$ is defined to be $\bigoplus_{j \leq i} V e_j$. Set $\operatorname{Gr}^i V = V^{\leq i}/V^{\leq i-1}$ and

$$S_i = \operatorname{Gr}^i P_i.$$

Then $\operatorname{Gr}^i S_i \cong K$ as $K$-vector spaces, $\operatorname{Gr}^j S_i = 0$ for any $j \neq i$, and $S_i$ is a simple right $A$-module. If we set $m_i = \dim_K \operatorname{Gr}^i V$, then $\operatorname{Gr}^i V \cong S_i^{\oplus m_i}$ as right $A$-modules.

It follows from Bondal's theorem [1, Theorem 6.2] that

$$\operatorname{RHom}(G, \bullet) : D^b(\mathbb{Q}^3) \to D^b(\operatorname{mod} A)$$

is an exact equivalence, and its quasi-inverse is

$$\bullet \otimes^{\mathrm{L}}{}_A G : D^b(\operatorname{mod} A) \to D^b(\mathbb{Q}^3).$$

For a coherent sheaf $\mathcal{F}$ on $\mathbb{Q}^3$, this fact can be rephrased in terms of a spectral sequence [15, Theorem 1]:

$$E_2^{p,q} = \mathcal{T}or_{-p}^A(\operatorname{Ext}^q(G, \mathcal{F}), G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{F} & \text{if} \quad p+q=0 \\ 0 & \text{if} \quad p+q \neq 0, \end{cases} \tag{2.1}$$

which is called the Bondal spectral sequence. Note that $E_2^{p,q}$ is the $p$th cohomology sheaf $\mathcal{H}^p(\operatorname{Ext}^q(G, \mathcal{F}) \otimes^{\mathrm{L}}{}_A G)$ of the complex $\operatorname{Ext}^q(G, \mathcal{F}) \otimes^{\mathrm{L}}{}_A G$. When we compute the spectral sequence, we consider the ascending filtration on the right $A$-module $\operatorname{Ext}^q(G, \mathcal{F})$ and apply the following

**Lemma 2.1.** *We have*

$$S_3 \otimes^{\mathrm{L}}{}_A G \cong \mathcal{O}(-1)[3]; \tag{2.2}$$

$$S_2 \otimes^{\mathrm{L}}{}_A G \cong T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}[2]; \tag{2.3}$$

$$S_1 \otimes^{\mathrm{L}}{}_A G \cong \mathcal{S}^\vee[1] \cong \mathcal{S}(-1)[1]; \tag{2.4}$$

$$S_0 \otimes^{\mathrm{L}}{}_A G \cong \mathcal{O}, \tag{2.5}$$

*where $T_{\mathbb{P}^4}$ denotes the tangent bundle of $\mathbb{P}^4$.*

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**Proof.** Since $\mathrm{RHom}(G, \mathcal{O}(-1)[3]) \cong S_3$, we obtain (2.2). Note that we have an isomorphism $\mathrm{RHom}(G, \mathcal{S}^\vee[1]) \cong S_1$ by [12, Lemma 8.2 (1)]. Hence we have (2.4). It is easy to see that the last isomorphism (2.5) holds. To see (2.3), first note that we have the following exact sequence:

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1) \otimes H^0(\mathcal{O}(1))^\vee \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to 0.$$

Serre duality shows that

$$H^3(\mathcal{O}(-4)) \to H^3(\mathcal{O}(-3)) \otimes H^0(\mathcal{O}(1))^\vee$$

is dual of the canonical isomorphism

$$H^0(\mathcal{O}) \otimes H^0(\mathcal{O}(1)) \to H^0(\mathcal{O}(1)).$$

Hence $H^q(T_{\mathbb{P}^4}(-4)|_{\mathbb{Q}^3}) = 0$ for all $q$. Moreover, $h^q(\mathcal{S}^\vee(-i)) = 0$ for $i = 0, 1, 2$ and all $q$. Therefore, we conclude that $\mathrm{RHom}(G, T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3})$ is isomorphic to $S_2[-2]$. □

**Remark 2.2.** As the referee pointed out, Lemma 2.1 shows that

$$(\mathcal{O}(-1), T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee, \mathcal{O}) \tag{2.6}$$

is the left dual exceptional collection of $(G_0, G_1, G_2, G_3)$ (see [1] and [5] for the definition and the characterization of the left dual exceptional collection). Moreover, the full exceptional collection above is strong by [4, Proposition 3.3] (or by showing directly that $\mathrm{Ext}^q(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee) = 0$ for any $q > 0$ through the Euler exact sequence).

Our proof of Theorem 1.1 relies on the following theorem [13, Theorem 1.1]:

**Theorem 2.3.** *Let $\mathcal{E}$ be a nef vector bundle of rank $r$ on a smooth quadric surface $\mathbb{Q}^2$ over an algebraically closed field $K$ of characteristic zero. Suppose that $\det \mathcal{E} \cong \mathcal{O}(2,2)$. Then $\mathcal{E}$ is isomorphic to one of the following vector bundles or fits in one of the following exact sequences:*

(1) $\mathcal{O}(2,2) \oplus \mathcal{O}^{\oplus r-1}$;
(2) $\mathcal{O}(2,1) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}^{\oplus r-2}$;
$\mathcal{O}(1,2) \oplus \mathcal{O}(1,0) \oplus \mathcal{O}^{\oplus r-2}$;
(*We do not exhibit the cases obtained by replacing $(a, b)$ with $(b, a)$ in the following:*)
(3) $\mathcal{O}(1,1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-2}$;
(4) $0 \to \mathcal{O} \xrightarrow{\iota} \mathcal{O}(1,1) \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}^{\oplus r-2} \to \mathcal{E} \to 0$;
(5) $0 \to \mathcal{O}(-1,-1) \to \mathcal{O}(1,1) \oplus \mathcal{O}^{\oplus r} \to \mathcal{E} \to 0$;
(6) $0 \to \mathcal{O}^{\oplus 2} \to \mathcal{O}(1,0)^{\oplus 2} \oplus \mathcal{O}(0,1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-2} \to \mathcal{E} \to 0$;
(7) $0 \to \mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1) \to \mathcal{O}^{\oplus r+3} \to \mathcal{E} \to 0$;
(8) $0 \to \mathcal{O}(-1,-2) \to \mathcal{O}(1,0) \oplus \mathcal{O}^{\oplus r} \to \mathcal{E} \to 0$;
(9) $0 \to \mathcal{O}(-1,-1)^{\oplus 2} \to \mathcal{O}^{\oplus r+2} \to \mathcal{E} \to 0$;
(10) $0 \to \mathcal{O}(-2,-2) \to \mathcal{O}^{\oplus r+1} \to \mathcal{E} \to 0$;

(11) $0 \to \mathcal{O}(-2,-2) \to \mathcal{O}^{\oplus r+1} \to \mathcal{E} \to k(p) \to 0$;
(12) $0 \to \mathcal{O}(-2,-2) \to \mathcal{O}^{\oplus r} \to \mathcal{E} \to \mathcal{O} \to 0$;
(13) $0 \to \mathcal{O}(-1,-1)^{\oplus 4} \to \mathcal{O}^{\oplus r} \oplus \mathcal{O}(-1,0)^{\oplus 2} \oplus \mathcal{O}(0,-1)^{\oplus 2} \to \mathcal{E} \to 0$.

## 3. Some basic properties of the spinor bundle $\mathcal{S}$ on $\mathbb{Q}^3$

We recall some basic facts and properties of the spinor bundle $\mathcal{S}$ on $\mathbb{Q}^3$ in our notation (see Ottaviani's result [16] and [12, Theorem 8.1]). First we have an exact sequence

$$0 \to \mathcal{S}^\vee \to \mathcal{O}^{\oplus 4} \to \mathcal{S} \to 0 \tag{3.1}$$

by [16, Theorem 2.8 (1)]. The restriction $\mathcal{S}|_{\mathbb{Q}^2}$ of $\mathcal{S}$ to a smooth hyperplane section $\mathbb{Q}^2$ of $\mathbb{Q}^3$ is isomorphic to $\mathcal{O}(1,0) \oplus \mathcal{O}(0,1)$, and $h^0(\mathcal{S}) = 4$. We have $\det \mathcal{S} = \mathcal{O}(1)$, and thus the canonical isomorphism

$$\mathcal{S}^\vee(1) \cong \mathcal{S}. \tag{3.2}$$

The zero locus $(s)_0$ of every non-zero element $s$ of $H^0(\mathcal{S})$ is a line $l$ in $\mathbb{Q}^3$. Thus $c_1(\mathcal{S}) \cap [\mathbb{Q}^3] = h$ and $c_2(\mathcal{S}) \cap [\mathbb{Q}^3] = l$. We have $h^q(\mathcal{S}) = 0$ for any $q > 0$ and $h^q(\mathcal{S}(-i)) = 0$ for all $q$ if $i = 1, 2$ or 3.

**Lemma 3.1.** *The natural map*

$$H^0(\mathcal{S}) \otimes H^0(\mathcal{S}) \to H^0(\mathcal{O}(1))$$

*sending $s \otimes t$ to $s \wedge t$ is surjective.*

**Proof.** Without loss of generality, we may assume that $\mathbb{Q}^3$ is defined by an equation $X_{01}^2 - X_{02}X_{13} + X_{03}X_{12} = 0$, where $[X_{01} : X_{02} : X_{03} : X_{12} : X_{13}]$ is the homogeneous coordinates of $\mathbb{P}^4$. We may also regard $\mathbb{Q}^3$ as a smooth hyperplane section $H \cap \mathbb{Q}^4$ of a smooth hyperquadric $\mathbb{Q}^4$ defined by an equation $X_{01}X_{23} - X_{02}X_{13} + X_{03}X_{12} = 0$, where $X_{ij}$ $(0 \le i < j \le 3)$ are homogeneous coordinates of $\mathbb{P}^5$, and $H$ is the hyperplane defined by $X_{01} = X_{23}$. Note that $\mathbb{Q}^4$ is the image of the Grassmannian $G(1,3)$ parametrizing lines in $\mathbb{P}^3$ by the Plücker embedding $\iota$. If we represent a point in $G(1,3)$ by a matrix $\begin{bmatrix} x_{10} & x_{11} & x_{12} & x_{13} \\ x_{20} & x_{21} & x_{22} & x_{23} \end{bmatrix}$, then $\iota^* X_{ij} = \begin{vmatrix} x_{1i} & x_{1j} \\ x_{2i} & x_{2j} \end{vmatrix}$. We will identify $\mathbb{Q}^4$ with $G(1,3)$ via $\iota$. Let $H^0(\mathbb{P}^3, \mathcal{O}(1)) \otimes \mathcal{O}_{G(1,3)} \to \mathcal{Q}$ be the universal quotient bundle on $G(1,3)$, which sends homogeneous coordinates $x_j$ of $\mathbb{P}^3$ to global sections $s_j$ of $\mathcal{Q}$ represented by $\begin{bmatrix} x_{1j} \\ x_{2j} \end{bmatrix}$. Recall that $\mathcal{S}$ is the restriction of $\mathcal{U}$ to the hyperplane section $H \cap \mathbb{Q}^4 = \mathbb{Q}^3$. By abuse of notation, we will denote by $s_j$ the restriction of $s_j$ to $\mathbb{Q}^3$. Since $h^0(\mathcal{S}) = 4$, $H^0(\mathcal{S})$ is spanned by $s_0, s_1, s_2, s_3$. Moreover, $H^0(\mathcal{O}(1))$ is spanned by $X_{i,j} = s_i \wedge s_j$, where $(i,j) = (0,1), (0,2), (0,3), (1,2)$ and $(1,3)$. This completes the proof. □

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## 4. Hirzebruch–Riemann–Roch formulas

Let $\mathcal{E}$ be a vector bundle of rank $r$ on $\mathbb{Q}^3$. Since the tangent bundle $T$ of $\mathbb{Q}^3$ fits in an exact sequence

$$0 \to T \to T_{\mathbb{P}^4}|_{\mathbb{Q}^3} \to \mathcal{O}_{\mathbb{Q}^3}(2) \to 0,$$

the Chern polynomial $c_t(T)$ of $T$ is

$$\frac{(1+ht)^5}{1+2ht} = 1 + 3ht + 4h^2t^2 + 2h^3t^3,$$

where $h$ denotes $c_1(\mathcal{O}_{\mathbb{Q}^3}(1))$. Then the Hirzebruch–Riemann–Roch formula implies that

$$\chi(\mathcal{E}) = r + \frac{13}{12}c_1h^2 + \frac{3}{4}(c_1^2 - 2c_2)h + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3),$$

where we set $c_i = c_i(\mathcal{E})$. To compute $\chi(\mathcal{E}(t))$, note that

$$\begin{aligned} c_1(\mathcal{E}(t)) &= c_1 + rth; \\ c_2(\mathcal{E}(t)) &= c_2 + (r-1)tc_1h + \binom{r}{2}t^2h^2; \\ c_3(\mathcal{E}(t)) &= c_3 + (r-2)tc_2h + \binom{r-1}{2}t^2c_1h^2 + \binom{r}{3}t^3h^3. \end{aligned}$$

Since $h^3 = 2$, we infer that

$$\begin{aligned} \chi(\mathcal{E}(t)) =& \frac{r}{3}t^3 + \frac{1}{2}(c_1h^2 + 3r)t^2 + \frac{1}{2}\{3c_1h^2 + (c_1^2 - 2c_2)h + \frac{13}{3}r\}t \\ &+ r + \frac{13}{12}c_1h^2 + \frac{3}{4}(c_1^2 - 2c_2)h + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3). \end{aligned} \tag{4.1}$$

Since $c_1(\mathcal{E}) = dh$ for some integer $d$, the formula above can be written as

$$\begin{aligned} \chi(\mathcal{E}(t)) =& \frac{r}{6}(2t+3)(t+2)(t+1) + dt^2 + (d^2 + 3d)t - c_2ht \\ &+ \frac{d}{6}(2d^2 + 9d + 13) + \frac{1}{2}\{c_3 - (d+3)c_2h\}. \end{aligned} \tag{4.2}$$

In this paper, we are dealing with the case $d = 2$:

$$\chi(\mathcal{E}(t)) = \frac{r}{6}(2t+3)(t+2)(t+1) + 2t^2 + 10t + 13 - c_2ht + \frac{1}{2}\{c_3 - 5c_2h\}. \tag{4.3}$$

In particular,

$$\chi(\mathcal{E}(-1)) = 5 - \frac{3}{2}c_2h + \frac{1}{2}c_3; \tag{4.4}$$

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$$\chi(\mathcal{E}(-2)) = 1 - \frac{1}{2}c_2h + \frac{1}{2}c_3. \tag{4.5}$$

Next we will compute $\chi(\mathcal{S}^\vee \otimes \mathcal{E}(t))$. Recall that $c_1(\mathcal{S}) = h$ and that $c_1(\mathcal{S})c_2(\mathcal{S}) = 1$. Note also that

$$\begin{aligned}
\operatorname{rank} \mathcal{S}^\vee \otimes \mathcal{E} &= 2r;\\
c_1(\mathcal{S}^\vee \otimes \mathcal{E}) &= 2c_1 - rh;\\
c_2(\mathcal{S}^\vee \otimes \mathcal{E}) &= 2c_2 - (2r-1)c_1h + c_1^2 + \binom{r}{2}h^2 + rc_2(\mathcal{S});\\
c_3(\mathcal{S}^\vee \otimes \mathcal{E}) &= 2c_3 - 2(r-1)c_2h + (r-1)^2c_1h^2 + 2(r-1)c_1c_2(\mathcal{S})\\
&\quad +2c_1c_2 - (r-1)c_1^2h - \frac{1}{3}r(r^2-1).
\end{aligned}$$

The formula (4.1) together with the formulas above implies the following formula:

$$\begin{aligned}
\chi(\mathcal{S}^\vee \otimes \mathcal{E}(t)) &= \frac{2}{3}rt^3 + (c_1h^2 + 2r)t^2 + \{2c_1h^2 + (c_1^2 - 2c_2)h + \frac{4}{3}r\}t\\
&\quad + \frac{7}{6}c_1h^2 + c_1^2h - 2c_2h + \frac{1}{3}c_1^3 + c_3 - c_1c_2 - c_1c_2(\mathcal{S}).
\end{aligned}$$

Since $c_1 = dh$, the formula above becomes the following formula:

$$\begin{aligned}
\chi(\mathcal{S}^\vee \otimes \mathcal{E}(t)) &= \frac{2}{3}rt(t+1)(t+2) + 2dt^2 + 2d(d+2)t\\
&\quad + \frac{2}{3}d(d+1)(d+2) - (2t+d+2)c_2h + c_3.
\end{aligned} \tag{4.6}$$

For the case $d = 2$, we have

$$\chi(\mathcal{S}^\vee \otimes \mathcal{E}(t)) = \frac{2}{3}rt(t+1)(t+2) + 4(t+2)^2 - 2(t+2)c_2h + c_3. \tag{4.7}$$

In particular,

$$\chi(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 4 - 2c_2h + c_3. \tag{4.8}$$

## 5. Key lemmas

**Lemma 5.1.** *We have the following exact sequence on* $\mathbb{Q}^3$*:*

$$0 \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to \mathcal{S}^\vee \otimes \operatorname{Hom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee)^\vee \to \Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3} \to 0, \tag{5.1}$$

*where the injection* $T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to \mathcal{S}^\vee \otimes \operatorname{Hom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee)^\vee$ *is the coevaluation morphism. Moreover,* $\dim \operatorname{Hom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee)^\vee = 4$.

