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THE STONE–WEIERSTRASS THEOREM FOR WALLMAN RINGS

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Abstract

Biles has called a subring A of the ring $C(X)$ a Wallman ring on X whenever $Z(A)$, the zero sets of function belonging to A, forms a normal base on X in the sense of Frink (1964). In the following, we are concerned with the uniform topology of $C(X)$. We formulate and prove some generalizations of the Stone–Weierstrass theorem in this setting.

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1. Introduction

Wallman (1938) gave a method for associating a compact T_1 space $w(F)$ with a distributive lattice F; $w(F)$ is the space of all F-ultrafilters and the topology of $w(F)$ has as a base for closed sets a lattice F^* which is isomorphic to the lattice F. Frink (1964) defined the concept of a normal base F on a Tychonoff space X and he applied Wallman's construction to obtain Hausdorff compactifications $w(F)$ of X. Throughout this paper, X will denote a Tychonoff space (= completely regular + Hausdorff).

1.1. Definition. A collection F of closed subsets of X is called a *lattice of closed subsets* of X provided that:

(1) $\emptyset, X \in F$; and

(2) if $A, B \in F$ then $A \cap B \in F$ and $A \cup B \in F$.

1.2. Definition. A base F for the closed subsets of X is called a *normal base* on X provided:

(1) F is a lattice of closed subsets of X.

(2) F is disjunctive (that is, if $A \in F$ and $x \in X - A$, then there exists $B \in F$ with $x \in B$ and $A \cap B = \phi$).

(3) F is normal (that is, if $A, B \in F$ with $A \cap B = \emptyset$, then there exist $C, D \in F$ with $A \cap D = \emptyset$, $B \cap C = \emptyset$ and $C \cup D = X$).

If F is a normal base on X, then $w(F)$ is the set of all F-ultrafilters which becomes a space as follows: If $A \in F$, let A^* be the set of all F-ultrafilters having F as a member. F^* then denotes the set of all A^* with $A \in F$. F^* is a base for the closed sets of a topology on $w(F)$. $w(F)$ with this topology is always a Hausdorff compactification of X. Here X is embedded into $w(F)$ by the map which sends each point $x \in X$ to the F-ultrafilter $\{A \in F \mid x \in A\}$.

Frink observed that the family $Z(X)$ of all zero sets of continuous real valued functions on X is a normal base on X which gives rise to a compactification $w(Z(X))$ equivalent to the Stone–Čech compactification βX of X. He also observed that if Y is any given compactification (all spaces are Hausdorff) of X, and if $E(X, Y)$ denotes the subset of $C(X)$ consisting of those real-valued continuous functions on X which are continuously extendible to all of Y, then $Z(E(X, Y))$, the zero sets of such functions, is a normal base on X. Biles (1970) later called a subring A of $C(X)$ a *Wallman ring* on X provided $Z(A)$, the zero sets of functions in A, is a normal base on X. Bentley and Taylor (1975) studied relationships between algebraic properties of a Wallman ring A and topological properties of the compactification $w(Z(A))$ of X.

We adopt our notation and terminology from our two earlier papers; these are mostly consistent with that of Gillman and Jerison (1960).

2. Generalizations of the Stone–Weierstrass Theorem

We investigate the consequences of having a Wallman ring which is uniformly closed; that is, closed in the uniform topology of $C(X)$. Two theorems motivate this work. One is Urysohn's Extension Theorem which states: "A subspace S of X is C^*-embedded in X if and only if any two completely separated sets in S are completely separated in X." The proof of this theorem as it appears in Gillman and Jerison uses the uniform closeness of $C^*(X)$ to construct a function in $C^*(X)$ whose restriction to S is a given function in $C^*(S)$. The other is the Stone–Weierstrass Theorem for real-valued functions which states: "If Y is compact and A is a closed subalgebra of $C(Y)$ which separates points and contains a non-zero constant function then $A = C(Y)$."

In generalizing the Stone–Weierstrass Theorem, we will consider a compactification Y of a space X and a Wallman ring A on X which is a closed subalgebra of $E(X, Y)$. This means each function $f \in A$ is extendible to Y. Therefore in much of what follows our Wallman rings will satisfy certain extendibility hypotheses.

We start by presenting a condition which implies that a Wallman ring A contains only functions which are extendible to $w(Z(A))$.

