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EXTENSION ALGEBRAS OF CUNTZ ALGEBRA, II

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Abstract

In this paper, we construct the unique (up to isomorphism) extension algebra, denoted by E_∞, of the Cuntz algebra $\mathcal{O}_\infty$ by the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space. We prove that two unital monomorphisms from E_∞ to a unital purely infinite simple C^*-algebra are approximately unitarily equivalent if and only if they induce the same homomorphisms in K-theory.

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1. Introduction and preliminaries

The study of extension theory for C^*-algebras was initiated by Busby [5] and further developed by Brown *et al.* [3, 4]. It provides an important way to construct new C^*-algebras.

Let A be a separable C^*-algebra. An extension of A by $\mathcal{K}$, the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space H, is a short exact sequence

$$0 \longrightarrow \mathcal{K} \xrightarrow{i} E \xrightarrow{q} A \longrightarrow 0$$

of C^*-algebras. The Busby invariant $\tau : A \to \mathcal{Q}(H) = B(H)/\mathcal{K}$ of the above extension is the unique homomorphism for which there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \arrow[r, "i"] \arrow[d, "id"'] & E \arrow[r, "q"] \arrow[d, "\sigma"'] & A \arrow[r] \arrow[d, "\tau"] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r, "i"] & B(H) \arrow[r, "\pi"] & \mathcal{Q}(H) \arrow[r] & 0
\end{tikzcd}$$

We also refer to τ as an extension of A by $\mathcal{K}$ and call E an extension algebra of A (see [1, 9] for details). The invariant τ is said to be *essential* if τ is injective, and

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trivial if τ lifts to a homomorphism from A to $B(H)$, which will be the case if and only if the associated exact sequence splits.

If A is unital, an extension $\tau : A \to \mathcal{Q}(H)$ is called unital if τ is unital. A trivial extension τ is called *strongly unital* if τ lifts to a unital homomorphism from A to $B(H)$.

Let $\tau_1, \tau_2 : A \to \mathcal{Q}(H)$ be two extensions of A. Then τ_1 and τ_2 are said to be *strongly equivalent* if there is a unitary $u \in B(H)$ such that $\tau_2(a) = \pi(u)\tau_1(a)\pi(u)^*$ for all $a \in A$. Let $\mathbf{Ext}_{\mathrm{s}}(A)$ be the set of strong equivalence classes of essential extensions of A. If A is unital, we denote by $\mathbf{Ext}_{\mathrm{s}}^{\mathrm{u}}(A)$ the set of strong equivalence classes of unital essential extensions of A. We define the sum of τ_1 and τ_2 by $\tau_1 \oplus \tau_2 : A \to \mathcal{Q}(H) \oplus \mathcal{Q}(H) \subset M_2(\mathcal{Q}(H)) \cong \mathcal{Q}(H)$. Then $\mathbf{Ext}_{\mathrm{s}}(A)$ and $\mathbf{Ext}_{\mathrm{s}}^{\mathrm{u}}(A)$ are semigroups. Let $\mathrm{Ext}(A)$ be the quotient of $\mathbf{Ext}_{\mathrm{s}}(A)$ by the subsemigroup of essential trivial extensions.

Extensions of Cuntz algebras were first studied by Cuntz [6, 7], and further investigations were undertaken by Paschke and Salinas [15], Lin [8, 11], and the authors [12]. In [12], we constructed and classified all extension algebras of the Cuntz algebra $\mathcal{O}_n$ for $n \geq 2$. In this paper, we consider the case of $\mathcal{O}_\infty$, and show that there is only one extension algebra (up to isomorphism) of $\mathcal{O}_\infty$, which we denote by E_∞. We construct this extension algebra and describe its K-theory. We also prove that two unital monomorphisms from E_∞ to a unital purely infinite simple C^*-algebra are approximately unitarily equivalent if and only if they induce the same homomorphisms in K-theory.

Throughout this paper, $\mathbb{Z}$ will denote the group of integers and $\mathbb{Z}_+$ the semigroup of nonnegative integers; we write $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}^n = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (n copies of $\mathbb{Z}$).

2. Extension algebras of $\mathcal{O}_\infty$

Recall that the Cuntz algebra $\mathcal{O}_\infty$ is the universal C^*-algebra generated by a sequence of isometries $\{s_i\}$ with mutually orthogonal range projections. It is well-known that $\mathcal{O}_\infty$ is a unital purely infinite simple C^*-algebra and that $K_0(\mathcal{O}_\infty) = \mathbb{Z}$ and $K_1(\mathcal{O}_\infty) = 0$. Also, $\mathcal{O}_\infty$ is unique in the sense that for any sequence of isometries $\{s_i'\}$ with mutually orthogonal range projections, the C^*-algebra generated by $\{s_i'\}$ is isomorphic to $\mathcal{O}_\infty$.