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**Proof.** The following simplified proof is due to the referee. As we have seen in Remark 2.2,

$$(\mathcal{O}(-1), T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee, \mathcal{O}) \tag{5.2}$$

is a full strong exceptional collection of $D^b(\mathbb{Q}^3)$. Since this is strong, the right mutation $\mathbf{R}_{\mathcal{S}^\vee}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3})$ of $T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}$ over $\mathcal{S}^\vee$ fits in the following distinguished triangle:

$$T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to \mathcal{S}^\vee \otimes \operatorname{Hom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee)^\vee \to \mathbf{R}_{\mathcal{S}^\vee}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}) \to .$$

Now consider the mutated full exceptional collection

$$(\mathcal{O}(-1), \mathcal{S}^\vee, \mathbf{R}_{\mathcal{S}^\vee}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}), \mathcal{O}). \tag{5.3}$$

Note here that

$$\operatorname{Ext}^q(\mathcal{S}^\vee, \mathbf{R}_{\mathcal{S}^\vee}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3})) = 0 \text{ for } q \neq 0. \tag{5.4}$$

Indeed, by taking $\operatorname{RHom}(\mathcal{S}^\vee, \bullet)$ with the triangle above, we see that $\operatorname{Hom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee)^\vee$ is isomorphic to $\operatorname{RHom}(\mathcal{S}^\vee, \mathbf{R}_{\mathcal{S}^\vee}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}))$. On the other hand, by dualizing the collection (5.2) (and reversing the order) and then twisting it by $\mathcal{O}(-1)$ gives the following full strong exceptional collection:

$$(\mathcal{O}(-1), \mathcal{S}^\vee, \Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3}, \mathcal{O}). \tag{5.5}$$

Comparing two full exceptional collections (5.3) and (5.5), we infer that

$$\langle \mathbf{R}_{\mathcal{S}^\vee}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3})\rangle = {}^\perp\langle \mathcal{O}(-1), \mathcal{S}^\vee\rangle \cap \langle \mathcal{O}\rangle^\perp = \langle \Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3}\rangle.$$

Thus, we have $\mathbf{R}_{\mathcal{S}^\vee}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}) \cong \Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3}[d]$ for some integer $d$, but the vanishing (5.4) implies that $d = 0$, namely

$$\mathbf{R}_{\mathcal{S}^\vee}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}) \cong \Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3}.$$

Hence we obtain the desired exact sequence (5.1). If follows immediately from the exact sequence (5.1) that $\dim \operatorname{Hom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee)^\vee = 4$. □

**Lemma 5.2.** *Let $\varphi : \mathcal{S}^\vee \to \Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3}$ be a morphism of $\mathcal{O}_{\mathbb{Q}^3}$-modules. If $\varphi \neq 0$, then $\varphi$ is injective, and there exists a line $L$ on $\mathbb{Q}^3$ such that the restriction* $\operatorname{Coker}(\varphi)|_L$ *to $L$ of the cokernel* $\operatorname{Coker}(\varphi)$ *of $\varphi$ admits a negative degree quotient.*

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**Proof.** We have an exact sequence

$$0 \to \Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3} \xrightarrow{i} H^0(\mathcal{O}(1)) \otimes \mathcal{O} \to \mathcal{O}(1) \to 0,$$

and the composite $i \circ \varphi$ can be written as

$$i \circ \varphi = \sum_{i=1}^{r} l_i \otimes s_i^\vee$$

for some $l_i \in H^0(\mathcal{O}(1))$ and $s_i \in H^0(\mathcal{S})$, where $s_i^\vee$ denotes the dual of the morphism $\mathcal{O} \to \mathcal{S}$ determined by $s_i$. We may assume that $l_i \neq 0$ for all $i$. By replacing $l_i$ if necessary, we may further assume that $s_1, \ldots, s_r$ are linearly independent. Since $h^0(\mathcal{S}) = 4$, we have $r \leq 4$. Note that $\sum_{i=1}^{r} l_i s_i^\vee = 0$ in $\mathrm{Hom}(\mathcal{S}^\vee, \mathcal{O}(1))$. Hence $r \geq 2$. Moreover, we have a surjective morphism

$$\psi : \mathrm{Coker}(i \circ \varphi) \to \mathcal{O}(1).$$

Note that the morphism $\mathcal{O}^{\oplus r} \to \mathcal{S}$ determined by $(s_1, \ldots, s_r)$ is generically surjective. Hence we see that $i \circ \varphi$ is injective. Therefore, $\varphi$ is injective and

$$\mathrm{Coker}(\varphi) \cong \mathrm{Ker}(\psi).$$

If $r = 2$, then $\mathrm{Coker}(i \circ \varphi) \cong \mathcal{T} \oplus \mathcal{O}^{\oplus 3}$ for some torsion sheaf $\mathcal{T}$ on $\mathbb{Q}^3$. Since $\mathcal{O}(1)$ is torsion-free, $\psi$ maps $\mathcal{T}$ to zero, and we have a surjective morphism $\bar{\psi} : \mathcal{O}^{\oplus 3} \to \mathcal{O}(1)$. On the other hand, $\bar{\psi} : \mathcal{O}^{\oplus 3} \to \mathcal{O}(1)$ cannot be surjecitve since three hyperplane sections of $\mathbb{Q}^3$ always meet at a point. This is a contradiction. Hence $r = 3$ or 4. Suppose that $r = 4$. Then it follows from the exact sequence (3.1) that $\mathrm{Coker}(i \circ \varphi) \cong \mathcal{S} \oplus \mathcal{O}$. Note that $\psi$ induces a morphism $\mathcal{S} \to \mathcal{O}(1)$, which factors through $\mathcal{I}_L(1)$ for some line $L$ in $\mathbb{Q}^3$. Since $L$ and a hyperplane in $\mathbb{Q}^3$ meet at a point, $\psi$ cannot be surjective. Hence the case $r = 4$ does not arise, and we have $r = 3$.

Now it follows from the exact sequence (3.1) that the cokernel of the morphism determined by ${}^t(s_1^\vee, s_2^\vee, s_3^\vee) : \mathcal{S}^\vee \to \mathcal{O}^{\oplus 3}$ is isomorphic to the cokernel of some non-zero morphism $\mathcal{O} \to \mathcal{S}$, and hence it is isomorphic to $\mathcal{I}_M(1)$ for some line $M$ on $\mathbb{Q}^3$. Therefore, $\mathrm{Coker}(i \circ \varphi) \cong \mathcal{I}_M(1) \oplus \mathcal{O}^{\oplus 2}$, and we have the following exact sequence:

$$0 \to \mathrm{Coker}(\varphi) \to \mathcal{I}_M(1) \oplus \mathcal{O}^{\oplus 2} \xrightarrow{\psi} \mathcal{O}(1) \to 0. \tag{5.6}$$

Let $\mathbb{Q}^2$ be a general hyperplane section of $\mathbb{Q}^3$ containing $M$. We may assume that $M$ is a divisor of type $(1, 0)$ of $\mathbb{Q}^2$. Then $\mathcal{I}_M(1)$ fits in the following exact sequence:

$$0 \to \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{I}_M(1) \to \mathcal{O}_{\mathbb{Q}^2}(0, 1) \to 0.$$

By pulling back the sequence above to a line $L$ of type $(0, 1)$ in $\mathbb{Q}^2$, we obtain the following exact sequence:

$$\mathcal{O}_L \to \mathcal{I}_M(1) \otimes \mathcal{O}_L \to \mathcal{O}_L \to 0.$$

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The image of $\mathcal{O}_L \to \mathcal{I}_M(1) \otimes \mathcal{O}_L$ is the torsion part of $\mathcal{I}_M(1) \otimes \mathcal{O}_L$. Therefore, $\psi \otimes 1_L$ factors through $\mathcal{O}_L^{\oplus 3}$ and induces a surjection $\mathcal{O}_L^{\oplus 3} \to \mathcal{O}_L(1)$. Hence $\operatorname{Coker}(\varphi) \otimes \mathcal{O}_L$ has $\mathcal{O}_L(-1) \oplus \mathcal{O}_L$ as a quotient. □

Lemma 5.3 will be applied to $\psi_a$ in (12.4) and (12.7) and plays a crucial role in our proof of Theorem 1.1.

**Lemma 5.3.** *For any positive integer $a$ and for any morphism $\psi_a : T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to \mathcal{S}^{\vee \oplus a}$, there exists a line $L$ in $\mathbb{Q}^3$ such that the cokernel* $\operatorname{Coker}(\psi_a)$ *of $\psi_a$ has $\mathcal{O}_L(-1)$ as a quotient. In case $a = 1$, there is a one-to-one correspondence between lines $L$ in $\mathbb{Q}^3$ and non-zero morphisms $\psi_1 : T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to \mathcal{S}^\vee$ up to scalar, and the correspondence is given by the following exact sequence:*

$$0 \to \mathcal{O}(-1)^{\oplus 2} \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \xrightarrow{\psi_1} \mathcal{S}^\vee \to \mathcal{O}_L(-1) \to 0. \tag{5.7}$$

**Proof.** The following brilliant proof is due to the referee. This proof is much shorter than the original and enlightens the meaning of the exact sequence (5.7) more clearly.

Denote by $\operatorname{Quot}(\mathcal{S}^\vee)$ the Quot-scheme parametrizing quotient sheaves of $\mathcal{S}^\vee$. Then we have a morphism

$$\Psi : \mathbb{P}(\operatorname{Hom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee)^\vee) \to \operatorname{Quot}(\mathcal{S}^\vee)$$

sending $[\psi_1]$ to $\operatorname{Coker}(\psi_1)$. Note that for any line $L \subset \mathbb{Q}^3$ we have $\mathcal{S}^\vee|_L \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L$ so that $\mathcal{S}^\vee$ admits $\mathcal{O}_L(-1)$ as a quotient. Note also that the Hilbert polynomial $\chi(\mathcal{O}_L(t-1))$ of $\mathcal{O}_L(-1)$ is $t$. Let $Z$ be the Hilbert scheme parametrizing lines in $\mathbb{Q}^3$. Then we have an inclusion

$$Z \hookrightarrow \operatorname{Quot}^t(\mathcal{S}^\vee)$$

sending $[L]$ to $\mathcal{O}_L(-1)$, where $\operatorname{Quot}^t(\mathcal{S}^\vee)$ is the Quot-scheme parametrizing quotients of $\mathcal{S}^\vee$ with Hilbert polynomial $t$. It is well-known that $Z \cong \mathbb{P}^3$. Note also that

$$\mathbb{P}(\operatorname{Hom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee)^\vee) \cong \mathbb{P}^3$$

by Lemma 5.1. We will show that $\Psi$ is an isomorphism onto $Z$.

We first claim that the image $\operatorname{Im}\Psi$ of $\Psi$ is $Z$. To see this, we first apply to $\mathcal{O}_L(-1)$ for any line $L \subset \mathbb{Q}^3$ the Bondal spectral sequence (2.1). We have the following:

$$\operatorname{ext}^q(\mathcal{O}, \mathcal{O}_L(-1)) = 0 \text{ for any } q;$$

$$\operatorname{ext}^q(\mathcal{S}, \mathcal{O}_L(-1)) = h^q(\mathcal{O}_L(-2) \oplus \mathcal{O}_L(-1)) = \begin{cases} 1 & \text{if } q = 1 \\ 0 & \text{if } q \neq 1 \end{cases}; \tag{5.8}$$

$$\operatorname{ext}^q(\mathcal{O}(1), \mathcal{O}_L(-1)) = h^q(\mathcal{O}_L(-2)) = \begin{cases} 1 & \text{if } q = 1 \\ 0 & \text{if } q \neq 1 \end{cases};$$

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$$\mathrm{ext}^q(\mathcal{O}(2), \mathcal{O}_L(-1)) = h^q(\mathcal{O}_L(-3)) = \begin{cases} 2 & \text{if } q = 1 \\ 0 & \text{if } q \neq 1 \end{cases}.$$

Thus, $\mathrm{Ext}^3(G, \mathcal{E}) = 0$, $\mathrm{Ext}^2(G, \mathcal{E}) = 0$, $\mathrm{Hom}(G, \mathcal{E}) = 0$, and $\mathrm{Ext}^1(G, \mathcal{E})$ has a filtration $S_1 \subset F \subset \mathrm{Ext}^1(G, \mathcal{E})$ of right $A$-modules such that the following sequences are exact:

$$0 \to F \to \mathrm{Ext}^1(G, \mathcal{E}) \to S_3^{\oplus 2} \to 0;$$

$$0 \to S_1 \to F \to S_2 \to 0.$$

These exact sequences induce the following distinguished triangles by Lemma 2.1:

$$F \otimes^{\mathrm{L}}{}_A G \to \mathrm{Ext}^1(G, \mathcal{E}) \otimes^{\mathrm{L}}{}_A G \to \mathcal{O}(-1)^{\oplus 2}[3] \to;$$

$$\mathcal{S}^\vee[1] \to F \otimes^{\mathrm{L}}{}_A G \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}[2] \to .$$

By taking cohomologies, we obtain the following exact sequences:

$$0 \to E_2^{-3,1} \to \mathcal{O}(-1) \to \mathcal{H}^{-2}(F \otimes^{\mathrm{L}}{}_A G) \to E_2^{-2,1} \to 0;$$

$$0 \to \mathcal{H}^{-2}(F \otimes^{\mathrm{L}}{}_A G) \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \xrightarrow{\psi_L} \mathcal{S}^\vee \to E_2^{-1,1} \to 0.$$

Moreover, we see that $E_2^{p,q} = 0$ unless $q = 1$ and that $E_2^{p,1} = 0$ unless $p = -3, -2$ or $-1$. Hence we infer that $E_2^{-3,1} = 0$, that $E_2^{-2,1} = 0$ and that $E_2^{-1,1} \cong \mathcal{O}_L(-1)$. Therefore, $\mathcal{O}_L(-1)$ is resolved as

$$0 \to \mathcal{O}(-1)^{\oplus 2} \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \xrightarrow{\psi_L} \mathcal{S}^\vee \to \mathcal{O}_L(-1) \to 0 \tag{5.9}$$

in terms of the full strong exceptional collection (2.6). This implies that the image $\mathrm{Im}\,\Psi$ of $\Psi$ contains $Z$. Since the source of $\Psi$ has the same dimension as $Z$, we conclude that $\mathrm{Im}\,\Psi = Z$.

Next we show that $\Psi$ is injective. Note that the exact sequence (5.9) splits into the following two exact sequences:

$$0 \to \mathcal{K} \to \mathcal{S}^\vee \to \mathcal{O}_L(-1) \to 0; \tag{5.10}$$

$$0 \to \mathcal{O}(-1)^{\oplus 2} \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to \mathcal{K} \to 0. \tag{5.11}$$

Since we have (5.8), the exact sequence (5.10) shows that $\mathcal{K}$ is the left mutation of $\mathcal{O}_L(-1)$ over $\mathcal{S}^\vee$. Moreover it follows from (5.11) that $\mathcal{O}(-1)^{\oplus 2}$ is the left mutation of $\mathcal{K}$ over $T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}$, since

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$$K \cong \mathrm{RHom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}) \cong \mathrm{RHom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, K).$$

Therefore, $\psi_L$ in (5.9) is uniquely determined by $L$ up to scalar. Hence $\Psi$ is injective.

Finally, if the composite of the morphism $\psi_a$ and some projection $\mathcal{S}^{\vee \oplus a} \to \mathcal{S}^\vee$ is zero, then $\mathrm{Coker}(\psi_a)$ admits $\mathcal{S}^\vee$ as a quotient, and the assertion follows. Hence we may assume that the composite cannot be zero for any projection $\mathcal{S}^{\vee \oplus a} \to \mathcal{S}^\vee$. Then the cokernel of the composite has $\mathcal{O}_L(-1)$ as a quotient, and so does $\mathrm{Coker}(\psi_a)$. □

Since the analyses of $\mathrm{Coker}(\psi_a)$ in case $a \geq 2$ in the original proof of Lemma 5.3 are indispensable for the proof of Lemma 5.4, we also provide that part of the proof as it is. Recall here that, for a coherent sheaf $\mathcal{F}$ of codimension $\geq p+1$ on a non-singular projective variety $X$, we have $c_i(\mathcal{F}) = 0$ for all $1 \leq i \leq p$ (see, e.g., [6, Example 15.3.6]).