2.1. Definition (Isbell, 1958). $A \subset C(X)$ is *closed under composition* if and only if for each $f \in A$ and $g \in C(R)$, $g \circ f \in A$.

2.2. Theorem. *Let $A \subset C(X)$ be closed under composition, then $Z(A) = \{f^{-1}[B]: B$ is closed in R and $f \in A\}$.*

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Proof. Let B be closed in R. B is a zero set of $C(R)$ so there is a function $g \in C(R)$ such that $B = Z(g)$. Let $f \in A$, then $f^{-1}[B] = Z(g \circ f) \in Z[A]$. Conversely if F is a zero set of A, $F = Z(f)$ for some $f \in A$ and $F = f^{-1}[\{0\}]$.

We will need to use the Taĭmanov Theorem.

Taĭmanov Theorem (Taĭmanov, 1952). *Let X be dense in Y and let $f\colon X \to T$ be a continuous map of X into a compact space T. Then f can be extended over Y if and only if for any two subsets B_1 and B_2 which are closed in T and disjoint, we have* $\mathrm{Cl}_Y(f^{-1}[B_1]) \cap \mathrm{Cl}_Y(f^{-1}[B_2]) = \varnothing$.

2.3. Theorem. *Let A be a Wallman ring on X such that A is closed under composition and $A \subset C^*(X)$, then $A \subset E(X, w(Z(A)))$.*

Proof. Let $f \in A$ and let F be a compact subset of R such that $f(X] \subset F$. Let B_1 and B_2 be disjoint closed subsets of F. Then $f^{-1}[B_1]$ and $f^{-1}[B_2]$ are disjoint zero sets of A and

$$\mathrm{Cl}_{w(Z(A))} f^{-1}[B_1] \cap \mathrm{Cl}_{w(Z(A))} f^{-1}[B_2] = \varnothing.$$

Therefore, by the Taĭmanov Theorem, f has an extension to $w(Z[A])$.

To further our investigation we make the following definitions which generalize the "completely separated" concept from Urysohn's Extension Theorem.

2.4. Definition. Let F be a family of subsets of X and let $L \subset C(X)$. Then L *discriminates* F-sets if and only if $F_1, F_2 \in F$, $F_1 \cap F_2 = \varnothing$ and $a, b \in R$ implies there is a function $f \in L$ such that $f[F_1] \subset \{a\}$ and $f[F_2] \subset \{b\}$.

2.5. Definition. If $L \subset C(X)$, then

(1) *L discriminates points* of X if and only if L discriminates $\{\{x\}\colon x \in X\}$-sets;

(2) *L discriminates compact sets* of X if and only if L discriminates

$$\{K \subset X\colon K \text{ is compact}\}\text{-sets.}$$

2.6. Theorem. *Let L be a sublattice of $C(X)$ which contains the real constants. If L discrimiantes points of X, then L discriminates compact sets of X.*

Proof. Let F_1 and F_2 be disjoint compact subsets of X and let $a, b \in R$. If $a = b$, then the constant function $f = a$ yields $f[F_1] = \{a\}$ and $f[F_2] = \{b\}$. Suppose $a \neq b$. Let $b > a$ and set $\varepsilon = b - a$. For each $x \in F_1$, $y \in F_2$ there is a function $f_{xy} \in L$ such that $f_{xy}(x) = a - \varepsilon$ and $f_{xy}(y) = b + \varepsilon$.

Let $G_{xy} = \{z \in X\colon f_{xy}(z) < a\}$. Then $x \in G_{xy}$ and so $F_1 \subset \bigcup_{x \in F_1} G_{xy}$. Since F_1 is compact, there exist $x_1, \ldots, x_n \in F_1$ such that $F_1 \subset \bigcup_{i=1}^n G_{x_i y}$. Let

$$g_y = (\inf\{f_{x_i y}\colon i = 1, \ldots, n\}) \vee a.$$

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If $z \in F_1$ then $z \in G_{x_i y}$ for some $i \in \{1, \ldots, n\}$ and $f_{x_i y}(z) < a$ which implies $g_y(z) = a$. Therefore $g_y[F_1] \subset \{a\}$.

Let $H_y = \{z \in H: g_y(z) > b\}$. $f_{x_i y}(y) = b + \varepsilon$ for $i = 1, \ldots, n$, and so

$$(\inf\{f_{x_i y}: i = 1, \ldots, n\})(y) > b \quad \text{and} \quad g_y(y) > b.$$

Therefore $y \in H_y$.