A separable nuclear C^*-algebra A is said to satisfy the *UCT* [1, 9] if for any separable C^*-algebra B there is the short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} KK^*(A, B) \xrightarrow{\gamma} \mathrm{Hom}(K_*(A), K_*(B)) \longrightarrow 0.$$

Here the map γ has degree zero and δ has degree one. In particular, taking $B = \mathcal{K}$, since $\mathrm{Ext}(A, B) = KK(A, SB)$ we have the following short exact sequence:

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}) \longrightarrow \mathrm{Ext}(A) \longrightarrow \mathrm{Hom}(K_1(A), \mathbb{Z}) \longrightarrow 0.$$

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PROPOSITION 2.1. $\mathbf{Ext}_{\mathrm{s}}^{\mathrm{u}}(\mathcal{O}_\infty) = 0$.

PROOF. Since $\mathcal{O}_\infty$ is a unital separable nuclear C^*-algebra, $\mathrm{Ext}(\mathcal{O}_\infty)$ is a group. Since $K_0(\mathcal{O}_\infty) = \mathbb{Z}$ and $K_1(\mathcal{O}_\infty) = 0$, $\mathrm{Ext}(\mathcal{O}_\infty) = 0$ by the *UCT*. Because $[\mathbf{1}_{\mathcal{O}_\infty}]_0$ is a generator of $K_0(\mathcal{O}_\infty) = \mathbb{Z}$,

$$\{h([\mathbf{1}_{\mathcal{O}_\infty}]_0) \mid h \in \mathrm{Hom}(K_0(\mathcal{O}_\infty), \mathbb{Z})\} = \mathbb{Z}.$$

Since $\mathcal{O}_\infty \otimes C(\mathbb{T})$ satisfies the *UCT*, it follows from [2, Proposition 1] that

$$\mathbf{Ext}_{\mathrm{s}}^{\mathrm{u}}(\mathcal{O}_\infty) = \mathbb{Z}/\{h([\mathbf{1}_{\mathcal{O}_\infty}]_0) \mid h \in \mathrm{Hom}(K_0(\mathcal{O}_\infty), \mathbb{Z})\} = 0. \qquad \square$$

The following lemma is well-known, but we give its proof for completeness.

LEMMA 2.2. *Let A be a (unital) separable C^*-algebra, and let*

$$0 \longrightarrow \mathcal{K} \xrightarrow{i_1} E_1 \xrightarrow{q_1} A \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{K} \xrightarrow{i_2} E_2 \xrightarrow{q_2} A \longrightarrow 0$$

be two (unital) essential extensions of A with Busby invariants τ_1 and τ_2, respectively. Then τ_1 and τ_2 are strongly equivalent if and only if there is an isomorphism $\varphi : E_1 \to E_2$ such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \arrow[r, "i_1"] \arrow[d, "\varphi|_{\mathcal{K}}"] & E_1 \arrow[r, "q_1"] \arrow[d, "\varphi"'] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r, "i_2"] & E2 \arrow[r, "q_2"] & A \arrow[r] & 0
\end{tikzcd}$$

PROOF. If τ_1 and τ_2 are strongly equivalent, then there is a unitary $u \in B(H)$ such that

$$\tau_2(x) = \pi(u)\tau_1(x)\pi(u)^* \quad \forall\, x \in A.$$

Define $\varphi : E_1 \to E_2$ by $\varphi(x) = uxu^*$. Now, for $x \in E_1$,

$$\begin{aligned}\pi(\varphi(x)) &= \pi(uxu^*) = \pi(u)\pi(x)\pi(u)^* \\ &= \pi(u)\tau_1(q_1(x))\pi(u)^* = \tau_2(q_1(x)),\end{aligned}$$

so $\varphi(x) \in E_2$. Therefore φ is well-defined and it is obvious that φ is an isomorphism. Since $\tau_2(q_2(\varphi(x))) = \pi(\varphi(x)) = \tau_2(q_1(x))$ and τ_2 is injective, we have $q_2(\varphi(x)) = q_1(x)$. Thus, the above diagram is commutative.

Conversely, suppose that there is an isomorphism $\varphi : E_1 \to E_2$ such that the diagram is commutative. It is well-known that $\varphi|_{\mathcal{K}}$ is an isomorphism from $\mathcal{K}$ onto $\mathcal{K}$ and there is a unitary $u \in B(H)$ such that $\varphi(x) = uxu^*$ for all $x \in \mathcal{K}$. For any $x \in \mathcal{K}$ and $y \in E_1$,

$$\varphi(y)(uxu^*) = \varphi(yx) = uyxu^* = (uyu^*)(uxu^*).$$

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Since the range of $u\mathcal{K}u^*$ is dense in H, $\varphi(y) = uyu^*$. For $x = q_1(y) \in A$ with $y \in E_1$,

$$\begin{aligned}\pi(u)\tau_1(x)\pi(u)^* &= \pi(u)\tau_1(q_1(y))\pi(u)^* \\ &= \pi(u)\pi(y)\pi(u)^* = \pi(\varphi(y)) \\ &= \tau_2(q_2(\varphi(y))) = \tau_2(q_1(y)) = \tau_2(x). \qquad \square\end{aligned}$$

By Proposition 2.1 and Lemma 2.2, there exists a unique (up to isomorphism) extension algebra of $\mathcal{O}_\infty$. We now construct it concretely.