**Proof. The original proof of Lemma 5.3 in case $a \geq 2$** If the composite of the morphism $\psi_a$ and some projection $\mathcal{S}^{\vee \oplus a} \to \mathcal{S}^\vee$ is zero, then $\mathrm{Coker}(\psi_a)$ admits $\mathcal{S}^\vee$ as a quotient, and the assertion follows. Hence we may assume that the composite cannot be zero for any projection $\mathcal{S}^{\vee \oplus a} \to \mathcal{S}^\vee$, and this implies that $a \leq 4$ by Lemma 5.1.

If $a = 4$, then Lemma 5.1 shows that $\mathrm{Coker}(\psi_4) \cong \Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3}$, and the assertion follows.

If $a = 3$, then $\psi_3$ can be regarded as the composite of the coevaluation morphism

$$\psi_4 : T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to \mathcal{S}^\vee \otimes \mathrm{Hom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee)^\vee$$

and some projection $\mathcal{S}^\vee \otimes \mathrm{Hom}(T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}, \mathcal{S}^\vee)^\vee \to \mathcal{S}^{\vee \oplus 3}$. Let $\mathcal{S}^\vee \to \mathcal{S}^{\vee \oplus 4}$ be the kernel of this projection, and let $\varphi$ be the composite of the inclusion $\mathcal{S}^\vee \to \mathcal{S}^{\vee \oplus 4}$ and the surjection $\mathcal{S}^{\vee \oplus 4} \to \Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3}$ in (5.1). Then

$$\mathrm{Coker}(\psi_3) \cong \mathrm{Coker}(\varphi) \tag{5.12}$$

and $\mathrm{Ker}(\psi_3) \cong \mathrm{Ker}(\varphi)$ by the snake lemma. Since $\mathrm{Hom}(\mathcal{S}^\vee, T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}) = 0$, $\varphi$ cannot be zero by (5.1). Lemma 5.2 then shows that $\varphi$ is injective and that the restriction $\mathrm{Coker}(\varphi)|_L$ to some line $L$ on $\mathbb{Q}^3$ admits a negative degree quotient. Hence the assertion holds, and $\psi_3$ is injective.

Suppose that $a = 2$. Then we can regard $\psi_2$ as the composite of some $\psi_3 : T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to \mathcal{S}^{\vee \oplus 3}$ and some projection $\mathcal{S}^{\vee \oplus 3} \to \mathcal{S}^{\vee \oplus 2}$. Let $\mathcal{S}^\vee \to \mathcal{S}^{\vee \oplus 3}$ be the kernel of this projection. Note here that we have an exact sequence

$$0 \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \xrightarrow{\psi_3} \mathcal{S}^{\vee \oplus 3} \to \mathrm{Coker}(\varphi) \to 0.$$

Denote by $\varphi_1 : \mathcal{S}^\vee \to \mathrm{Coker}(\varphi)$ the composite of the inclusion $\mathcal{S}^\vee \to \mathcal{S}^{\vee \oplus 3}$ and the surjection $\mathcal{S}^{\vee \oplus 3} \to \mathrm{Coker}(\varphi)$. Then $\varphi_1$ cannot be zero, since $\mathrm{Hom}(\mathcal{S}^\vee, T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}) = 0$. Moreover, the snake lemma implies that

$$\mathrm{Coker}(\psi_2) \cong \mathrm{Coker}(\varphi_1) \quad \text{and that} \quad \mathrm{Ker}(\psi_2) \cong \mathrm{Ker}(\varphi_1).$$

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Recall the inclusion $i : \mathrm{Coker}(\varphi) \hookrightarrow \mathcal{I}_M(1) \oplus \mathcal{O}^{\oplus 2}$ in (5.6) and consider the composite $i \circ \varphi_1$. We have the following exact sequence:

$$0 \to \mathrm{Coker}(\varphi_1) \to \mathrm{Coker}(i \circ \varphi_1) \to \mathcal{O}(1) \to 0. \tag{5.13}$$

Let $i \circ \varphi_1$ be equal to $(t^\vee, s_1^\vee, s_2^\vee)$, where $t^\vee \in \mathrm{Hom}(\mathcal{S}^\vee, \mathcal{I}_M(1))$, $t \in H^0(\mathcal{S}(1))$, $s_1^\vee$, $s_2^\vee \in \mathrm{Hom}(\mathcal{S}^\vee, \mathcal{O})$ and $s_1$, $s_2 \in H^0(\mathcal{S})$. Since we have an exact sequence (5.6), we have $t^\vee + h_1 s_1^\vee + h_2 s_2^\vee = 0$ for some $h_1$, $h_2 \in H^0(\mathcal{O}(1))$. Now we have two cases:

(1) $s_1$ and $s_2$ are linearly independent;
(2) $s_1$ and $s_2$ are linearly dependent.

(1) If $s_1$ and $s_2$ are linearly independent, then $\varphi_1$ is injective, and $\mathrm{Coker}(i \circ \varphi_1)$ has rank one. Thus we see that $\mathrm{Coker}(\varphi_1)$ is a torsion sheaf. Moreover, we claim that $\mathrm{Coker}(\varphi_1)$ is pure by [8, Prop. 1.1.6]: first note that $\mathcal{E}xt^q_{\mathbb{Q}^3}(\mathrm{Coker}(\varphi), \omega_{\mathbb{Q}^3}) = 0$ for all $q \geq 2$; thus $\mathcal{E}xt^q_{\mathbb{Q}^3}(\mathrm{Coker}(\varphi_1), \omega_{\mathbb{Q}^3}) = 0$ for all $q \geq 2$, and hence $\mathrm{Coker}(\varphi_1)$ satisfies the generalized Serre's condition $S_{1,1}$ in [8, Section 1.1]. Now we compute the Chern polynomial of $\mathrm{Coker}(\varphi_1)$. First note that $c_t(\mathrm{Coker}(\varphi)) = c_t(\Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3})/c_t(\mathcal{S}^\vee) = 1 + lt^2 - t^3$. Hence

$$c_t(\mathrm{Coker}(\varphi_1)) = c_t(\mathrm{Coker}(\varphi))/c_t(\mathcal{S}^\vee) = 1 + ht + 2lt^2.$$

Since $\mathrm{Coker}(\varphi_1)$ is a torsion sheaf, this implies that $\mathrm{Coker}(\varphi_1)$ is supported on a hyperplane section $H$ of $\mathbb{Q}^3$, and the length of $\mathrm{Coker}(\varphi_1)$ at the generic point of $H$ is one. Since $\mathrm{Coker}(\varphi_1)$ is pure, this implies that $\mathrm{Coker}(\varphi_1)$ is of the form $\mathcal{I}_{Z,H}(D)$, where $D$ is a divisor on $H$ and $\mathcal{I}_{Z,H}$ denotes the ideal sheaf of some zero-dimensional closed subscheme $Z$ in $H$. Note here that $c_t(\mathcal{O}_H) = 1 + ht + 2lt^2 + 2t^3$, that $c_t(\mathcal{O}_L) = (c_t(\mathcal{S}^\vee)/c_t(\mathcal{O}(-1)))^{-1} = 1 - lt^2 - t^3$ and that $c_t(k(p)) = 1 + 2t^3$, where $k(p)$ is the residue field at a point $p$ (see also [6, Example 15.3.1] for the formula $c_t(k(p)) = 1 + 2t^3$). Hence we see that $[D] = 0 \cdot l$ in $A^2\mathbb{Q}^3$. Moreover, if $D$ is of type $(d, -d)$, then $c_t(\mathcal{I}_{Z,H}(D)) = 1 + ht + 2lt^2 + (2 - 2d^2 - 2\,\mathrm{length}\,Z)t^3$. Hence $(d, \mathrm{length}\,Z) = (0, 1)$ or $(\pm 1, 0)$. Therefore, $\mathrm{Coker}(\varphi_1)$ is isomorphic to either $\mathcal{I}_{p,H}$ or $\mathcal{O}_H(d, -d)$ where $d = \pm 1$. Thus the assertion holds.

(2) If $s_1$ and $s_2$ are linearly dependent, by replacing $s_i$ and $h_i$ if necessary, we may assume that $s_2 = 0$, and we have $t^\vee + h_1 s_1^\vee = 0$. Set $\varphi_1' := (t^\vee, s_1^\vee) : \mathcal{S}^\vee \to \mathcal{I}_M(1) \oplus \mathcal{O}_{\mathbb{Q}^3}$. Then $\mathrm{Coker}(i \circ \varphi_1) \cong \mathrm{Coker}(\varphi_1') \oplus \mathcal{O}_{\mathbb{Q}^3}$ and $\mathrm{Ker}(\varphi_1) \cong \mathrm{Ker}(\varphi_1')$. Note that $\varphi_1' \neq 0$ since $\varphi_1 \neq 0$. Hence $s_1 \neq 0$. Let $L$ be the zero locus $(s_1)_0$ of $s_1$. Then the composite of $\varphi_1'$ and the inclusion $\mathcal{I}_M(1) \oplus \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{O}(1) \oplus \mathcal{O}_{\mathbb{Q}^3}$ factors through the morphism $(-h_1, 1) : \mathcal{O} \to \mathcal{O}(1) \oplus \mathcal{O}_{\mathbb{Q}^3}$, and we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & \mathcal{S}^\vee & \xrightarrow{s_1^\vee} & \mathcal{O}_{\mathbb{Q}^3} & \longrightarrow & \mathcal{O}_L & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_1'} & & \downarrow{\scriptstyle (-h_1,1)} & & \downarrow{\scriptstyle -\bar{h}_1} & & \\
0 & \longrightarrow & \mathcal{I}_M(1) \oplus \mathcal{O}_{\mathbb{Q}^3} & \longrightarrow & \mathcal{O}(1) \oplus \mathcal{O}_{\mathbb{Q}^3} & \longrightarrow & \mathcal{O}_M(1) & \longrightarrow & 0
\end{array} \tag{5.14}$$

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We see that $\mathrm{Im}(\varphi_1') \cong \mathcal{I}_L$ and that $\mathrm{Ker}(\varphi_1') \cong \mathcal{O}(-1)$. We claim here that $\bar{h}_1 \neq 0$. Assume, to the contrary, that $\bar{h}_1 = 0$. Then the snake lemma implies that $\mathrm{Coker}(\varphi_1')$ fits in the following exact sequence:

$$0 \to \mathcal{O}_L \to \mathrm{Coker}(\varphi_1') \to \mathcal{O}(1) \to \mathcal{O}_M(1) \to 0.$$

Since $\mathcal{O}_L$ is a torsion sheaf, the surjection $\mathrm{Coker}(\varphi_1') \oplus \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{O}(1)$ induces a surjection $\mathcal{I}_M(1) \oplus \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{O}(1)$. On the other hand, the morphism $\mathcal{I}_M(1) \oplus \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{O}(1)$ cannot be surjective since a line $M$ and a hyperplane meets at least at one point. This is a contradiction. Hence $\bar{h}_1 \neq 0$, and thus $L = M$. Moreover, the commutative diagram (5.14) induces the following exact sequence by the snake lemma:

$$0 \to \mathrm{Coker}(\varphi_1') \to \mathcal{O}(1) \to k(p) \to 0,$$

where $p = (\bar{h}_1)_0$. Therefore, $\mathrm{Coker}(\varphi_1') = \mathcal{I}_p(1)$. The exact sequence (5.13), i.e. the sequence

$$0 \to \mathrm{Coker}(\varphi_1) \to \mathcal{I}_p(1) \oplus \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{O}(1) \to 0$$

then shows that $\mathrm{Coker}(\varphi_1) = \mathcal{I}_p$. Thus the assertion also holds if $s_1$ and $s_2$ are linearly dependent. □

Lemma 5.4 will be applied to $\pi$ in (12.8) and plays a crucial role in the proof of Theorem 1.1.

**Lemma 5.4.** *Let $\psi_a : T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to \mathcal{S}^{\vee \oplus a}$ be a morphism of $\mathcal{O}_{\mathbb{Q}^3}$-modules where $a$ is a positive integer, and let $\pi : \mathcal{O}_{\mathbb{Q}^3}(-1) \to \mathrm{Coker}(\psi_a)$ be a morphism of $\mathcal{O}_{\mathbb{Q}^3}$-modules. If* $\mathrm{Coker}(\pi)$ *does not admit a negative degree quotient, then $a = 1$,* $\mathrm{Coker}(\pi) = 0$ *and* $\mathrm{Ker}(\pi)$ *is isomorphic to $\mathcal{I}_L(-1)$ for some line $L$ in $\mathbb{Q}^3$.*

**Proof.** We may assume that $\pi \neq 0$.

Suppose that $\mathrm{Coker}(\psi_a)$ admits $\mathcal{S}^\vee$ as a quotient; let $p : \mathrm{Coker}(\psi_a) \to \mathcal{S}^\vee$ be the surjection. Note that $\mathrm{Coker}(\pi)$ admits $\mathrm{Coker}(p \circ \pi)$ as a quotient. If $p \circ \pi = 0$, then $\mathrm{Coker}(p \circ \pi) \cong \mathcal{S}^\vee$, and if $p \circ \pi \neq 0$, then $\mathrm{Coker}(p \circ \pi) \cong \mathcal{I}_L$ for some line $L$ in $\mathbb{Q}^3$. Therefore, the restriction of $\mathrm{Coker}(\pi)$ to a line admits a negative degree quotient.

In the following, we assume that $\mathrm{Coker}(\psi_a)$ does not admit $\mathcal{S}^\vee$ as a quotient. Hence $a \leq 4$ by Lemma 5.1.

Suppose that $a = 4$. Then $\mathrm{Coker}(\psi_4) \cong \Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3}$ by Lemma 5.1. Since $\Omega_{\mathbb{P}^4}(1)|_L \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L^{\oplus 3}$ for any line $L$ in $\mathbb{Q}^3$, if $\mathrm{Coker}(\pi)|_L$ does not admit a negative degree quotient for any line $L$ in $\mathbb{Q}^3$, we see that $\mathrm{Coker}(\pi)|_L \cong \mathcal{O}_L^{\oplus 3}$ for any line $L$ in $\mathbb{Q}^3$. This implies that $\mathrm{Coker}(\pi) \cong \mathcal{O}_{\mathbb{Q}^3}^{\oplus 3}$ by [18, (3.6.1) Lemma]. Thus $\Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3} \cong \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus 3}$, which contradicts $H^0(\Omega_{\mathbb{P}^4}(1)|_{\mathbb{Q}^3}) = 0$. Therefore, $\mathrm{Coker}(\pi)|_L$ admits a negative degree quotient for some line $L$ in $\mathbb{Q}^3$.

Suppose that $a = 3$. Recall that $\mathrm{Coker}(\psi_3) \cong \mathrm{Coker}(\varphi)$ in (5.12). Recall also the inclusion $i : \mathrm{Coker}(\varphi) \hookrightarrow \mathcal{I}_M(1) \oplus \mathcal{O}^{\oplus 2}$ in (5.6) and consider the composite $i \circ \pi$. We have the following exact sequence:

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$$0 \to \mathrm{Coker}(\pi) \to \mathrm{Coker}(i \circ \pi) \xrightarrow{\rho} \mathcal{O}(1) \to 0. \tag{5.15}$$

Let $i \circ \pi$ be equal to $(t, g_1, g_2)$, where $t \in \mathrm{Hom}(\mathcal{O}(-1), \mathcal{I}_M(1)) \cong H^0(\mathcal{I}_M(2))$, $g_1$, $g_2 \in \mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}) \cong H^0(\mathcal{O}(1))$. Since we have an exact sequence (5.6), we have $t + h_1 g_1 + h_2 g_2 = 0$ for some $h_1$, $h_2 \in H^0(\mathcal{O}(1))$. Now we have two cases:

(1) $g_1$ and $g_2$ are linearly independent;
(2) $g_1$ and $g_2$ are linearly dependent.

(1) If $g_1$ and $g_2$ are linearly independent, then the cokernel of the morphism $(g_1, g_2) : \mathcal{O}(-1) \to \mathcal{O}^{\oplus 2}$ is of the form $\mathcal{I}_C(1)$, where $C$ is the conic defined by $g_1$ and $g_2$. Hence $\mathrm{Coker}(i \circ \pi)$ fits in the following exact sequence:

$$0 \to \mathcal{I}_M(1) \to \mathrm{Coker}(i \circ \pi) \to \mathcal{I}_C(1) \to 0.$$

Now consider the composite of the injection $\mathcal{I}_M \to \mathrm{Coker}(i \circ \pi)(-1)$ and the surjection $\rho(-1) : \mathrm{Coker}(i \circ \pi)(-1) \to \mathcal{O}$. The composite is nothing but the inclusion $\mathcal{I}_M \hookrightarrow \mathcal{O}$ and its cokernel is $\mathcal{O}_M$. Thus the surjection $\rho(-1)$ induces a surjection $\bar{\rho}(-1) : \mathcal{I}_C \to \mathcal{O}_M$. This implies that $C \cap M = \emptyset$. Moreover $\mathrm{Coker}(\pi)(-1) \cong \mathrm{Ker}(\bar{\rho}(-1)) \cong \mathcal{I}_{C \sqcup M}$. Hence $\mathrm{Coker}(\pi) \cong \mathcal{I}_{C \sqcup M}(1)$. Note that the conic $C$ and the line $M$ can be joined by a line $L$ in $\mathbb{Q}^3$. Indeed, any hyperplane section $H$ containing $M$ intersects $C$ at some point $p$, and the point $p$ and $M$ can be joined by a line $L$ in $H$. Now we see that $\mathrm{Coker}(\pi)|_L$ admits a negative degree quotient.