Now we let y vary. $F_2 \subset \bigcup_{y \in F_2} H_y$. Since F_2 is compact, there are $y_1, \ldots, y_m \in F_2$ such that $F_2 \subset \bigcup_{j=1}^{m} H_{y_j}$. Let $h = (\sup\{g_{y_j}: j = 1, \ldots, m\}) \wedge b$. If $z \in F_1$ then $g_y(z) = a$ for each $y \in F_2$ and $(\sup\{g_{y_j}: j = 1, \ldots, m\})(z) = a$ which implies $h(z) = a$. Therefore $h[F_1] \subset \{a\}$. If $z \in F_2$ then there exists $k \in \{1, \ldots, m\}$ such that $z \in H_{y_k}$ and so $g_{y_k}(z) > b$ which implies that $(\sup\{g_{y_j}: j = 1, \ldots, m\})(z) > b$ and finally that $h(z) = b$. Therefore $h[F_2] \subset \{b\}$. $h \in L$ since L is a lattice and L contains the constant functions.

2.7. THEOREM. *If F is a normal base on X, then $E(X, w(F))$ is a sublattice of $C(X)$ which contains all real constants.*

PROOF. If $f, g \in E(X, w(F))$, then there are f' and $g' \in C(w(f))$ such that $f = f' | X$ and $g = g' | X$. $f' \wedge g'$ and $f' \vee g' \in C(w(F))$ so $f \wedge g = (f' \wedge g') | X \in E(X, w(F))$ and $f \vee g = (f' \vee g') | X \in E(X, w(F))$. Therefore $E(X, w(F))$ is a sublattice of $C(X)$. Obviously the real constants are in $E(X, w(F))$.

Since $E(X, w(F))$ is a lattice, we can consider sublattices of $E(X, w(F))$. We find that a sublattice of $E(X, w(F))$ which contains the real constants discriminates F-sets if and only if the extensions of functions from this sublattice discriminate points of $w(F)$.

2.8. THEOREM. *If F is a normal base on X, L is a sublattice of $E(X, w(F))$ which contains the real constants, and $H = \{f \in C(w(F)): f | X \in L\}$, then H discriminates points of $w(F)$ if and only if L discriminates F-sets.*

PROOF. Assume H discriminates points of X. Let F_1 and $F_2 \in F$ such that $F_1 \cap F_2 = \varnothing$, and let $a, b \in R$. $\mathrm{Cl}_{w(F)} F_1$ and $\mathrm{Cl}_{w(F)} F_2$ are disjoint, compact subsets of $w(F)$. By Theorem 2.6, H discriminates compact sets of $w(F)$, so there exists a function $g \in H$ such that

$$(g | X)[F_1] \subset g[\mathrm{Cl}_{w(F)} F_1] \subset \{a\} \quad \text{and} \quad (g | X)[F_2] \subset g[\mathrm{Cl}_{w(F)} F_2] \subset \{b\}.$$

$g | X \in L$ and so L discriminates F-sets.

Now assume L discriminates F-sets. Let $x, y \in w(F)$ such that $x \neq y$ and let $a, b \in R$. There exist F_1 and F_2 in F such that $x \in \mathrm{Cl}_{w(F)} F_1$, $y \in \mathrm{Cl}_{w(F)} F_2$ and $F_1 \cap F_2 = \varnothing$. Then there exists $f \in L$ such that $f[F_1] \subset \{a\}$ and $f[F_2] \subset \{b\}$. Also, there is a function $g \in H$ such that $g | X = f$. Then $g(x) \in \mathrm{Cl}_{w(F)}(f[F_1]) \subset \{a\}$ and $g(y) \in \mathrm{Cl}_{w(F)}(f[F_2]) \subset \{b\}$.

We are interested in subsets of $C(X)$ which discriminate their own zero sets so we make the following definition.

2.9. DEFINITION. Let A be a subset of $C(X)$, then A is *discriminating* if and only if the following condition is satisfied: $F_1, F_2 \in Z(A)$, $F_1 \cap F_2 = \emptyset$ and $a, b \in R$ implies there is a function $f \in A$ such that $f[F_1] \subset \{a\}$ and $f[F_2] \subset \{g\}$.