Let $\{t_i \mid i = 1, 2, \ldots\}$ be a sequence of partial isometries with mutually orthogonal range projections such that $t_1^* t_1 < 1$, $t_i^* t_i = 1$ for $i \geq 2$, and $p + \sum_{i=1}^n t_i t_i^* < 1$ for any positive integer n, where $p = 1 - t_1^* t_1$ is a projection of rank one. Let E_∞ be the universal C^*-algebra generated by $\{t_i \mid i = 1, 2, \ldots\}$, and let $I(E_\infty)$ be the (only closed) ideal of E_∞ generated by p.

Theorem 2.3. *$I(E_\infty) \cong \mathcal{K}$ and $E_\infty / I(E_\infty) \cong \mathcal{O}_\infty$; that is, E_∞ is an essential unital extension of $\mathcal{O}_\infty$ by $\mathcal{K}$.*

Proof. Let l be a positive integer and W_l the set of all l-tuples $(i_1, \ldots, i_l)$ with $i_j \in \mathbb{N} = \{1, 2, \ldots,\}$, $j = 1, \ldots, l$. We assume that $W_0 = \{0\}$ and $W = \bigcup_{l \geq 0} W_l$. Let $T_0 = 1$ and, for $\alpha = (i_1, \ldots, i_l) \in W_l$, denote by T_α the partial isometry $T_\alpha = t_{i_1} \cdots t_{i_l}$. Note that $pT_\alpha = 0$ for any $\alpha \in W$.

Let $\mathcal{F}$ be the set of all linear combinations of elements of the form $T_\alpha p T_\beta^*$ for any $\alpha, \beta \in W$. It is easy to see that the closure of $\mathcal{F}$ is an ideal of E_∞, and that $\mathcal{F}$ is contained in every ideal containing p. It follows that the closure of $\mathcal{F}$ is the ideal of E_∞ generated by p, that is, $I(E_\infty)$.

Let $X = (T_\alpha p T_\mu^*)(T_\nu p T_\beta^*)$. It is easy to see that $X \neq 0$ if and only if $\mu = \nu$. One can check that

$$(T_\alpha p T_\mu^*)(T_\nu p T_\beta^*) = \delta_{\mu\nu} T_\alpha p T_\beta^*$$

and

$$(T_\alpha p T_\beta^*)^* = T_\beta p T_\alpha^*.$$

Then the set $\{T_\alpha p T_\beta^* \mid \alpha, \beta \in W\}$ is a self-adjoint system of matrix units generating $\mathcal{F}$. It follows that $I(E_\infty)$ is isometric to $\mathcal{K}$.

Let $\pi : E_\infty \to E_\infty / I(E_\infty)$ be the quotient map. It is easy to see that $\{\pi(t_i)\}$ is a sequence of isometries in $E_\infty / I(E_\infty)$ with orthogonal ranges. Since $E_\infty / I(E_\infty)$ is a unital C^*-algebra generated by $\{\pi(t_i)\}$, $E_\infty / I(E_\infty) \cong \mathcal{O}_\infty$ by the uniqueness of $\mathcal{O}_\infty$. Therefore E_∞ is an essential unital extension of $\mathcal{O}_\infty$ by $\mathcal{K}$, since $\mathcal{O}_\infty$ is a unital simple C^*-algebra. $\square$

Remark 2.4. Let H be a separable infinite-dimensional Hilbert space. It is easy to see that there exist partial isometries $t_1, t_2, \ldots$ in $B(H)$ satisfying the conditions in the definition of E_∞.

Corollary 2.5 (Uniqueness of E_∞). *Let $\{t_i' \mid i = 1, 2, \ldots\}$ be a sequence of partial isometries with mutually orthogonal range projections such that $t_1'^* t_1' < 1$,*

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$t'^*_i t'_i = 1$ for $i \geq 2$, and $p' + \sum_{i=1}^n t'_i t'^*_i < 1$ for any positive integer n, where $p' = 1 - t'^*_1 t'_1$ is a projection of rank one. Then the C^*-algebra E generated by $\{t'_i \mid i = 1, 2, \ldots\}$ is isomorphic to E_∞. In other words, E_∞ is independent of the choice of the generators.