(2) If $g_1$ and $g_2$ are linearly dependent, by replacing $g_i$ and $h_i$ if necessary, we may assume that $g_2 = 0$, and we have $t + h_1 g_1 = 0$. Set $\pi_1' := (t, g_1) : \mathcal{O}(-1) \to \mathcal{I}_M(1) \oplus \mathcal{O}_{\mathbb{Q}^3}$. Then $\mathrm{Coker}(i \circ \pi) \cong \mathrm{Coker}(\pi') \oplus \mathcal{O}_{\mathbb{Q}^3}$. Note that $\pi' \neq 0$ since $\pi \neq 0$. Hence $g_1 \neq 0$. Let $H$ be the hyperplane defined by $g_1$. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-1) & \xrightarrow{g_1} & \mathcal{O}_{\mathbb{Q}^3} & \longrightarrow & \mathcal{O}_H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle (-h_1, 1)} & & \downarrow{\scriptstyle -\bar{h}_1} & & \\ 0 & \longrightarrow & \mathcal{I}_M(1) \oplus \mathcal{O}_{\mathbb{Q}^3} & \longrightarrow & \mathcal{O}(1) \oplus \mathcal{O}_{\mathbb{Q}^3} & \longrightarrow & \mathcal{O}_M(1) & \longrightarrow & 0 \end{array} \tag{5.16}$$

We claim here that $\bar{h}_1 \neq 0$. Assume, to the contrary, that $\bar{h}_1 = 0$. Then the snake lemma shows that we have the following exact sequence:

$$0 \to \mathcal{O}_H \to \mathrm{Coker}(\pi') \to \mathcal{O}(1) \to \mathcal{O}_M(1) \to 0.$$

Since $\mathcal{O}_H$ is a torsion sheaf, the surjection $\rho : \mathrm{Coker}(\pi') \oplus \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{O}(1)$ sends $\mathcal{O}_H$ to zero, and thus $\rho$ induces a surjection $\mathcal{I}_M(1) \oplus \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{O}(1)$. On the other hand, the morphism $\mathcal{I}_M(1) \oplus \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{O}(1)$ cannot be surjective since a line $M$ and a hyperplane meets at least at one point. This is a contradiction. Hence $\bar{h}_1 \neq 0$. Then the kernel of the morphism $-\bar{h}_1 : \mathcal{O}_H \to \mathcal{O}_M(1)$ is $\mathcal{O}_H(-M)$ and the cokernel of $-\bar{h}_1$ is $k(p)$ for some point $p \in M$.

Hence the commutative diagram (5.16) induces the following exact sequence by the snake lemma:

$$0 \to \mathcal{O}_H(-M) \to \mathrm{Coker}(\pi') \to \mathcal{O}(1) \to k(p) \to 0.$$

Since $\mathcal{O}_H(-M)$ is a torsion sheaf, the surjection $\rho : \mathrm{Coker}(\pi') \oplus \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{O}(1)$ sends $\mathcal{O}_H(-M)$ to zero, and thus the inclusion $\mathcal{O}_H(-M) \hookrightarrow \mathrm{Coker}(\pi') \oplus \mathcal{O}_{\mathbb{Q}^3}$ induces an inclusion $\mathcal{O}_H(-M) \hookrightarrow \mathrm{Coker}(\pi)$. The exact sequence (5.15) induces the following exact sequence:

$$0 \to \mathrm{Coker}(\pi)/\mathcal{O}_H(-M) \to \mathcal{I}_p(1) \oplus \mathcal{O}_{\mathbb{Q}^3} \to \mathcal{O}(1) \to 0.$$

This shows that $\mathrm{Coker}(\pi)/\mathcal{O}_H(-M) = \mathcal{I}_p$.

Suppose that $a = 2$. As we have seen in the original proof of Lemma 5.3, $\mathrm{Coker}(\psi_2)$ is isomorphic to $\mathrm{Coker}(\varphi_1)$, and $\mathrm{Coker}(\varphi_1)$ is one of the following: $\mathcal{I}_{p,H}$; $\mathcal{O}_H(d,-d)$ where $d = \pm 1$; $\mathcal{I}_p$. If $\mathrm{Coker}(\varphi_1) = \mathcal{I}_{p,H}$, then $\mathrm{Coker}(\pi)$ admits $\mathcal{O}_C(-p)$ as a quotient, where $C$ is a conic on $H$. If $\mathrm{Coker}(\varphi_1) = \mathcal{O}_H(d,-d)$ with $d = \pm 1$, then $\mathrm{Coker}(\pi)$ admits $\mathcal{O}_L(-1)$ as a quotient, where $L$ is a line on $H$. If $\mathrm{Coker}(\varphi_1) = \mathcal{I}_p$, then $\mathrm{Coker}(\pi)$ admits $\mathcal{I}_{p,H}$ as a quotient. Hence the assertion follows if $a = 2$.

Suppose that $a = 1$. Then $\mathrm{Coker}(\psi_1) \cong \mathcal{O}_L(-1)$ by Lemma 5.3. Since $\pi \neq 0$, the morphism $\pi : \mathcal{O}(-1) \to \mathcal{O}_L(-1)$ is surjective, and $\mathrm{Ker}(\pi) \cong \mathcal{I}_L(-1)$. This completes the proof. □

## 6. A lower bound for the third Chern class

Note that

$$c_3 \geq 2c_1c_2 - c_1^3 \tag{6.1}$$

for a nef vector bundle $\mathcal{E}$ on a complete threefold $X$, since $H(\mathcal{E})^{r+2} = c_3 - 2c_1c_2 + c_1^3 \geq 0$ for a nef line bundle $H(\mathcal{E})$. If there exists an injection $\mathcal{L} \to \mathcal{E}$ from a line bundle $\mathcal{L}$, then we have a lower bound, which is better if $\mathcal{L} \cong \mathcal{O}(D)$ for some effective divisor $D$, as the following lemma shows:

**Lemma 6.1.** *Let $\mathcal{E}$ be a nef vector bundle of rank $r$ on a complete variety $X$ of dimension three. Let $\mathcal{L}$ be a line bundle on $X$ such that $H^0(\mathcal{E} \otimes \mathcal{L}^{-1}) \neq 0$. Then we have the following inequality:*

$$c_3 \geq 2c_1c_2 - c_1^3 + (c_1^2 - c_2)c_1(\mathcal{L}).$$

**Proof.** The following short proof is due to the referee. Let $p : \mathbb{P}(\mathcal{E}) \to X$ be the projection. Then $H^0(H(\mathcal{E}) \otimes p^*\mathcal{L}^{-1}) \cong H^0(\mathcal{E} \otimes \mathcal{L}^{-1}) \neq 0$. Hence $H(\mathcal{E})^{r+1}(H(\mathcal{E}) - p^*c_1(\mathcal{L})) \geq 0$. This yields the desired inequality. □

Lemma 6.1 will be applied to $\mathcal{E}$ in § 12.1.

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## 7. Set-up for the proof of Theorem 1.1

Let $\mathcal{E}$ be a nef vector bundle of rank $r$ on $\mathbb{Q}^3$ with $c_1 = 2h$. It follows from [12, Lemma 4.1 (1)] that

$$h^q(\mathcal{E}(t)) = 0 \text{ for } q > 0 \text{ and } t \geq 0. \tag{7.1}$$

Moreover, if $H(\mathcal{E})^{r+2} = c_3 - 2c_1c_2 + c_1^3 = c_3 - 4c_2h + 16 > 0$, then

$$h^q(\mathcal{E}(-1)) = 0 \text{ for } q > 0 \tag{7.2}$$

by [12, Lemma 4.1 (2)]. Note here that

$$c_3 \geq 0 \tag{7.3}$$

by [11, Theorem 8.2.1], since $\mathcal{E}$ is nef. Hence we see that

$$h^q(\mathcal{E}(-1)) = 0 \text{ for } q > 0 \text{ if } c_2h \leq 3. \tag{7.4}$$

It follows from [12, Lemma 4.3] that

$$\mathrm{Ext}^q(\mathcal{S}, \mathcal{E}(2)) = 0 \text{ for } q > 0. \tag{7.5}$$

The exact sequence (3.1) together with the isomorphism (3.2) implies that $\mathcal{S}^\vee \otimes \mathcal{E}(2)$ fits in an exact sequence

$$0 \to \mathcal{S}^\vee \otimes \mathcal{E}(1) \to \mathcal{E}(1)^{\oplus 4} \to \mathcal{S}^\vee \otimes \mathcal{E}(2) \to 0.$$

It then follows from (7.1) and (7.5) that

$$\mathrm{Ext}^q(\mathcal{S}, \mathcal{E}(1)) = 0 \text{ for } q \geq 2. \tag{7.6}$$

If $h^0(\mathcal{E}(-2)) \neq 0$, then $\mathcal{E} \cong \mathcal{O}(2) \oplus \mathcal{O}^{\oplus r-1}$ by [12, Proposition 5.1 and Remark 5.3]. Thus, we will always assume that

$$h^0(\mathcal{E}(-2)) = 0 \tag{7.7}$$

in the following. It follows from Theorem 2.3 that

$$h^q(\mathcal{E}|_{\mathbb{Q}^2}) = 0 \text{ for } q \geq 2. \tag{7.8}$$

Moreover

$$h^1(\mathcal{E}|_{\mathbb{Q}^2}) = \begin{cases} 1 & \text{if } \mathcal{E}|_{\mathbb{Q}^2} \text{ belongs to Case (11) of Theorem 2.3;} \\ 0 & \text{otherwise.} \end{cases} \tag{7.9}$$

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The vanishing (7.1) then shows that

$$h^3(\mathcal{E}(-1)) = 0. \tag{7.10}$$

Moreover

$$h^2(\mathcal{E}(-1)) = 0 \,\text{unless}\; \mathcal{E}|_{\mathbb{Q}^2} \;\text{belongs to Case (11) of Theorem 2.3.} \tag{7.11}$$

It follows from Theorem 2.3 that

$$h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0 \,\text{for}\; q \geq 2. \tag{7.12}$$

The vanishing (7.10) then shows that

$$h^3(\mathcal{E}(-2)) = 0. \tag{7.13}$$

The exact sequence (3.1) together with (3.2) also induces the following exact sequence

$$0 \to \mathcal{S}^\vee \otimes \mathcal{E}(-1) \to \mathcal{E}(-1)^{\oplus 4} \to \mathcal{S}^\vee \otimes \mathcal{E} \to 0. \tag{7.14}$$

This exact sequence (7.14) and an exact sequence

$$0 \to \mathcal{S}^\vee \otimes \mathcal{E}(-1) \to \mathcal{S}^\vee \otimes \mathcal{E} \to \mathcal{S}^\vee \otimes \mathcal{E}|_{\mathbb{Q}^2} \to 0 \tag{7.15}$$

will be used to compute $\operatorname{Ext}^q(\mathcal{S}, \mathcal{E})$.

## 8. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (1) of Theorem 2.3

The assumption (7.7) implies that this case does not arise. Indeed, if $\mathcal{E}|_{\mathbb{Q}^2} \cong \mathcal{O}(2,2) \oplus \mathcal{O}^{\oplus r-1}$, then $h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$ for $q > 0$. Moreover $c_2 h = 0$. Hence $h^q(\mathcal{E}(-1)) = 0$ for $q > 0$ by (7.4). This implies that $h^q(\mathcal{E}(-2)) = 0$ for $q \geq 2$. The assumption (7.7) then shows that

$$0 \geq -h^1(\mathcal{E}(-2)) = \chi(\mathcal{E}(-2)) = 1 + \frac{1}{2}c_3$$

by (4.5). This contradicts (7.3). Hence this case does not arise.

## 9. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (2) of Theorem 2.3

Suppose that

$$\mathcal{E}|_{\mathbb{Q}^2} \cong \mathcal{O}(2,1) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}^{\oplus r-2}.$$

Then $h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 2$ and $h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$ for $q > 0$. Moreover $c_2 h = 2$. Hence

$$h^q(\mathcal{E}(-1)) = 0 \,\text{for}\; q > 0$$

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by (7.4). It then follows from (4.4) and (7.3) that $h^0(\mathcal{E}(-1)) = \chi(\mathcal{E}(-1)) = 2 + \frac{1}{2}c_3 \geq 2$. On the other hand, we have $h^0(\mathcal{E}(-1)) \leq h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 2$ by (7.7). Therefore, the restriction map $H^0(\mathcal{E}(-1)) \to H^0(\mathcal{E}(-1)|_{\mathbb{Q}^2})$ is an isomorphism,

$$h^0(\mathcal{E}(-1)) = 2 \text{ and } c_3 = 0.$$

Hence we see that

$$h^q(\mathcal{E}(-2)) = 0 \text{ for all } q.$$

Since $\mathcal{E}(-2)|_{\mathbb{Q}^2} \cong \mathcal{O}(0,-1) \oplus \mathcal{O}(-2,-1) \oplus \mathcal{O}(-2,-2)^{\oplus r-2}$, we have $h^q(\mathcal{E}(-2)|_{\mathbb{Q}^2}) = 0$ for $q < 2$ and $h^2(\mathcal{E}(-2)|_{\mathbb{Q}^2}) = r-2$. Therefore

$$h^q(\mathcal{E}(-3)) = 0 \text{ for } q < 3 \text{ and } h^3(\mathcal{E}(-3)) = r-2.$$

Next we will compute $\operatorname{Ext}^q(\mathcal{S}, \mathcal{E}(-1))$. Since

$$\mathcal{S}^\vee \otimes \mathcal{E}(t)|_{\mathbb{Q}^2} \cong (\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)) \otimes (\mathcal{O}(2+t, 1+t) \oplus \mathcal{O}(t, 1+t) \oplus \mathcal{O}(t,t)^{\oplus r-2}),$$

we see that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}(t)|_{\mathbb{Q}^2}) = 0$ for $q > 0$ and $t \geq 0$. Hence it follows from (7.6) that

$$\operatorname{Ext}^q(\mathcal{S}, \mathcal{E}(-1)) = 0 \text{ for } q \geq 2.$$

Since $c_2 h = 2$ and $c_3 = 0$, the formula (4.8) shows that

$$h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)).$$

Set $a = h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1))$. Note that $\mathcal{S}^\vee \otimes \mathcal{E}(-1)$ fits in an exact sequence

$$0 \to \mathcal{S}^\vee \otimes \mathcal{E}(-2) \to \mathcal{E}(-2)^{\oplus 4} \to \mathcal{S}^\vee \otimes \mathcal{E}(-1) \to 0$$

by (3.1) and (3.2). Since $h^q(\mathcal{E}(-2)) = 0$ for all $q$, this exact sequence shows that

$$h^q(\mathcal{S}^\vee \otimes \mathcal{E}(-2)) = \begin{cases} 0 & \text{if } q = 0, 3 \\ a & \text{otherwise.} \end{cases}$$

On the other hand, we have an exact sequence

$$0 \to \mathcal{S}^\vee \otimes \mathcal{E}(-2) \to \mathcal{S}^\vee \otimes \mathcal{E}(-1) \to (\mathcal{S}^\vee \otimes \mathcal{E}(-1))|_{\mathbb{Q}^2} \to 0. \tag{9.1}$$

Since

$$\mathcal{S}^\vee \otimes \mathcal{E}(-1)|_{\mathbb{Q}^2} \cong (\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)) \otimes (\mathcal{O}(1,0) \oplus \mathcal{O}(-1,0) \oplus \mathcal{O}(-1,-1)^{\oplus r-2}),$$

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we see that

$$h^q(\mathcal{S}^\vee \otimes \mathcal{E}(-1)|_{\mathbb{Q}^2}) = \begin{cases} 1 & \text{if } q = 0, 1 \\ 0 & \text{if } q = 2, 3. \end{cases}$$

Hence the exact sequence (9.1) implies that $a = 1$.