2.10. THEOREM. *Let A be a subset of $C(X)$, then A is discriminating if and only if A discriminates $Z(A)$-sets.*

2.11. THEOREM. *If A is an inverse closed Wallman ring on X which contains all the real constants, then A is discriminating.*

PROOF. Let F_1 and $F_2 \in Z(A)$ such that $F_1 \cap F_2 = \emptyset$, and let $a, b \in R$. There are functions f_1 and $f_2 \in A$ such that $F_1 = Z(f_1)$ and $F_2 = Z(f_2)$. Let

$$g = (b-a)[f_1^2/(f_1^2+f_2^2)]+a.$$

Then $g \in A$, $g[F_1] \subset \{a\}$ and $g[F_2] \subset \{b\}$.

If we consider what happens when $E(X, Y)$ is discrimintaing we obtain the following theorem.

2.12. THEOREM. *Let Y be a compactification of X, then $Y \cong w(Z[E(X, Y)])$ if and only if $E(X, Y)$ is discriminating.*

PROOF. Assume $Y \cong w(Z(E(X, Y)))$. Let $H_1, H_2 \in Z(E(X, Y))$ such that $H_1 \cap H_2 = \emptyset$, and let $a, b \in R$. Y is a normal space and $\mathrm{Cl}_Y H_1 \cap \mathrm{Cl}_Y H_2 = \emptyset$, so there is a function $f \in C(Y)$ such that $f[\mathrm{Cl}_Y H_1] \subset \{a\}$ and $f[\mathrm{Cl}_Y H_2] \subset)b\}$. Let $g = f | X$. Then $g \in E(X, Y)$, $g[H_1] \subset \{a\}$ and $g[H_2] \subset \{b\}$.

Assume $E(X, Y)$ is discriminating. Let H_1 and H_2 be disjoint closed subsets of X. If $\mathrm{Cl}_Y H_1 \cap \mathrm{Cl}_Y H_2 = \emptyset$, then there is a function $h \in C(Y)$ such that $h[\mathrm{Cl}_Y H_1] \subset \{0\}$ and $h[\mathrm{Cl}_Y H_2] \subset \{1\}$. Let $g = h | X$. Then $g \in E(X, Y)$, $H_1 \subset Z(g)$ and $H_2 \subset Z(g-1)$. Therefore $Y \leqslant w(Z(E(X, Y)))$.

If $\mathrm{Cl}_{w(Z[E(X,Y)])} H_1 \cap \mathrm{Cl}_{w(Z[E(X,Y)])} H_2 = \emptyset$ then there are $F_1, F_2 \in Z(E(X, Y))$ such that $H_1 \subset F_1$, $H_2 \subset F_2$ and $F_1 \cap F_2 = \emptyset$. Since $E(X, Y)$ is discriminating there is a function $g \in E(X, Y)$ such that $g[F_1] \subset \{0\}$ and $g[F_2] \subset \{1\}$. There is a function $h \in C(X)$ such that $h | X = g$. Then $h[F_1] \subset \{0\}$ and $h[F_2] \subset \{1\}$. Therefore

$$\mathrm{Cl}_Y F_1 \cap \mathrm{Cl}_Y F_2 = \emptyset \quad \text{and} \quad w(Z(E(X, Y))) \leqslant Y.$$

2.13. COROLLARY. *$C^*(X)$ is discriminating.*

PROOF. $C^*(X) = E(X, \beta X)$ and $\beta X = w(Z(X))$.

2.14. THEOREM. *If $A \subset C(X)$ and $S \subset X$, then $\{f | S : f \in A\}$ is discriminating if and only if A discriminates $\{S \cap H : H \in Z(A))\}$-sets.*

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PROOF. Let $\{f|S: f\in A\}$ be discriminating. Let H_1 and $H_2\in Z[A]$ such that $S\cap H_1\cap H_2=\emptyset$, and let $a,b\in R$. $S\cap H_1$ and $S\cap H_2$ are disjoint zero sets of $\{f|S: f\in A\}$, so there is a function $g\in A$ such that $(g|S)[H_1\cap S]\subset\{a\}$ and $(g|S)[H_2\cap S]\subset\{b\}$ so A discriminates $\{S\cap H: H\in Z[A]\}$-sets.