We apply to $\mathcal{E}(-1)$ the Bondal spectral sequence (2.1). We have $\mathrm{Ext}^3(G, \mathcal{E}(-1)) \cong S_3^{\oplus r-2}$, $\mathrm{Ext}^2(G, \mathcal{E}(-1)) = 0$ and $\mathrm{Ext}^1(G, \mathcal{E}(-1)) \cong S_1$. Moreover, $\mathrm{Hom}(G, \mathcal{E}(-1))$ fits in an exact sequence

$$0 \to S_0^{\oplus 2} \to \mathrm{Hom}(G, \mathcal{E}(-1)) \to S_1 \to 0.$$

Now Lemma 2.1 shows that $E_2^{p,3} = 0$ unless $p = -3$, that $E_2^{-3,3} \cong \mathcal{O}(-1)^{\oplus r-2}$, that $E_2^{p,2} = 0$ for all $p$, that $E_2^{p,1} = 0$ unless $p = -1$, that $E_2^{-1,1} \cong \mathcal{S}(-1)$ and that a distinguished triangle

$$\mathcal{O}^{\oplus 2} \to \mathrm{Hom}(G, \mathcal{E}(-1)) \otimes^{\mathrm{L}}{}_A G \to \mathcal{S}(-1)[1] \to$$

exists. Hence we have the following exact sequence:

$$0 \to E_2^{-1,0} \to \mathcal{S}(-1) \to \mathcal{O}^{\oplus 2} \to E_2^{0,0} \to 0. \tag{9.2}$$

Note here that $E_2^{-1,0} \cong E_\infty^{-1,0} = 0$. Hence we see that $E_2^{0,0}$ is a non-zero torsion sheaf. On the other hand, $\mathcal{E}(-1)$ has $E_2^{0,0}$ as a subsheaf, so that $E_2^{0,0}$ must be torsion-free. This is a contradiction. Therefore, this case does not arise.

## 10. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (3) of Theorem 2.3

Suppose that $\mathcal{E}|_{\mathbb{Q}^2} \cong \mathcal{O}(1,1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-2}$. Then $c_2 \cdot h = 2$. Hence $h^q(\mathcal{E}(-1)) = 0$ for $q > 0$ by (7.4). Since $h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$ for $q > 0$, this implies that $h^q(\mathcal{E}(-2)) = 0$ for $q \geq 2$. The assumption (7.7) together with (4.5) and (7.3) shows that

$$0 \geq -h^1(\mathcal{E}(-2)) = \chi(\mathcal{E}(-2)) = \frac{1}{2}c_3 \geq 0.$$

Hence $h^1(\mathcal{E}(-2)) = 0$ and $c_3 = 0$. Thus $h^0(\mathcal{E}(-1)) = h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 2$. Since $h^q(\mathcal{E}(-2)) = 0$ for any $q$, we see that $h^q(\mathcal{E}(-3)) = h^{q-1}(\mathcal{E}(-2)|_{\mathbb{Q}^2})$ for all $q$. Hence $h^q(\mathcal{E}(-3)) = 0$ unless $q = 3$ and $h^3(\mathcal{E}(-3)) = r - 2$. Since

$$\mathcal{S}^\vee \otimes \mathcal{E}(t)|_{\mathbb{Q}^2} \cong (\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)) \otimes (\mathcal{O}(1+t, 1+t)^{\oplus 2} \oplus \mathcal{O}(t,t)^{\oplus r-2}),$$

we see that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}(t)|_{\mathbb{Q}^2}) = 0$ for $q > 0$ and $t \geq -1$. Hence it follows from (7.6) that $\mathrm{Ext}^q(\mathcal{S}, \mathcal{E}(-t)) = 0$ for $q \geq 2$ and $t = 0, 1, 2$. Since the exact sequence (3.1) together with (3.2) induces an exact sequence

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$$0 \to \mathcal{S}^\vee \otimes \mathcal{E}(-2) \to \mathcal{E}(-2)^{\oplus 4} \to \mathcal{S}^\vee \otimes \mathcal{E}(-1) \to 0,$$

the vanishing $h^1(\mathcal{E}(-2)) = 0$ implies that $h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$. Since $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$, this implies that $h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-2)) = 0$. Hence $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$. We apply to $\mathcal{E}(-1)$ the Bondal spectral sequence (2.1). We see that $\mathrm{Hom}(G, \mathcal{E}(-1)) \cong S_0^{\oplus 2}$, that $\mathrm{Ext}^q(G, \mathcal{E}(-1)) = 0$ for $q = 1, 2$ and that $\mathrm{Ext}^3(G, \mathcal{E}(-1)) \cong S_3^{\oplus r-2}$. Hence $E_2^{p,q} = 0$ unless $q = 0$ or $q = 3$, $E_2^{p,0} = 0$ unless $p = 0$, $E_2^{0,0} = \mathcal{O}^{\oplus 2}$, $E_2^{p,3} = 0$ unless $p = -3$ and $E_2^{-3,3} = \mathcal{O}(-1)^{\oplus r-2}$ by Lemma 2.1. Therefore, $\mathcal{E}(-1)$ fits in an exact sequence

$$0 \to \mathcal{O}^{\oplus 2} \to \mathcal{E}(-1) \to \mathcal{O}(-1)^{\oplus r-2} \to 0.$$

Hence $\mathcal{E} \cong \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-2}$. This is Case (2) of Theorem 1.1.

## 11. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (4) of Theorem 2.3

Suppose that $\mathcal{E}|_{\mathbb{Q}^2}$ fits in an exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1,1) \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}^{\oplus r-2} \to \mathcal{E}|_{\mathbb{Q}^2} \to 0.$$

Then $c_2 h = 3$. Hence $h^q(\mathcal{E}(-1)) = 0$ for $q > 0$ by (7.4). Note that $h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$ for $q > 0$ and that $h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 1$. Hence $h^q(\mathcal{E}(-2)) = 0$ for $q \geq 2$. The assumption (7.7) together with (4.5) and (7.3) shows that

$$0 \geq -h^1(\mathcal{E}(-2)) = \chi(\mathcal{E}(-2)) = -\frac{1}{2} + \frac{1}{2}c_3 \geq -\frac{1}{2}.$$

Hence $h^1(\mathcal{E}(-2)) = 0$ and $c_3 = 1$. Now that $h^q(\mathcal{E}(-2)) = 0$ for any $q$, we have $h^q(\mathcal{E}(-3)) = h^{q-1}(\mathcal{E}(-2)|_{\mathbb{Q}^2})$ for any $q$. Set $a = h^1(\mathcal{E}(-2)|_{\mathbb{Q}^2})$. Then $a = 0$ or $1$, and $h^2(\mathcal{E}(-2)|_{\mathbb{Q}^2}) = r - 3 + a$. Hence we see that $h^q(\mathcal{E}(-3)) = 0$ for $q \leq 1$, that $h^2(\mathcal{E}(-3)) = a$ and that $h^3(\mathcal{E}(-3)) = r - 3 + a$. Moreover, the assumption (7.7) implies that $h^0(\mathcal{E}(-1)) = h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 1$. Since $\mathcal{E}|_{\mathbb{Q}^2}(-2,-1)$ fits in an exact sequence

$$\begin{aligned}0 \to \mathcal{O}(-2,-1) \to \mathcal{O}(-1,0) \oplus \mathcal{O}(-1,-1) \oplus \mathcal{O}(-2,0) \oplus \quad & \mathcal{O}(-2,-1)^{\oplus r-2} \\ & \to \mathcal{E}|_{\mathbb{Q}^2}(-2,-1) \to 0,\end{aligned}$$

we see that $h^q(\mathcal{E}|_{\mathbb{Q}^2}(-2,-1)) = 0$ unless $q = 1$. Hence $h^q(\mathcal{S}^\vee \otimes \mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$ unless $q = 1$. Note that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}(t)|_{\mathbb{Q}^2}) = 0$ for $t \geq 0$ and $q \geq 1$. Hence it follows from (7.6) that $\mathrm{Ext}^q(\mathcal{S}, \mathcal{E}(-t)) = 0$ for $q \geq 2$ and $t = 0, 1$. Note that $\mathcal{S}^\vee \otimes \mathcal{E}(-2)$ is a subbundle of $\mathcal{E}(-2)^{\oplus 4}$ by (3.1). Since $h^0(\mathcal{E}(-2)) = 0$, this implies that $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-2)) = 0$. Since we have an exact sequence

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$$0 \to \mathcal{S}^\vee \otimes \mathcal{E}(-2) \to \mathcal{S}^\vee \otimes \mathcal{E}(-1) \to \mathcal{S}^\vee \otimes \mathcal{E}(-1)|_{\mathbb{Q}^2} \to 0$$

and $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$, we infer that $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$. Now, from (4.8), it follows that

$$-h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = \chi(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 4 - 2 \cdot 3 + 1 = -1.$$

We apply to $\mathcal{E}(-1)$ the Bondal spectral sequence (2.1). We have the following isomorphisms: $\mathrm{Ext}^3(G, \mathcal{E}(-1)) \cong S_3^{\oplus r-3+a}$; $\mathrm{Ext}^2(G, \mathcal{E}(-1)) \cong S_3^{\oplus a}$; $\mathrm{Ext}^1(G, \mathcal{E}(-1)) \cong S_1$; $\mathrm{Hom}(G, \mathcal{E}(-1)) \cong S_0$. Lemma 2.1 then shows that $E_2^{p,q} = 0$ unless $(p, q) = (-3, 3)$, $(-3, 2)$, $(-1, 1)$ or $(0, 0)$, that $E_2^{-3,3} = \mathcal{O}(-1)^{\oplus r-3+a}$, that $E_2^{-3,2} = \mathcal{O}(-1)^{\oplus a}$, that $E_2^{-1,1} = \mathcal{S}(-1)$ and that $E_2^{0,0} = \mathcal{O}$. Hence $E_3^{-3,2} = 0$ and $E_3^{-1,1}$ fits in the following exact sequence:

$$0 \to \mathcal{O}(-1)^{\oplus a} \to \mathcal{S}(-1) \to E_3^{-1,1} \to 0.$$

Moreover $\mathcal{E}(-1)$ has a filtration $\mathcal{O} \subset F(\mathcal{E}(-1)) \subset \mathcal{E}(-1)$ such that $F(\mathcal{E}(-1))$ fits in the following exact sequences:

$$0 \to F(\mathcal{E}(-1)) \to \mathcal{E}(-1) \to \mathcal{O}(-1)^{\oplus r-3} \to 0;$$

$$0 \to \mathcal{O} \to F(\mathcal{E}(-1)) \to E_3^{-1,1} \to 0.$$

In particular, we see that $F(\mathcal{E}(-1))$ is a vector bundle, since so is $\mathcal{E}(-1)$. On the other hand, since $\mathrm{Ext}^1(\mathcal{S}(-1), \mathcal{O}) = 0$, $F(\mathcal{E}(-1))$ fits in the following exact sequence:

$$0 \to \mathcal{O}(-1)^{\oplus a} \to \mathcal{O} \oplus \mathcal{S}(-1) \to F(\mathcal{E}(-1)) \to 0.$$

This implies that $a = 0$. Indeed, if $a = 1$, then $F(\mathcal{E}(-1))$ cannot be a vector bundle, since the intersection of a line and a hyperplane section cannot be empty. Therefore $F(\mathcal{E}(-1)) \cong \mathcal{O} \oplus \mathcal{S}(-1)$, and thus $\mathcal{E} \cong \mathcal{O}(1) \oplus \mathcal{S} \oplus \mathcal{O}^{\oplus r-3}$. This is Case (3) of Theorem 1.1.

## 12. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (5) of Theorem 2.3

Suppose that $\mathcal{E}|_{\mathbb{Q}^2}$ fits in an exact sequence

$$0 \to \mathcal{O}(-1, -1) \to \mathcal{O}(1, 1) \oplus \mathcal{O}^{\oplus r} \to \mathcal{E}|_{\mathbb{Q}^2} \to 0.$$

Then $c_2 h = 4$. Note that

$$h^q(\mathcal{S}^\vee \otimes \mathcal{E}|_{\mathbb{Q}^2}) = \begin{cases} 4 & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, \end{cases} \tag{12.1}$$

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and that

$$h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = \begin{cases} 1 & \text{if} \quad q = 0, 1 \\ 0 & \text{if} \quad q \neq 0, 1. \end{cases} \tag{12.2}$$

Hence we have

$$h^0(\mathcal{E}(-1)) \leq 1$$

by (7.7).

### 12.1. Suppose that $h^0(\mathcal{E}(-1)) = 1$.

Lemma 6.1 then shows that $c_3 \geq 4$. Hence $H^q(\mathcal{E}(-1))$ vanishes for $q > 0$ by (7.2). The formula (4.4) then shows that

$$h^0(\mathcal{E}(-1)) = -1 + \frac{1}{2}c_3.$$

Thus we have $c_3 = 4$. We also see that $h^q(\mathcal{E}(-2)) = 0$ unless $q = 2$ and that $h^2(\mathcal{E}(-2)) = 1$ by (12.2) and (7.7). We have $h^0(\mathcal{E}) = r + 5$. Since we have an exact sequence

$$0 \to \mathcal{S}^\vee \otimes \mathcal{E}(-2) \to \mathcal{E}(-2)^{\oplus 4} \to \mathcal{S}^\vee \otimes \mathcal{E}(-1) \to 0,$$

we see that $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-2)) = 0$ and that $h^3(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$. Note that $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$. Since we have an exact sequence

$$0 \to \mathcal{S}^\vee \otimes \mathcal{E}(-2) \to \mathcal{S}^\vee \otimes \mathcal{E}(-1) \to \mathcal{S}^\vee \otimes \mathcal{E}(-1)|_{\mathbb{Q}^2} \to 0,$$

we infer that $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$. Since we have an exact sequence (7.14), we see that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}) = 0$ for $q \geq 2$. The exact sequence (7.15) together with (12.1) shows that $h^2(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$. Now the formula (4.8) shows that

$$-h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = \chi(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0,$$

since $c_3 = 4$ and $c_2 h = 4$. The exact sequence (7.14) then implies that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}) = 0$ unless $q = 0$ and that $h^0(\mathcal{S}^\vee \otimes \mathcal{E}) = 4$. Since $h^0(\mathcal{E}(-1)) = 1$, we have an injection $\mathcal{O}(1) \to \mathcal{E}$. Let $\mathcal{F}$ be its cokernel: we have the following exact sequence:

$$0 \to \mathcal{O}(1) \to \mathcal{E} \to \mathcal{F} \to 0.$$

We apply to $\mathcal{F}$ the Bondal spectral sequence (2.1). We see that $h^q(\mathcal{F}) = 0$ unless $q = 0$ and that $h^0(\mathcal{F}) = r$. Moreover $h^q(\mathcal{F}(-1)) = 0$ for any $q$, $h^q(\mathcal{F}(-2)) = 0$ unless $q = 2$ and $h^2(\mathcal{F}(-2)) = 1$. Finally, we have $h^q(\mathcal{S}^\vee \otimes \mathcal{F}) = 0$ for all $q$. Therefore $\operatorname{Ext}^q(G, \mathcal{F}) = 0$ for $q = 3$ and 1, $\operatorname{Ext}^2(G, \mathcal{F}) \cong S_3$ and $\operatorname{Hom}(G, \mathcal{F}) \cong S_0^{\oplus r}$. Hence $E_2^{p,q} = 0$ unless $(p \cdot q) =$

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$(-3,2)$ or $(0,0)$, $E_2^{-3,2} = \mathcal{O}(-1)$ and $E_2^{0,0} = \mathcal{O}^{\oplus r}$ by Lemma 2.1. Thus, we have an exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus r} \to \mathcal{F} \to 0.$$

Therefore $\mathcal{E}$ belongs to Case (4) of Theorem 1.1.

### 12.2. Suppose that $h^0(\mathcal{E}(-1)) = 0$.

Then $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$ by (7.14). Note that $H^q(\mathcal{E}|_{\mathbb{Q}^2})$ vanishes for all $q > 0$. Since $h^q(\mathcal{E}) = 0$ for all $q > 0$ by (7.1), we have $h^q(\mathcal{E}(-1)) = 0$ for all $q \geq 2$. Hence (4.4) and (7.3) imply that

$$0 \geq -h^1(\mathcal{E}(-1)) = \chi(\mathcal{E}(-1)) = -1 + \frac{1}{2}c_3 \geq -1.$$

Therefore, $(h^1(\mathcal{E}(-1)), c_3)$ is either $(0,2)$ or $(1,0)$. Since $h^3(\mathcal{E}(-1)) = 0$, we first have $h^3(\mathcal{S}^\vee \otimes \mathcal{E}) = 0$ by (7.14). Secondly, we have $h^3(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$ by (12.1) and (7.15). Thirdly, we have $h^2(\mathcal{S}^\vee \otimes \mathcal{E}) = 0$ by (7.14) since $h^2(\mathcal{E}(-1)) = 0$. Finally, we have $h^2(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$ by (12.1) and (7.15). Hence

$$-h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = \chi(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = -4 + c_3 \tag{12.3}$$

by (4.8). We apply to $\mathcal{E}$ the Bondal spectral sequence (2.1).

*12.2.1. Suppose that* $(h^1(\mathcal{E}(-1)), c_3) = (0,2)$.

Then $h^1(\mathcal{S}^\vee \otimes \mathcal{E}) = 0$ by (7.14). Moreover $h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 2$ by (12.3). Hence we have $h^0(\mathcal{S}^\vee \otimes \mathcal{E}) = 2$ by (7.14). Since $h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 1$ for $q = 0$, 1 and $h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$ for $q = 2$, 3, we infer that $h^q(\mathcal{E}(-2)) = 1$ for $q = 1$, 2, and that $h^q(\mathcal{E}(-2)) = 0$ unless $q = 1$ or 2. Since $h^0(\mathcal{E}|_{\mathbb{Q}^2}) = r + 4$, we see that $h^0(\mathcal{E}) = r + 4$. Therefore, we have an exact sequence

$$0 \to S_0^{\oplus r+4} \to \mathrm{Hom}(G, \mathcal{E}) \to S_1^{\oplus 2} \to 0$$

and the following: $\mathrm{Ext}^1(G, \mathcal{E}) \cong S_3$; $\mathrm{Ext}^2(G, \mathcal{E}) \cong S_3$ and $\mathrm{Ext}^3(G, \mathcal{E}) = 0$. Therefore, Lemma 2.1 implies that $E_2^{p,q} = 0$ unless $(p,q) = (-3,1)$, $(-3,2)$, $(-1,0)$ or $(0,0)$, that $E_2^{-3,1} \cong \mathcal{O}(-1)$, that $E_2^{-3,2} \cong \mathcal{O}(-1)$ and that there is an exact sequence

$$0 \to E_2^{-1,0} \to \mathcal{S}(-1)^{\oplus 2} \to \mathcal{O}^{\oplus r+4} \to E_2^{0,0} \to 0.$$

It follows from the Bondal spectral sequence (2.1) that $E_2^{-3,1} \cong E_2^{-1,0}$, that $E_2^{-3,2} \cong E_3^{-3,2}$, that $E_2^{0,0} \cong E_3^{0,0}$ and that there is an exact sequence

$$0 \to E_3^{-3,2} \to E_3^{0,0} \to \mathcal{E} \to 0.$$

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Hence we obtain the following exact sequences:

$$0 \to \mathcal{O}(-1) \to \mathcal{S}(-1)^{\oplus 2} \to \mathcal{O}^{\oplus r+4} \to E_3^{0,0} \to 0;$$

$$0 \to \mathcal{O}(-1) \to E_3^{0,0} \to \mathcal{E} \to 0.$$

The latter exact sequence shows that $E_3^{0,0}$ is a vector bundle since so is $\mathcal{E}$. The former exact sequence then splits into the following two exact sequences with $\mathcal{G}$ a vector bundle of rank three:

$$0 \to \mathcal{O}(-1) \to \mathcal{S}(-1)^{\oplus 2} \to \mathcal{G} \to 0;$$

$$0 \to \mathcal{G} \to \mathcal{O}^{\oplus r+4} \to E_3^{0,0} \to 0.$$

The latter exact sequence shows that the dual $\mathcal{G}^\vee$ of $\mathcal{G}$ is globally generated. The injection $\mathcal{O}(-1) \to \mathcal{S}(-1)^{\oplus 2}$ in the former exact sequence gives rise to two global sections $s_0$, $s_1$ of $\mathcal{S}$, and we infer that $(s_0)_0 \cap (s_1)_0 = \emptyset$ since $\mathcal{G}$ is a vector bundle. Hence $s_0$ and $s_1$ are linearly independent. We also see that $\mathcal{G}^\vee$ fits in the following exact sequence:

$$0 \to \mathcal{G}^\vee \to \mathcal{S}^{\oplus 2} \to \mathcal{O}(1) \to 0.$$

Note that the induced map $H^0(\mathcal{S})^{\oplus 2} \to H^0(\mathcal{O}(1))$ sends $(t_0, t_1)$ to $s_0 \wedge t_0 + s_1 \wedge t_1$, and Lemma 3.1 implies that it is surjective. Therefore $h^0(\mathcal{G}^\vee) = 3$. Since $\mathcal{G}^\vee$ is a globally generated vector bundle of rank three, this implies that $\mathcal{G}^\vee \cong \mathcal{O}^{\oplus 3}$. On the other hand, the exact sequence above shows that $c_1(\mathcal{G}^\vee) = 1$. This is a contradiction. Hence the case $(h^1(\mathcal{E}(-1)), c_3) = (0, 2)$ does not arise.

*12.2.2. Suppose that* $(h^1(\mathcal{E}(-1)), c_3) = (1, 0)$.

Then $h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 4$ by (12.3). Set $a := h^0(\mathcal{S}^\vee \otimes \mathcal{E})$. Then $h^1(\mathcal{S}^\vee \otimes \mathcal{E}) = a$ by (7.14). From (12.2), it follows that $h^q(\mathcal{E}(-2)) = 0$ unless $q = 1$ or 2 and that $(h^1(\mathcal{E}(-2)), h^2(\mathcal{E}(-2))) = (1, 0)$ or $(2, 1)$. Note also that $h^0(\mathcal{E}) = r + 3$.

12.2.2.1. Suppose that $(h^1(\mathcal{E}(-2)), h^2(\mathcal{E}(-2))) = (1, 0)$. Then we see that $\operatorname{Ext}^3(G, \mathcal{E}) = 0$, that $\operatorname{Ext}^2(G, \mathcal{E}) = 0$, that $\operatorname{Ext}^1(G, \mathcal{E})$ has a filtration $S_1^{\oplus a} \subset F \subset \operatorname{Ext}^1(G, \mathcal{E})$ of right $A$-modules such that the following sequences are exact:

$$0 \to F \to \operatorname{Ext}^1(G, \mathcal{E}) \to S_3 \to 0;$$

$$0 \to S_1^{\oplus a} \to F \to S_2 \to 0,$$

and that $\operatorname{Hom}(G, \mathcal{E})$ fits in the following exact sequence of right $A$-modules:

$$0 \to S_0^{\oplus r+3} \to \operatorname{Hom}(G, \mathcal{E}) \to S_1^{\oplus a} \to 0.$$

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These exact sequences induce the following distinguished triangles by Lemma 2.1:

$$F \otimes^{\mathrm{L}}{}_A G \to \mathrm{Ext}^1(G, \mathcal{E}) \otimes^{\mathrm{L}}{}_A G \to \mathcal{O}(-1)[3] \to;$$

$$\mathcal{S}(-1)[1]^{\oplus a} \to F \otimes^{\mathrm{L}}{}_A G \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}[2] \to;$$

$$\mathcal{O}^{\oplus r+3} \to \mathrm{Hom}(G, \mathcal{E}) \otimes^{\mathrm{L}}{}_A G \to \mathcal{S}(-1)[1]^{\oplus a} \to .$$

By taking cohomologies, we obtain the following exact sequences by (3.2):

$$0 \to E_2^{-3,1} \to \mathcal{O}(-1) \to \mathcal{H}^{-2}(F \otimes^{\mathrm{L}}{}_A G) \to E_2^{-2,1} \to 0;$$

$$0 \to \mathcal{H}^{-2}(F \otimes^{\mathrm{L}}{}_A G) \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \xrightarrow{\psi_a} \mathcal{S}^{\vee \oplus a} \to E_2^{-1,1} \to 0; \tag{12.4}$$

$$0 \to E_2^{-1,0} \to \mathcal{S}^{\vee \oplus a} \to \mathcal{O}^{\oplus r+3} \to E_2^{0,0} \to 0.$$

Moreover, we have the following exact sequences:

$$0 \to E_2^{-2,1} \to E_2^{0,0} \to E_3^{0,0} \to 0;$$

$$0 \to E_2^{-3,1} \to E_2^{-1,0} \to 0;$$

$$0 \to E_3^{0,0} \to \mathcal{E} \to E_2^{-1,1} \to 0.$$

Since $\mathcal{E}$ is nef, $E_2^{-1,1}$ cannot admit negative degree quotients. Hence it follows from Lemma 5.3 that $a = 0$. Then $E_2^{-1,1} = 0$, $E_2^{-3,1} = E_2^{-1,0} = 0$, $E_2^{0,0} = \mathcal{O}^{\oplus r+3}$, and we have the following exact sequence:

$$0 \to \mathcal{O}(-1) \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to E_2^{-2,1} \to 0.$$

Hence $\mathcal{E}$ fits in the following exact sequence:

$$0 \to \mathcal{O}(-1) \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to \mathcal{O}^{\oplus r+3} \to \mathcal{E} \to 0. \tag{12.5}$$

Since $T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}$ fits in an exact sequence

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^{\oplus 5} \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to 0,$$

the exact sequence (12.5) induces the following exact sequence:

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^{\oplus 4} \to \mathcal{O}^{\oplus r+3} \to \mathcal{E} \to 0.$$

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This is Case (9) of Theorem 1.1.

12.2.2.2. Suppose that $(h^1(\mathcal{E}(-2)), h^2(\mathcal{E}(-2))) = (2,1)$. Then we see that $\mathrm{Ext}^3(G,\mathcal{E}) = 0$, that $\mathrm{Ext}^2(G,\mathcal{E}) \cong S_3$, that $\mathrm{Ext}^1(G,\mathcal{E})$ has a filtration $S_1^{\oplus a} \subset F \subset \mathrm{Ext}^1(G,\mathcal{E})$ of right $A$-modules such that the following sequences are exact:

$$0 \to F \to \mathrm{Ext}^1(G,\mathcal{E}) \to S_3^{\oplus 2} \to 0;$$

$$0 \to S_1^{\oplus a} \to F \to S_2 \to 0,$$

and that $\mathrm{Hom}(G,\mathcal{E})$ fits in the following exact sequence of right $A$-modules:

$$0 \to S_0^{\oplus r+3} \to \mathrm{Hom}(G,\mathcal{E}) \to S_1^{\oplus a} \to 0.$$

Lemma 2.1 implies that $\mathrm{Ext}^2(G,\mathcal{E}) \otimes^{\mathrm{L}}{}_A G \cong \mathcal{O}(-1)[3]$ and that the three exact sequences above induce the following distinguished triangles:

$$F \otimes^{\mathrm{L}}{}_A G \to \mathrm{Ext}^1(G,\mathcal{E}) \otimes^{\mathrm{L}}{}_A G \to \mathcal{O}(-1)^{\oplus 2}[3] \to;$$

$$\mathcal{S}(-1)[1]^{\oplus a} \to F \otimes^{\mathrm{L}}{}_A G \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}[2] \to;$$

$$\mathcal{O}^{\oplus r+3} \to \mathrm{Hom}(G,\mathcal{E}) \otimes^{\mathrm{L}}{}_A G \to \mathcal{S}(-1)[1]^{\oplus a} \to .$$

By taking cohomologies, we see that $E_2^{p,2} = 0$ unless $p = -3$, that $E_2^{-3,2} \cong \mathcal{O}(-1)$, and that we have the following exact sequences by (3.2):

$$0 \to E_2^{-3,1} \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{H}^{-2}(F \otimes^{\mathrm{L}}{}_A G) \to E_2^{-2,1} \to 0; \tag{12.6}$$

$$0 \to \mathcal{H}^{-2}(F \otimes^{\mathrm{L}}{}_A G) \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \xrightarrow{\psi_a} \mathcal{S}^{\vee \oplus a} \to E_2^{-1,1} \to 0; \tag{12.7}$$

$$0 \to E_2^{-1,0} \to \mathcal{S}^{\vee \oplus a} \to \mathcal{O}^{\oplus r+3} \to E_2^{0,0} \to 0.$$

Moreover, we have the following exact sequences:

$$0 \to E_3^{-3,2} \to E_2^{-3,2} \xrightarrow{\pi} E_2^{-1,1} \to E_3^{-1,1} \to 0; \tag{12.8}$$

$$0 \to E_2^{-2,1} \to E_2^{0,0} \to E_3^{0,0} \to 0;$$

$$0 \to E_2^{-3,1} \to E_2^{-1,0} \to 0;$$

$$0 \to E_3^{-3,2} \to E_3^{0,0} \to E_4^{0,0} \to 0;$$

$$0 \to E_4^{0,0} \to \mathcal{E} \to E_3^{-1,1} \to 0.$$

Since $\mathcal{E}$ is nef, $E_3^{-1,1}$ cannot admit negative degree quotients. If $a > 0$, it follows from Lemmas 5.4 and 5.3 that $a = 1$, that $E_3^{-1,1} = 0$, that $E_3^{-3,2} \cong \mathcal{I}_L(-1)$ for some line $L \subset \mathbb{Q}^3$, that $E_2^{-1,1} \cong \mathcal{O}_L(-1)$ and that $\mathcal{H}^{-2}(F \otimes^{\mathrm{L}}{}_A G) \cong \mathcal{O}(-1)^{\oplus 2}$. Therefore, $\mathcal{E} \cong E_4^{0,0}$ and the exact sequence (12.6) becomes the following exact sequence:

$$0 \to E_2^{-3,1} \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}(-1)^{\oplus 2} \to E_2^{-2,1} \to 0.$$

Set $\mathcal{O}(-1)^{\oplus b} \cong E_2^{-3,1}$ for some non-negative integer $b \le 2$. Then $E_2^{-2,1} \cong \mathcal{O}(-1)^{\oplus b}$ and we have the following exact sequences:

$$0 \to \mathcal{O}(-1)^{\oplus b} \to \mathcal{S}^\vee \to \mathcal{O}^{\oplus r+3} \to E_2^{0,0} \to 0;$$

$$0 \to \mathcal{O}(-1)^{\oplus b} \to E_2^{0,0} \to E_3^{0,0} \to 0;$$

$$0 \to \mathcal{I}_L(-1) \to E_3^{0,0} \to \mathcal{E} \to 0.$$

Since $\mathcal{O}^{\oplus r+3}$ is torsion-free and $\mathcal{S}^\vee$ is not isomorphic to $\mathcal{O}^{\oplus 2}$, we see that $b \le 1$. Note here that $E_3^{0,0}$ is torsion-free, and so is $E_2^{0,0}$. If $b = 1$, we get an exact sequence

$$0 \to \mathcal{I}_M \to \mathcal{O}^{\oplus r+3} \to E_2^{0,0} \to 0$$

for some line $M$ in $\mathbb{Q}^3$. Since we can extend $\mathcal{I}_M \to \mathcal{O}^{\oplus r+3}$ to an injection $\mathcal{O} \to \mathcal{O}^{\oplus r+3}$ by taking double duals, we infer that $E_2^{0,0}$ contains a torsion sheaf $\mathcal{O}_M$. This is a contradiction. Hence $b = 0$, and $E_2^{0,0}$ fits in the following exact sequences:

$$0 \to \mathcal{S}^\vee \to \mathcal{O}^{\oplus r+3} \to E_2^{0,0} \to 0;$$

$$0 \to \mathcal{I}_L(-1) \to E_2^{0,0} \to \mathcal{E} \to 0.$$

Since $\mathcal{I}_L(-1)$ is torsion-free but not locally free, so is $E_2^{0,0}$. Hence the former exact sequence together with (3.1) implies that $E_2^{0,0} \cong \mathcal{I}_M(1) \oplus \mathcal{O}^{\oplus r}$ for some line $M$ in $\mathbb{Q}^3$. This can be shown by the similar argument as in the proof of Lemma 5.2. Indeed, by taking a free basis of $\mathcal{O}^{\oplus r+3}$ suitably, we may assume that the injection $\mathcal{S}^\vee \to \mathcal{O}^{\oplus r+3}$ is written as ${}^t(s_1^\vee, \dots, s_m^\vee, 0, \dots, 0)$ for some linearly independent elements $s_1, \dots, s_m$ of $H^0(\mathcal{S})$, where $s_i^\vee$ denotes the dual of the morphism $\mathcal{O} \to \mathcal{S}$ defined by $s_i$. We have $2 = \operatorname{rank} \mathcal{S}^\vee \le m \le h^0(\mathcal{S}) = 4$. Since $E_2^{0,0}$ is torsion-free, we have $3 \le m$. Since $E_2^{0,0}$ is not

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locally free, it follows from the exact sequence (3.1) that $m \neq 4$. Hence $m=3$. Moreover, the exact sequence (3.1) shows that if we extend $(s_1, s_2, s_3)$ to a basis $(s_1, s_2, s_3, s_4)$ of $H^0(\mathcal{S})$ then there exists a basis $(t_1, t_2, t_3, t_4)$ of $H^0(\mathcal{S})$ such that $\sum_{i=1}^{4} t_i s_i^{\vee} = 0$ and that the cokernel of the morphism ${}^t(s_1^{\vee}, s_2^{\vee}, s_3^{\vee})$ is isomorphic to the cokernel of the morphism $t_4 : \mathcal{O} \to \mathcal{S}$. Hence the cokernel of ${}^t(s_1^{\vee}, s_2^{\vee}, s_3^{\vee})$ is isomorphic to $\mathcal{I}_M(1)$ for some line $M$ on $\mathbb{Q}^3$. Therefore $E_2^{0,0} \cong \mathcal{I}_M(1) \oplus \mathcal{O}^{\oplus r}$. By taking the double dual of the injection $\mathcal{I}_L(-1) \to E_2^{0,0}$ in the latter exact sequence, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_L(-1) & \longrightarrow & E_2^{0,0} & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}(-1) & \longrightarrow & \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

for some coherent sheaf $\mathcal{F}$. Note that $\mathrm{Tor}_q^{\mathcal{O}_p}(\mathcal{F}_p, k(p)) = 0$ for $q \geq 2$ and any point $p$. Since $\mathcal{E}$ is torsion-free, the snake lemma implies that $L=M$ and that we have an exact sequence

$$0 \to \mathcal{O}_L(-1) \to \mathcal{O}_M(1) \to \mathcal{O}_Z \to 0$$

for some closed subscheme $Z$ of length two. Moreover, $\mathcal{E}$ fits in the following exact sequence:

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{O}_Z \to 0.$$

For an associated point $p$ of $Z$, the exact sequence above induces a coherent sheaf $\mathcal{G}$ and the following exact sequence:

$$0 \to \mathcal{E} \to \mathcal{G} \to k(p) \to 0.$$

Since $\mathrm{Tor}_3^{\mathcal{O}_p}(\mathcal{F}_p, k(p)) = 0$, we have $\mathrm{Tor}_3^{\mathcal{O}_p}(\mathcal{G}_p, k(p)) = 0$. Note that $\mathrm{Tor}_q^{\mathcal{O}_p}(\mathcal{E}_p, k(p)) = 0$ for $q \geq 1$. Hence $\mathrm{Tor}_3^{\mathcal{O}_p}(k(p), k(p)) = 0$, which contradicts the fact that $\mathrm{Tor}_3^{\mathcal{O}_p}(k(p), k(p)) = 1$. Therefore, $a$ cannot be positive: $a=0$. Thus $0 = E_2^{-1,1} = E_3^{-1,1}$, $0 = E_2^{-1,0} = E_2^{-3,1}$, $\mathcal{O}^{\oplus r+3} \cong E_2^{0,0}$, $E_3^{-3,2} \cong E_2^{-3,2} \cong \mathcal{O}(-1)$, $E_4^{0,0} \cong \mathcal{E}$, and we have the following exact sequences:

$$0 \to \mathcal{O}(-1)^{\oplus 2} \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to E_2^{-2,1} \to 0;$$

$$0 \to E_2^{-2,1} \to \mathcal{O}^{\oplus r+3} \to E_3^{0,0} \to 0;$$

$$0 \to \mathcal{O}(-1) \to E_3^{0,0} \to \mathcal{E} \to 0.$$

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Since $T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3}$ fits in an exact sequence

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^{\oplus 5} \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to 0,$$

$E_2^{-2,1}$ has a resolution of the following form:

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^{\oplus 3} \to E_2^{-2,1} \to 0.$$

Therefore, we see that $\mathcal{E}$ belongs to Case (9) of Theorem 1.1.

## 13. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (6) of Theorem 2.3

Suppose that $\mathcal{E}|_{\mathbb{Q}^2}$ fits in the following exact sequence:

$$0 \to \mathcal{O}^{\oplus 2} \to \mathcal{O}(1,0)^{\oplus 2} \oplus \mathcal{O}(0,1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-2} \to \mathcal{E}|_{\mathbb{Q}^2} \to 0.$$

Then $h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$ for any $q$, and $c_2h = 4$. Since $h^q(\mathcal{E}|_{\mathbb{Q}^2}) = 0$ for any $q > 0$, the vanishing (7.1) shows that $h^q(\mathcal{E}(-t)) = 0$ for $q \geq 2$ and $t = 1, 2$. The assumption (7.7) together with (4.5) and (7.3) shows that

$$0 \geq -h^1(\mathcal{E}(-2)) = \chi(\mathcal{E}(-2)) = -1 + \frac{1}{2}c_3 \geq -1.$$

Therefore we have two cases: $(h^1(\mathcal{E}(-2)), c_3) = (0,2)$ or $(1,0)$. Note here that $h^q(\mathcal{E}(-1)) = h^q(\mathcal{E}(-2))$ for any $q$. In particular, $h^0(\mathcal{E}(-1)) = h^0(\mathcal{E}(-2)) = 0$ by (7.7).

We claim here that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}(t)|_{\mathbb{Q}^2}) = 0$ for $q > 0$ and $t \geq 0$. Indeed, we see that

$$h^q((\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)) \otimes (\mathcal{O}(1+t,t)^{\oplus 2} \oplus \mathcal{O}(t,1+t)^{\oplus 2} \oplus \mathcal{O}(t,t)^{\oplus r-3})) = 0$$

for $q > 0$ and $t \geq 0$. Hence we obtain the claim. Then it follows from (7.6) that

$$h^q(\mathcal{S}^\vee \otimes \mathcal{E}(t)) = 0 \text{ for } q \geq 2 \text{ and } t = 0, -1. \tag{13.1}$$

Since $h^0(\mathcal{E}(-1)) = 0$, the exact sequence (7.14) together with (13.1) shows that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$ unless $q = 1$. Hence

$$-h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = \chi(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = -4 + c_3 \tag{13.2}$$

by (4.8).

### 13.1. Suppose that $(h^1(\mathcal{E}(-2)), c_3) = (0,2)$.

Then $h^q(\mathcal{E}(-2)) = 0$ for any $q$. Hence $h^q(\mathcal{E}(-1)) = 0$ for any $q$. Set $a = h^1(\mathcal{E}(-2)|_{\mathbb{Q}^2})$. Then $a \leq 2$ and $h^2(\mathcal{E}(-2)|_{\mathbb{Q}^2}) = r - 4 + a$. Thus $h^2(\mathcal{E}(-3)) = a$, $h^3(\mathcal{E}(-3)) = r - 4 + a$ and $h^q(\mathcal{E}(-3)) = 0$ unless $q = 2$ or 3. It follows from (13.2) that $h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 2$. We

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apply to $\mathcal{E}(-1)$ the Bondal spectral sequence (2.1). We have $\mathrm{Ext}^3(G,\mathcal{E}(-1)) \cong S_3^{\oplus r-4+a}$, $\mathrm{Ext}^2(G,\mathcal{E}(-1)) \cong S_3^{\oplus a}$, $\mathrm{Ext}^1(G,\mathcal{E}(-1)) \cong S_1^{\oplus 2}$ and $\mathrm{Hom}(G,\mathcal{E}(-1)) = 0$. Lemma 2.1 then shows that $E_2^{-3,3} \cong \mathcal{O}(-1)^{\oplus r-4+a}$, that $E_2^{-3,2} \cong \mathcal{O}(-1)^{\oplus a}$, that $E_2^{-1,1} \cong \mathcal{S}(-1)^{\oplus 2}$ and that $E_2^{p,q} = 0$ unless $(p,q) = (-3,3)$, $(-3,2)$ or $(-1,1)$. Then $\mathcal{E}(-1)$ fits in the $(-1)$-twist of the following exact sequence:

$$0 \to \mathcal{O}^{\oplus a} \to \mathcal{S}^{\oplus 2} \to \mathcal{E} \to \mathcal{O}^{\oplus r-4+a} \to 0. \tag{13.3}$$

This sequence splits into the following two exact sequences:

$$0 \to \mathcal{O}^{\oplus a} \to \mathcal{S}^{\oplus 2} \to \mathcal{F} \to 0;$$

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O}^{\oplus r-4+a} \to 0,$$

where $\mathcal{F}$ is a globally generated vector bundle of rank $4-a$. We claim here that $a \leq 1$. Indeed, if $a=2$, then we have the following exact sequences:

$$0 \to \mathcal{O} \to \mathcal{S}^{\oplus 2} \to \mathcal{G} \to 0;$$

$$0 \to \mathcal{O} \to \mathcal{G} \to \mathcal{F} \to 0,$$

where $\mathcal{G}$ is a globally generated vector bundle of rank 3. Since $\mathcal{F}$ is a vector bundle, $\mathcal{G}$ must have a nowhere vanishing global section, and thus $c_3(\mathcal{G}) = 0$. On the other hand, $c_3(\mathcal{G}) = c_3(\mathcal{S}^{\oplus 2}) = 2c_2(\mathcal{S})h = 2$. This is a contradiction. Hence the case $a=2$ does not arise. Now note that $\mathcal{E}$ is isomorphic to $\mathcal{F} \oplus \mathcal{O}^{\oplus r-4+a}$ since $h^1(\mathcal{F}) = 0$. Therefore, $\mathcal{E}$ fits in an exact sequence

$$0 \to \mathcal{O}^{\oplus a} \to \mathcal{S}^{\oplus 2} \oplus \mathcal{O}^{\oplus r-4+a} \to \mathcal{E} \to 0,$$

where the composite of the inclusion $\mathcal{O}^{\oplus a} \to \mathcal{S}^{\oplus 2} \oplus \mathcal{O}^{\oplus r-4+a}$ and the projection $\mathcal{S}^{\oplus 2} \oplus \mathcal{O}^{\oplus r-4+a} \to \mathcal{O}^{\oplus r-4+a}$ is zero. This is Case (5) of Theorem 1.1.

### 13.2. Suppose that $(h^1(\mathcal{E}(-2)), c_3) = (1,0)$.

Then $h^1(\mathcal{E}(-1)) = 1$. Hence $h^0(\mathcal{E}) = h^0(\mathcal{E}|_{\mathbb{Q}^2}) - 1 = r+3$. It follows from (13.2) that $h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 4$. Set $a = h^0(\mathcal{S}^\vee \otimes \mathcal{E})$. Then the exact sequence (7.14) shows that $a \leq 4$, that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}) = 0$ unless $q=0$ or 1 and that $h^1(\mathcal{S}^\vee \otimes \mathcal{E}) = a$. Hence we have $\mathrm{Ext}^q(G,\mathcal{E}) = 0$ for $q=2$ and 3, and $\mathrm{Hom}(G,\mathcal{E})$ fits in an exact sequence

$$0 \to S_0^{\oplus r+3} \to \mathrm{Hom}(G,\mathcal{E}) \to S_1^{\oplus a} \to 0.$$

Moreover, $\mathrm{Ext}^1(G,\mathcal{E})$ has a filtration $S_1^{\oplus a} \subset F \subset \mathrm{Ext}^1(G,\mathcal{E})$ of right $A$-modules such that the following sequences are exact:

$$0 \to F \to \mathrm{Ext}^1(G,\mathcal{E}) \to S_3 \to 0;$$

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$$0 \to S_1^{\oplus a} \to F \to S_2 \to 0.$$

Now the structures of right $A$-modules $\mathrm{Ext}^q(G, \mathcal{E})$'s are the same as those of $\mathrm{Ext}^q(G, \mathcal{E})$'s in § 12.2.2.1, and we conclude that $\mathcal{E}$ belongs to Case (9) of Theorem 1.1.

## 14. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (7) of Theorem 2.3

Suppose that $\mathcal{E}|_{\mathbb{Q}^2}$ fits in the following exact sequence:

$$0 \to \mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1) \to \mathcal{O}^{\oplus r+3} \to \mathcal{E}|_{\mathbb{Q}^2} \to 0.$$

Then $c_2h = 5$. It then follows from (6.1) that $c_3 \geq 4$. Note that

$$h^q(\mathcal{S}^\vee \otimes \mathcal{E}|_{\mathbb{Q}^2}) = \begin{cases} 2 & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, \end{cases} \tag{14.1}$$

and that

$$h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = \begin{cases} 1 & \text{if} \quad q = 1 \\ 0 & \text{if} \quad q \neq 1. \end{cases} \tag{14.2}$$

Hence we have

$$h^0(\mathcal{E}(-1)) = 0$$

by (7.7). Note that $H^q(\mathcal{E}|_{\mathbb{Q}^2})$ vanishes for any $q > 0$. Since $h^q(\mathcal{E}) = 0$ for any $q > 0$ by (7.1), we have $h^q(\mathcal{E}(-1)) = 0$ for any $q \geq 2$. Hence it follows from (4.4) that

$$0 \geq -h^1(\mathcal{E}(-1)) = \chi(\mathcal{E}(-1)) = -\frac{5}{2} + \frac{1}{2}c_3 \geq -\frac{1}{2}.$$

Therefore $c_3 = 5$ and $h^1(\mathcal{E}(-1)) = 0$. Now it follows from (7.14) that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}) = h^{q+1}(\mathcal{S}^\vee \otimes \mathcal{E}(-1))$ for any $q$. In particular, $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$. Moreover $h^{q+1}(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = h^{q+1}(\mathcal{S}^\vee \otimes \mathcal{E})$ for $q \geq 1$ by (14.1) and (7.15). Hence $h^q(\mathcal{S}^\vee \otimes \mathcal{E}) = 0$ for $q \geq 1$ and $h^q(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$ for $q \geq 2$. Therefore

$$-h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = \chi(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = -6 + c_3 = -1$$

by (4.8). Thus $h^0(\mathcal{S}^\vee \otimes \mathcal{E}) = 1$. We apply to $\mathcal{E}$ the Bondal spectral sequence (2.1). From (14.2), it follows that $h^q(\mathcal{E}(-2)) = 0$ unless $q = 2$ and that $h^2(\mathcal{E}(-2)) = 1$. Since $h^0(\mathcal{E}|_{\mathbb{Q}^2}) = r + 3$, we see that $h^0(\mathcal{E}) = r + 3$. Hence we have an exact sequence

$$0 \to S_0^{\oplus r+3} \to \mathrm{Hom}(G, \mathcal{E}) \to S_1 \to 0,$$

and the following: $\mathrm{Ext}^q(G, \mathcal{E}) = 0$ for $q = 1, 3$; $\mathrm{Ext}^2(G, \mathcal{E}) \cong S_3$. Therefore, Lemma 2.1 implies that $E_2^{p,q} = 0$ unless $(p, q) = (-3, 2)$ or $(0, 0)$, that $E_2^{-3,2} \cong \mathcal{O}(-1)$, and that there is the following exact sequence:

$$0 \to \mathcal{S}(-1) \to \mathcal{O}^{\oplus r+3} \to E_2^{0,0} \to 0.$$

Note that we have the following exact sequence:

$$0 \to E_2^{-3,2} \to E_2^{0,0} \to \mathcal{E} \to 0.$$

Since $\mathrm{Ext}^1(\mathcal{O}(-1), \mathcal{S}(-1)) = 0$, this implies that $\mathcal{E}$ fits in the following exact sequence:

$$0 \to \mathcal{S}(-1) \oplus \mathcal{O}(-1) \to \mathcal{O}^{\oplus r+3} \to \mathcal{E} \to 0.$$

This is Case (6) of Theorem 1.1.