Let A discriminate $\{S\cap H: H\in Z[A]\}$-sets. Let F_1 and F_2 be disjoint zero sets of $\{f|S: f\in A\}$ and let $a,b\in R$. There are zero sets H_1 and $H_2\in Z(A)$ such that $F_1=H_1\cap S$ and $F_2=H_2\cap S$ and there is a function $f\in A$ such that $f[H_1\cap S]\subset\{a\}$ and $f[H_2\cap S]\subset\{b\}$. Therefore $(f|S)[F_1]\subset\{a\}$ and $(f|S)[F_2]\subset\{b\}$ so $\{f|S: f\in A\}$ is discriminating.

Since "discriminating" is a generalization of the "completely separated" concept from Urysohn's Extension Theorem and $Z(A)$-embedding is a generalization of C^*-embedding, it is logical that there be some relationship between the two concepts. In the following theorems we investigate this relationship.

2.15. THEOREM. *Let $A\subset C(X)$ be discriminating, let $S\subset X$ and let S be $Z(A)$-embedded in X, then $\{f|S: f\in A\}$ is discriminating.*

PROOF. Let F_1 and F_2 be disjoint zero sets of $\{f|S: f\in A\}$ and let $a,b\in R$. Then there are functions g_1, and $g_2\in A$ such that $F_1=Z(g_1)\cap S$ and $F_2=Z(g_2)\cap S$. Since S is $Z[A]$-embedded in X, there are functions f_1 and $f_2\in A$ such that $F_1=Z(f_1)\cap S$, $F_2=Z(f_2)\cap S$ and $Z(f_1)\cap Z(f_2)=\emptyset$. Since A is discriminating, there is a function $h\in A$ such that $g[Z(f_1)]\subset\{a\}$ and $h(Z(f_2)]\subset\{b\}$. Therefore $(h|S)[F_1]\subset\{a\}$ and $(h|S)[F_2]\subset\{b\}$. Hence $\{f|S: f\in A\}$ is discriminating.

2.16. THEOREM. *Let A be a subring of $C(X)$ which contains a non-zero constant function a and let $S\subset X$ be such that $\{f|S: f\in A\}$ is discriminating, then S is $Z[A]$-embedded in X.*

PROOF. Let f_1 and $f_2\in A$ such that $Z(f_1)\cap Z(f_2)\cap S$ is empty. Then there is a function $g\in A$ such that $(g|S)[Z(f_1)\cap S]\subset\{0\}$ and $(g|S)[Z(f_2)\cap S]\subset\{a\}$. Let $h=g-a$, then $h\in A$, $Z(f_1)\cap S\subset Z(g)$, $Z(f_2)\cap S\subset Z(h)$ and $Z(g)\cap Z(h)=\emptyset$. Therefore S is $Z[A]$-embedded in X.

2.17. COROLLARY. *Let A be a subring of $C(X)$ such that A is discriminating and A contains a nonzero constant function. If $S\subset X$, then X is $Z(A)$-embedded in X if and only if $\{f|S: f\in A\}$ is discriminating.*

A closed sublattice of $E(X, w(F))$ which discriminates F-sets actually equals $E(X, w(F))$. To prove this we will use the following lemma as stated by Simmons (1963), p. 158.

2.18. LEMMA. *Let X be a compact space, and let L be a closed sublattice of $C(X)$ with the following property: if x and y are distinct points of X and a and b are any* *two real numbers, then there exists a function f in L such that* $f(x) = a$ and $f(y) = b$. *Then* $L = C(X)$.

2.19. THEOREM. *If F is a normal base on X and L is a closed sublattice of* $E(X, w(F))$ *such that L discriminates F-sets, then* $L = E(X, w(F))$.

PROOF. Let $H = \{f \in C(w(F)) : f | X \in L\}$.

(1) *H is closed.* Let $f_n \in H$, and $g = \lim_n f_n$. Then

$$g \in C(w(F)) \quad \text{and} \quad g | X = \lim_n (f_n | X) \in L.$$

(2) *H is a sublattice of* $C(w(F))$. Let $f, g \in H$. $(f \vee g) | X = (f | X) \vee (g | X) \in L$ and $(f \wedge g) | X = (f | X) \wedge (g | X) \in L$. Therefore $f \vee g$ and $f \wedge g \in H$.

(3) *If* $x, y \in w(F)$, $x \neq y$ *and* $a, b \in R$, *then there is a function* $f \in H$ *such that* $f(x) = a$ *and* $f(y) = b$. There exist F_1 and $F_2 \in F$ such that $x \in \mathrm{Cl}_{w(F)} F_1$, $y \in \mathrm{Cl}_{w(F)} F_2$ and $F_1 \cap F_2 = \emptyset$. Then there exists $g \in L$ such that $g[F_1] \subset \{a\}$ and $g[F_2] \subset \{b\}$. $L \subset E(X, w(F))$ so there is a function f in $C(w(F))$ such that $g = f | X$. Then $f(x) \in \mathrm{Cl}_R f[F_1] = \mathrm{Cl}_R g[F_1] \subset \{a\}$ and $f(y) \in \mathrm{Cl}_R f[F_2] \subset \{b\}$.

Therefore by the previous lemma $H = C(w(F))$. If $f \in E(X, w(F))$, then there is a function g in $C(w(F))$ such that $g | X = f$. $g \in H$ so $f \in L$. Therefore $L = E(X, w(F))$.

Simmons (1963), p. 159 also has a proof of the lemma which states:

2.20. LEMMA. *Every closed subring of* $C(X)$ *is a closed sublattice.*

Therefore Theorem 2.19 could also have been stated as follows:

2.21. THEOREM. *Let F be a normal base on X. Let A be a closed subring of* $E(X, w(F))$ *which discriminates F-sets, then* $A = E(X, w(F))$.

Conversely, if $A = E(X, w(F))$, then A discriminates F-sets.

2.22. THEOREM. *Let F be a normal base on X, then* $E(X, w(F))$ *discriminates F-sets.*

PROOF. Let $F_1, F_2 \in F$ such that $F_1 \cap F_2 = \emptyset$ and let $a, b \in R$. $\mathrm{Cl}_{w(F)} F_1$ and $\mathrm{Cl}_{w(F)} F_2$ are disjoint closed subsets of the normal space $w(F)$; so by Urysohn's Lemma there is a function $h \in C(w(F))$ such that

$$h[\mathrm{Cl}_{w(F)} F_1] \subset \{a\} \quad \text{and} \quad h[\mathrm{Cl}_{w(F)} F_2] \subset \{b\}.$$

If $g = h | X$, then $g \in E(X, w(F))$, $g[F_1] \subset \{a\}$ and $g[F_2] \subset \{b\}$. Therefore $E(X, w(F))$ discriminates F-sets.

Combining the results of previous theorems we obtain the following necessary and sufficient conditions for a subset of $E(X, w(F))$ to be all of $E(X, w(F))$.

2.23. THEOREM. *Let F be a normal base on X. Let $L \subset E(X, w(F))$. Then $L = E(X, w(F))$ if and only if*

(1) *L is closed in $C(X)$;*

(2) *L is a sublattice of $C(X)$; and*

(3) *L discriminates F-sets.*

PROOF. If $L = E(X, w(F))$, L is closed since $C(w(F))$ is closed. L is a sublattice of $C(X)$ by Theorem 2.7. L discriminates F-sets by Theorem 2.22.

If L satisfies the three conditions then $L = E(X, w(F))$ by Theorem 2.19.

By Lemma 2.20, L is a closed sublattice of $C(X)$ if and only if L is a closed subring of $C(X)$. Therefore Theorem 2.23 could also have been stated as follows.

2.24. THEOREM. *Let F be a normal base on X. Let $A \subset E(X, w(F))$, then $A = E(X, w(F))$ if and only if*

(1) *A is closed in $C(X)$;*

(2) *A is a subring of $C(X)$; and*

(3) *A discriminates F-sets.*

By Theorem 2.11 we know that an inverse closed Wallman ring A which contains all the real constant functions discriminates $Z(A)$-sets. Therefore as a corollary to Theorem 2.24 we have the following.

2.25. THEOREM. *Let A be a Wallman ring on X such that $A \subset E(X, w(Z(A)))$. If A is uniformly closed, and inverse closed then $A = E(X, w(Z(A)))$.*

PROOF. As was noted in Bentley and Taylor (1975), Corollary 3.4, an inverse closed Wallman ring contains all the rationals. Therefore a Wallman ring which is both inverse closed and uniformly closed contains all the real constants.

The next theorem generalizes the Stone–Weierstrass Theorem so we call it the Stone–Weierstrass Theorem for Wallman lattices.