## 15. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (8) of Theorem 2.3

Suppose that $\mathcal{E}|_{\mathbb{Q}^2}$ fits in the following exact sequence:

$$0 \to \mathcal{O}(-1, -2) \to \mathcal{O}(1, 0) \oplus \mathcal{O}^{\oplus r} \to \mathcal{E}|_{\mathbb{Q}^2} \to 0.$$

Then $c_2 h = 6$. It then follows from (6.1) that $c_3 \geq 8$. Note that

$$h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = \begin{cases} 2 & \text{if} \quad q = 1 \\ 0 & \text{if} \quad q \neq 1, \end{cases} \tag{15.1}$$

and that

$$h^q(\mathcal{S}^\vee \otimes \mathcal{E}|_{\mathbb{Q}^2}) = \begin{cases} 1 & \text{if} \quad q = 0, 1 \\ 0 & \text{if} \quad q \neq 0, 1. \end{cases} \tag{15.2}$$

Hence we have

$$h^0(\mathcal{E}(-1)) = 0$$

by (7.7). Note that $H^q(\mathcal{E}|_{\mathbb{Q}^2})$ vanishes for all $q > 0$. Since $h^q(\mathcal{E}) = 0$ for all $q > 0$ by (7.1), we have $h^q(\mathcal{E}(-1)) = 0$ for all $q \geq 2$. It follows from (4.4) that

$$0 \geq -h^1(\mathcal{E}(-1)) = \chi(\mathcal{E}(-1)) = -4 + \frac{1}{2} c_3 \geq 0.$$

Therefore $c_3 = 8$ and $h^1(\mathcal{E}(-1)) = 0$. Now it follows from (7.14) that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}) = h^{q+1}(\mathcal{S}^\vee \otimes \mathcal{E}(-1))$ for any $q$. In particular, $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$. Moreover $h^{q+1}(\mathcal{S}^\vee \otimes$

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$\mathcal{E}(-1)) = h^{q+1}(\mathcal{S}^\vee \otimes \mathcal{E})$ for $q \geq 2$ by (7.15) and (15.2). Hence $h^q(\mathcal{S}^\vee \otimes \mathcal{E}) = 0$ for $q \geq 2$ and $h^3(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$. Hence

$$-h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) + h^2(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = \chi(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = -8 + c_3 = 0$$

by (4.8). Set $a = h^0(\mathcal{S}^\vee \otimes \mathcal{E})$. Then $a = h^1(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = h^2(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = h^1(\mathcal{S}^\vee \otimes \mathcal{E})$. We see that $a = 1$ by (7.15) and (15.2). We apply to $\mathcal{E}$ the Bondal spectral sequence (2.1). It follows from (15.1) that $h^q(\mathcal{E}(-2))$ vanishes unless $q = 2$ and that $h^2(\mathcal{E}(-2)) = 2$. Since $h^0(\mathcal{E}|_{\mathbb{Q}^2}) = r+2$, we see that $h^0(\mathcal{E}) = r+2$. Therefore, $\mathrm{Ext}^3(G, \mathcal{E}) = 0$, $\mathrm{Ext}^2(G, \mathcal{E}) \cong S_3^{\oplus 2}$, $\mathrm{Ext}^1(G, \mathcal{E}) \cong S_1$ and $\mathrm{Hom}(G, \mathcal{E})$ fits in the following exact sequence:

$$0 \to S_0^{\oplus r+2} \to \mathrm{Hom}(G, \mathcal{E}) \to S_1 \to 0.$$

Therefore, Lemma 2.1 implies that $E_2^{p,q} = 0$ unless $(p, q) = (-3, 2)$, $(-1, 1)$, $(-1, 0)$ or $(0, 0)$, that $E_2^{-3,2} \cong \mathcal{O}(-1)^{\oplus 2}$, that $E_2^{-1,1} \cong \mathcal{S}(-1)$ and that there exists the following exact sequence:

$$0 \to E_2^{-1,0} \to \mathcal{S}(-1) \to \mathcal{O}^{\oplus r+2} \to E_2^{0,0} \to 0.$$

The Bondal spectral sequence implies that $E_2^{-1,0} = 0$, that $E_2^{0,0} \cong E_3^{0,0}$ and that we have the following exact sequences:

$$0 \to E_3^{-3,2} \to \mathcal{O}(-1)^{\oplus 2} \xrightarrow{\varphi} \mathcal{S}(-1) \to E_3^{-1,1} \to 0;$$

$$0 \to E_3^{-3,2} \to E_3^{0,0} \to E_4^{0,0} \to 0;$$

$$0 \to E_4^{0,0} \to \mathcal{E} \to E_3^{-1,1} \to 0.$$

Since $\mathcal{E}$ is nef, $E_3^{-1,1}$ cannot admit a negative degree quotient. Hence $\varphi \neq 0$. Thus, there exists an inclusion $\iota : \mathcal{O}(-1) \to \mathcal{O}(-1)^{\oplus 2}$ such that $\varphi \circ \iota \neq 0$. Now we have a morphism $\bar{\varphi} : \mathcal{O}(-1) \cong \mathrm{Coker}(\iota) \to \mathrm{Coker}(\varphi \circ \iota) \cong \mathcal{I}_L$ for some line $L$ in $\mathbb{Q}^3$ and $\bar{\varphi}$ fits in the following exact sequence:

$$0 \to E_3^{-3,2} \to \mathcal{O}(-1) \xrightarrow{\bar{\varphi}} \mathcal{I}_L \to E_3^{-1,1} \to 0.$$

This shows that $E_3^{-1,1}|_M$ admits a negative degree quotient for some line $M$ in $\mathbb{Q}^3$. This is a contradiction. Therefore, $\mathcal{E}|_{\mathbb{Q}^2}$ cannot belong to Case (8) of Theorem 2.3.

## 16. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (9) of Theorem 2.3

Suppose that $\mathcal{E}|_{\mathbb{Q}^2}$ fits in the following exact sequence:

$$0 \to \mathcal{O}(-1, -1)^{\oplus 2} \to \mathcal{O}^{\oplus r+2} \to \mathcal{E}|_{\mathbb{Q}^2} \to 0.$$

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Then $c_2 h = 6$. It then follows from (6.1) that $c_3 \geq 8$. Note that

$$h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = \begin{cases} 2 & \text{if} \quad q = 1 \\ 0 & \text{if} \quad q \neq 1, \end{cases} \tag{16.1}$$

and that

$$h^q(\mathcal{S}^\vee \otimes \mathcal{E}|_{\mathbb{Q}^2}) = 0 \, \text{for all } q. \tag{16.2}$$

Hence we have

$$h^0(\mathcal{E}(-1)) = 0$$

by (7.7). Note that $H^q(\mathcal{E}|_{\mathbb{Q}^2})$ vanishes for all $q > 0$. Since $h^q(\mathcal{E}) = 0$ for all $q > 0$ by (7.1), we have $h^q(\mathcal{E}(-1)) = 0$ for all $q \geq 2$. It follows from (4.4) that

$$0 \geq -h^1(\mathcal{E}(-1)) = \chi(\mathcal{E}(-1)) = -4 + \frac{1}{2} c_3 \geq 0.$$

Therefore $c_3 = 8$ and $h^1(\mathcal{E}(-1)) = 0$. Now it follows from (7.14) that $h^q(\mathcal{S}^\vee \otimes \mathcal{E}) = h^{q+1}(\mathcal{S}^\vee \otimes \mathcal{E}(-1))$ for any $q$. Moreover $h^{q+1}(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = h^{q+1}(\mathcal{S}^\vee \otimes \mathcal{E})$ for any $q$ by (7.15) and (16.2). Hence $h^q(\mathcal{S}^\vee \otimes \mathcal{E}) = 0$ for any $q$. We apply to $\mathcal{E}$ the Bondal spectral sequence (2.1). It follows from (16.1) that $h^q(\mathcal{E}(-2))$ vanishes unless $q = 2$ and that $h^2(\mathcal{E}(-2)) = 2$. Since $h^0(\mathcal{E}|_{\mathbb{Q}^2}) = r + 2$, we see that $h^0(\mathcal{E}) = r + 2$. Therefore, $\mathrm{Hom}(G, \mathcal{E}) \cong S_0^{\oplus r+2}$, $\mathrm{Ext}^1(G, \mathcal{E}) = 0$, $\mathrm{Ext}^2(G, \mathcal{E}) \cong S_3^{\oplus 2}$ and $\mathrm{Ext}^3(G, \mathcal{E}) = 0$. Therefore, Lemma 2.1 implies that $E_2^{p,q} = 0$ unless $(p, q) = (-3, 2)$ or $(0, 0)$, that $E_2^{-3,2} \cong \mathcal{O}(-1)^{\oplus 2}$ and that $E_2^{0,0} \cong \mathcal{O}^{\oplus r+2}$. It follows from the Bondal spectral sequence that $\mathcal{E}$ fits in the following exact sequence:

$$0 \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}^{\oplus r+2} \to \mathcal{E} \to 0.$$

This is Case (7) of Theorem 1.1.

## 17. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (10) of Theorem 2.3

Suppose that $\mathcal{E}|_{\mathbb{Q}^2}$ fits in the following exact sequence:

$$0 \to \mathcal{O}(-2, -2) \to \mathcal{O}^{\oplus r+1} \to \mathcal{E}|_{\mathbb{Q}^2} \to 0.$$

Then $c_2 h = 8$. It then follows from (6.1) that $c_3 \geq 16$. Note that

$$h^q(\mathcal{E}|_{\mathbb{Q}^2}) = \begin{cases} r + 1 & \text{if} \quad q = 0 \\ 1 & \text{if} \quad q = 1 \\ 0 & \text{if} \quad q = 2, \end{cases} \tag{17.1}$$

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that

$$h^q(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = \begin{cases} 4 & \text{if} \quad q = 1 \\ 0 & \text{if} \quad q \neq 1, \end{cases} \tag{17.2}$$

that

$$h^q(\mathcal{S}^\vee \otimes \mathcal{E}(1)|_{\mathbb{Q}^2}) = \begin{cases} 4r + 4 & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, \end{cases} \tag{17.3}$$

and that

$$h^q(\mathcal{S}^\vee \otimes \mathcal{E}|_{\mathbb{Q}^2}) = \begin{cases} 4 & \text{if} \quad q = 1 \\ 0 & \text{if} \quad q \neq 1. \end{cases} \tag{17.4}$$

Hence we have

$$h^0(\mathcal{E}(-1)) = 0$$

by (7.7). Then $h^0(\mathcal{S}^\vee \otimes \mathcal{E}(-1)) = 0$ by (7.14). Since $h^q(\mathcal{E}) = 0$ for all $q > 0$ by (7.1), we have $h^2(\mathcal{E}(-1)) = 1$ and $h^3(\mathcal{E}(-1)) = 0$ by (17.1). It then follows from (4.4) that

$$1 \geq 1 - h^1(\mathcal{E}(-1)) = \chi(\mathcal{E}(-1)) = -7 + \frac{1}{2}c_3 \geq 1.$$

Therefore $c_3 = 16$ and $h^1(\mathcal{E}(-1)) = 0$. Hence $h^0(\mathcal{E}) = r + 1$ since $h^0(\mathcal{E}|_{\mathbb{Q}^2}) = r + 1$ by (17.1). Moreover $h^2(\mathcal{E}(-2)) = 5$ and $h^q(\mathcal{E}(-2)) = 0$ unless $q = 2$ by (17.2). It follows from (7.6) and (17.3) that

$$h^q(\mathcal{S}^\vee \otimes \mathcal{E}) = 0 \text{ for } q \geq 2.$$

Moreover $h^0(\mathcal{S}^\vee \otimes \mathcal{E}) = 0$ since $h^0(\mathcal{S}^\vee \otimes \mathcal{E}|_{\mathbb{Q}^2}) = 0$ by (17.4). Hence it follows from (4.7)

$$-h^1(\mathcal{S}^\vee \otimes \mathcal{E}) = \chi(\mathcal{S}^\vee \otimes \mathcal{E}) = 16 - 4c_2h + c_3 = 0.$$

We apply to $\mathcal{E}$ the Bondal spectral sequence (2.1). We see that $\operatorname{Hom}(G, \mathcal{E}) \cong S_0^{\oplus r+1}$, that $\operatorname{Ext}^q(G, \mathcal{E}) = 0$ for $q = 1, 3$ and that $\operatorname{Ext}^2(G, \mathcal{E})$ fits in the following exact sequence of right $A$-modules:

$$0 \to S_2 \to \operatorname{Ext}^2(G, \mathcal{E}) \to S_3^{\oplus 5} \to 0.$$

Therefore, Lemma 2.1 implies that $E_2^{p,q} = 0$ unless $(p, q) = (-3, 2)$ $(-2, 2)$ or $(0, 0)$, that $E_2^{0,0} \cong \mathcal{O}^{\oplus r+1}$ and that $E_2^{-3,2}$ and $E_2^{-2,2}$ fit in the following exact sequence:

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$$0 \to E_2^{-3,2} \to \mathcal{O}(-1)^{\oplus 5} \to T_{\mathbb{P}^4}(-2)|_{\mathbb{Q}^3} \to E_2^{-2,2} \to 0. \tag{17.5}$$

The Bondal spectral sequence induces the following isomorphisms and exact sequences:

$$E_2^{-3,2} \cong E_3^{-3,2};$$

$$E_2^{0,0} \cong E_3^{0,0};$$

$$0 \to E_3^{-3,2} \to E_3^{0,0} \to E_4^{0,0} \to 0;$$

$$0 \to E_4^{0,0} \to \mathcal{E} \to E_2^{-2,2} \to 0.$$

Note here that $E_2^{-2,2}|_L$ cannot admit a negative degree quotient for any line $L \subset \mathbb{Q}^3$ since $\mathcal{E}$ is nef. We will show that $E_2^{-2,2} = 0$; first note that the exact sequence (17.5) induces the following exact sequence:

$$0 \to E_2^{-3,2} \to \mathcal{O}(-1)^{\oplus 5} \oplus \mathcal{O}(-2) \xrightarrow{p} \mathcal{O}(-1)^{\oplus 5} \to E_2^{-2,2} \to 0.$$

Consider the composite of the inclusion $\mathcal{O}(-1)^{\oplus 5} \to \mathcal{O}(-1)^{\oplus 5} \oplus \mathcal{O}(-2)$ and the morphism $p$ above, and let $\mathcal{O}(-1)^{\oplus a}$ be the cokernel of this composite. Then we have the following exact sequence:

$$\mathcal{O}(-2) \xrightarrow{\pi} \mathcal{O}(-1)^{\oplus a} \to E_2^{-2,2} \to 0.$$

We claim here that $a = 0$. Suppose, to the contrary, that $a > 0$. Since $E_2^{-2,2}$ cannot be isomorphic to $\mathcal{O}(-1)^{\oplus a}$, the morphism $\pi$ above is not zero. Therefore, the composite of $\pi$ and some projection $\mathcal{O}(-1)^{\oplus a} \to \mathcal{O}(-1)$ is not zero, whose quotient is of the form $\mathcal{O}_H(-1)$ for some hyperplane $H$ in $\mathbb{Q}^3$. Hence $E_2^{-2,2}$ admits $\mathcal{O}_H(-1)$ as a quotient. This is a contradiction. Thus $a = 0$ and $E_2^{-2,2} = 0$. Moreover, we see that $E_2^{-3,2} \cong \mathcal{O}(-2)$. Therefore, $\mathcal{E}$ fits in the following exact sequence:

$$0 \to \mathcal{O}(-2) \to \mathcal{O}^{\oplus r+1} \to \mathcal{E} \to 0.$$

This is Case (8) of Theorem 1.1.

## 18. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (11) of Theorem 2.3

Suppose that $\mathcal{E}|_{\mathbb{Q}^2}$ fits in the following exact sequence:

$$0 \to \mathcal{O}(-2,-2) \to \mathcal{O}^{\oplus r+1} \to \mathcal{E}|_{\mathbb{Q}^2} \to k(p) \to 0.$$

Then $c_2 h = 7$. It then follows from (6.1) that

$$c_3 \geq 12.$$

We claim here that $h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$. Indeed, if $h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) \neq 0$, then

$$c_2h \leq c_1(\mathcal{E}|_{\mathbb{Q}^2})(c_1(\mathcal{E}|_{\mathbb{Q}^2}) - c_1(\mathcal{O}_{\mathbb{Q}^2}(1,1))) = 4$$

by [12, Lemma 10.1]. This is a contradiction. Hence $h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$. Thus, we have $h^0(\mathcal{E}(-1)) = 0$ by (7.7). It follows from (4.4) that

$$\chi(\mathcal{E}(-1)) = -\frac{11}{2} + \frac{1}{2}c_3.$$

In particular $c_3$ is odd, and thus $c_3 > 12$. Therefore $h^q(\mathcal{E}(-1)) = 0$ for all $q > 0$ by (7.2). This implies that $\chi(\mathcal{E}(-1)) = 0$, which is a contradiction. Therefore, $\mathcal{E}|_{\mathbb{Q}^2}$ cannot belong to Case (11) of Theorem 2.3.

## 19. The case where $\mathcal{E}|_{\mathbb{Q}^2}$ belongs to Case (12) or (13) of Theorem 2.3

Suppose that $\mathcal{E}|_{\mathbb{Q}^2}$ fits in either of the following exact sequences:

$$0 \to \mathcal{O}(-2,-2) \to \mathcal{O}^{\oplus r} \to \mathcal{E}|_{\mathbb{Q}^2} \to \mathcal{O} \to 0;$$

$$0 \to \mathcal{O}(-1,-1)^{\oplus 4} \to \mathcal{O}^{\oplus r} \oplus \mathcal{O}(-1,0)^{\oplus 2} \oplus \mathcal{O}(0,-1)^{\oplus 2} \to \mathcal{E}|_{\mathbb{Q}^2} \to 0.$$

Then $c_2h = 8$. It then follows from (6.1) that

$$c_3 \geq 16.$$

We claim here that $h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$. Indeed, if $h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) \neq 0$, then

$$c_2h \leq c_1(\mathcal{E}|_{\mathbb{Q}^2})(c_1(\mathcal{E}|_{\mathbb{Q}^2}) - c_1(\mathcal{O}_{\mathbb{Q}^2}(1,1))) = 4$$

by [12, Lemma 10.1]. This is a contradiction. Hence $h^0(\mathcal{E}(-1)|_{\mathbb{Q}^2}) = 0$. Thus, we have $h^0(\mathcal{E}(-1)) = 0$ by (7.7). Note that $h^q(\mathcal{E}|_{\mathbb{Q}^2}) = 0$ for all $q > 0$. Since $h^q(\mathcal{E}) = 0$ for all $q > 0$ by (7.1), this implies that $h^q(\mathcal{E}(-1)) = 0$ for all $q \geq 2$. It follows from (4.4) that

$$0 \geq -h^1(\mathcal{E}(-1)) = \chi(\mathcal{E}(-1)) = -7 + \frac{1}{2}c_3 \geq 1.$$

This is a contradiction. Therefore, $\mathcal{E}|_{\mathbb{Q}^2}$ cannot belong to Case (12) or (13) of Theorem 2.3.

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