2.26. THEOREM. *Let A be a subset of $C(X)$ such that $Z[A]$ is a normal base on X and $A \subset E(X, w(Z(A)))$. Let L be a sublattice of $C(X)$ such that L is closed in A and L discriminates $Z[A]$-sets. Then $L = A$.*

PROOF. Let $H = \mathrm{Cl}_{E(X,w(Z(A)))} L$. $L \subset H$ and H is a closed sublattice of $E(X, w(Z(A)))$. Since L discriminates $Z[A]$-sets, H discriminates $Z(A)$-sets. Therefore $H = E(X, w(Z(A)))$. L is closed in A so $H \cap A = L$. Also $A \subset H$, so $H \cap A = A$. Therefore $L = A$.

Similarly we have the Stone–Weierstrass Theorem for Wallman rings.

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2.27. THEOREM. *Let A be a sublattice of $C(X)$ such that $Z(A)$ is a normal base on X and $A \subset E(X, w(Z(A)))$. Let L be closed in A, let L be a subring of $C(X)$ which contains the real constants, and let L discriminate $Z(A)$-sets. Then $L = A$.*

PROOF. The hypotheses of Theorem 2.27 include all the hypotheses of Theorem 2.26 except L being a sublattice of $C(X)$. To show L is a sublattice of $C(X)$, it suffices to show $|f| \in L$ for each $f \in L$.

Let $t = \sup\{|f(x)|: x \in X\}$ and let $\varepsilon > 0$. There is a polynomial $p: [-t, t] \to R$ such that p has real coefficients and $||r| - p(r)| < \varepsilon$ for all $r \in [-t, t]$ (Weierstrass Approximation Theorem). Then $||f|(x) - p(f(x))| = ||f(x)| - p(f(x))| < \varepsilon$ for all $x \in X$. $p \circ f \in L$ and $|f| \in A$ so $|f| \in \mathrm{Cl}_A L = L$. L is a sublattice of $C(X)$.

If we let $A \subset C^*(X)$ be an algebra on X we find $A = E(X, w(Z[A]))$ and also obtain some interesting results involving (B, A)-embedding.

2.28. DEFINITION (Hager, 1969). A is an *algebra* on X if and only if:

(1) A is a subring of $C(X)$;

(2) A contains all real valued constant functions;

(3) A separates points and closed sets;

(4) A is closed under uniform convergence; and

(5) A is inverse closed.

We will show that every algebra on X is a Wallman subring of $C(X)$. Lemma 2.20 established that every closed subring of $C(X)$ is a closed sublattice of $C(X)$ and so we have the following result which was observed by Mrówka (1964).

2.29. THEOREM. *If A is an algebra on X, then A is a lattice.*

Biles (1970) established the following.

2.30. THEOREM. *Let A be a subring of $C(X)$ such that $Z[A]$ is a base for the closed sets of X and if $f \in A$, then $|f| \in A$. Then A is a Wallman ring on X.*

If A is a lattice and $f \in A$, then $|f| \in A$. So $|f| \in A$ for each f in an algebra A. Therefore we have proven that every algebra on X is a Wallman ring on X.

2.31. THEOREM. *Every algebra on X is a Wallman subring of $C(X)$.*

In fact, if $A \subset C^*(X)$ is an algebra on X, then A is the Wallman ring $E(X, w(Z(A)))$. To prove this we will use the following theorem which is due to Isbell (1958).

2.32. THEOREM. *If A is an algebra on X, then A is closed under composition.*

2.33. THEOREM. *If $A \subset C^*(X)$ is an algebra on X, then $A = E(X, w(Z(A)))$.*

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PROOF. A is a Wallman ring which is closed under composition so by Theorem 2.3, $A \subset E(X, w(Z(A)))$. By Theorem 2.11, A discriminates $Z(A)$-sets. Therefore the hypotheses of Theorem 2.24 are satisfied and $A = E(X, w(Z(A)))$.

From this we are able to show that if A is an algebra of bounded functions on X and B is an algebra of bounded functions on S, where $S \subset X$, then S is (B, A)-embedded in X if and only if $B = \{f|S : f \in A\}$.

2.34. THEOREM. *Let A be an algebra on X such that $A \subset C^*(X)$. Let $S \subset X$. Let B be an algebra on S such that $B \subset C^*(S)$. Then S is (B, A)-embedded in X if and only if $B = \{f|S : f \in A\}$.*

PROOF. $A = E(X, w(Z(A)))$ and $B = E(S, w(Z(B)))$. Let S be (B, A)-embedded in X. If $f \in A$, then there is a function $g \in C(w(Z(A)))$ such that $f = g|X$. If $h' = g|\mathrm{Cl}_{w(Z(A))} S$, then since $\mathrm{Cl}_{w(Z(A))} S \cong w(Z(B))$ there is a function $h \in C(w(Z(B)))$ such that $h|S = h'|S = f|S$. Therefore $f|S \in B$ and $\{f|S : f \in A\} \subset B$.

If $f \in B$, then there is a function $g \in C(w(Z(B)))$ such that $f = g|S$, and consequently a function $h \in C(\mathrm{Cl}_{w(Z(A))} S)$ such that $h|S = f$. Since $\mathrm{Cl}_{w(Z(A))} S$ is compact, it is C^*-embedded in $w(Z(A))$ and h has a continuous extension h' to $w(Z(A))$. Then $h'|X \in A$, and $(h'|X)|S = f$, so $B \subset \{f|S : f \in A\}$.

Conversely, if $B = \{f|S : f \in A\}$, then by Theorem 2.39 of Bentley and Taylor (1978), S is $Z(A)$-embedded in X and by Theorem 2.40 of Bentley and Taylor (1978), S is (B, A)-embedded in X.

The next two theorems are corollaries to this theorem.

2.35. THEOREM. *If A is a sublattice of $C(X)$, $Z(A)$ is a normal base on X, A is discriminating, $A \subset E(X, w(Z(A)))$, $S \subset X$, S is $Z(A)$-embedded in X, $B \subset C(S)$, $Z(B)$ is a normal base on S, $B \subset E(S, w(Z[B]))$ and $\{f|S : f \in A\}$ is closed in B, then S is (B, A)-embedded in X if and only if $B = \{f|S : f \in A\}$.*

PROOF. Let $L = \{f|S : f \in A\}$. L is a sublattice of $C(S)$ and L is closed in B. Since S is $Z(A)$-embedded in X, $A \cong_S L$, by Corollary 2.37 of Bentley and Taylor (1978). If S is (B, A)-embedded in X, then by Theorem 2.23 of Bentley and Taylor (1978), $A \cong_S B$. Therefore $L \cong B$. Since A is discriminating and S is $Z(A)$-embedded in X, L is discriminating. Therefore L discriminates $Z(B)$-sets. Now we have satisfied the hypotheses of Theorem 2.26 so $L = B$.

Conversely, if $L = B$, then since $\{f|S : f \in A\} \cong_X A$, $B \cong_S A$. So by Theorem 2.23 of Bentley and Taylor (1978), S is (B, A)-embedded in X.

2.36. THEOREM. *If $A \subset C(X)$, $Z(A)$ is a normal base on X, $A \subset E(X, w(Z(A)))$, $S \subset X$, S is $Z(A)$-embedded in X, B is closed in $\{f|S : f \in A\}$, B is a sublattice of $C(S)$ and B is discriminating, then S is (B, A)-embedded in X if and only if $B = \{f|S : f \in A\}$.*

PROOF. Let S be (B, A)-embedded in X. Let $H = \{f | S : f \in A\}$. $Z(H)$ is a normal base on S, by Theorem 2.34 of Bentley and Taylor (1978). Since S is (B, A)-embedded in X, $A \cong_S B$. But $H \cong_S A$ so $H \cong B$. $h \in H$ implies h has an extension to a function in A, hence to a function in $C(w(Z(A)))$. But $\mathrm{Cl}_{w(Z(A))} S \cong w(Z(H))$, so h has an extension to a function in $C(w(Z(H)))$. Therefore $H \subset E(S, w(Z(H)))$. B discriminates $Z(B)$-sets, consequently $Z(H)$-sets. Now by Theorem 2.26, $H = B$.

If $B = \{f | S : f \in A\}$, then $B \cong_S A$. So, by Theorem 2.23 of Bentley and Taylor (1978), S is (B, A)-embedded in X.

REFERENCES

All references refer to those listed at the end of the immediately preceding paper:

H. L. Bentley and B. J. Taylor (1978), "On generalizations of C^*-embedding for Wallman rings", *J. Austral. Math. Soc.* **25** (Ser. A), 215–229.

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