PROOF. As in Theorem 2.3, we may prove that the ideal I of E generated by the projection p' is isomorphic to $\mathcal{K}$ and that $E/I \cong \mathcal{O}_\infty$. Since there is only one extension algebra of $\mathcal{O}_\infty$, we have $E \cong E_\infty$. □

THEOREM 2.6. *$K_0(E_\infty) = \mathbb{Z} \oplus \mathbb{Z}$ and $K_1(E_\infty) = 0$.*

PROOF. By Theorem 2.3, we have the short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow E_\infty \longrightarrow \mathcal{O}_\infty \longrightarrow 0.$$

By the six-term exact sequence of K-theory, we have the following commutative diagram:

$$\begin{tikzcd}
K_0(\mathcal{K}) \arrow[r] & K_0(E_\infty) \arrow[r] & K_0(\mathcal{O}_\infty) \arrow[d] \\
K_1(\mathcal{O}_\infty) \arrow[u] & K_1(E_\infty) \arrow[l] & K_1(\mathcal{K}) \arrow[l]
\end{tikzcd}$$

Since $K_0(\mathcal{O}_\infty) = \mathbb{Z}$ and $K_1(\mathcal{O}_\infty) = 0$, $K_1(E_\infty) = 0$ and the sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow K_0(E_\infty) \longrightarrow \mathbb{Z} \longrightarrow 0$$

is exact. Therefore $K_0(E_\infty) \cong \mathbb{Z} \oplus \mathbb{Z}$. □

Let A be a unital C^*-algebra. Two projections p and q in A are said to be (Murray–von Neumann) *equivalent*, denoted by $p \sim q$, if there is a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* = q$; we let $[p]$ denote the equivalence class of projections in A containing p. The projections p and q are called *unitarily equivalent*, denoted by $p \sim_{\mathrm{u}} q$, if there is a unitary u in A such that $u^*pu = q$; $[p]_{\mathrm{u}}$ denotes the unitary equivalence class of projections in A containing p. Moreover, p and q are called *homotopically equivalent*, denoted by $p \sim_{\mathrm{h}} q$, if p and q are in the same path component of projections in A; $[p]_{\mathrm{h}}$ denotes the homotopic equivalence class of projections in A containing p.

The following proposition can be obtained immediately from results in [13] and [14].

PROPOSITION 2.7.

(1) $$\begin{aligned} K_0(E_\infty) &= \{[p] \mid p \in E_\infty \backslash \mathcal{K} \text{ is a projection}\} \\ &= \{[p] \mid p \in E_\infty \backslash \mathcal{K} \text{ is a projection and } 1 - p \in E_\infty \backslash \mathcal{K}\} \\ &= \{[p]_{\mathrm{h}} \mid p \in E_\infty \backslash \mathcal{K} \text{ is a projection and } 1 - p \in E_\infty \backslash \mathcal{K}\} \\ &= \{[p]_{\mathrm{u}} \mid p \in E_\infty \backslash \mathcal{K} \text{ is a projection and } 1 - p \in E_\infty \backslash \mathcal{K}\}. \end{aligned}$$

(2) *Let p and q be projections in $E_\infty \backslash \mathcal{K}$ such that $1 - p$ and $1 - q \in E_\infty \backslash \mathcal{K}$. Then $p \sim q$, $p \sim_{\mathrm{u}} q$ and $p \sim_{\mathrm{h}} q$ are equivalent.*

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Recall that E_n is the universal C^*-algebra generated by isometries $t_1, t_2, \ldots, t_n$ with $\sum_{i=1}^n t_i t_i^* < 1$. It is well-known that E_n is a unital essential extension of $\mathcal{O}_n$ by $\mathcal{K}$ and that $\mathcal{O}_\infty = \varinjlim E_n$.

Let F_n ($n \geq 2$) be the C^*-subalgebra of E_∞ generated by $t_1, t_2, \ldots, t_n$, and let $I(F_n)$ be the ideal of F_n generated by $p = 1 - t_1^* t_1$. We can prove the following results as above, so the details are omitted.

PROPOSITION 2.8.

(1) $\mathbf{Ext}_s^{\mathrm{u}}(E_n) = 0$.
(2) $I(F_n) \cong \mathcal{K}$ *and* $F_n / I(F_n) \cong E_n$*; moreover,* F_n *is the only unital essential extension algebra of* E_n *by* $\mathcal{K}$*.*
(3) $K_0(F_n) = \mathbb{Z} \oplus \mathbb{Z}$ *and* $K_1(F_n) = 0$*.*
(4) $E_\infty = \varinjlim F_n$*, where the map* $F_n \to F_{n+1}$ *of the system sends* t_i *to* t_i *for* $i = 1, 2, \ldots, n$*.*

COROLLARY 2.9. *Let H be a separable infinite-dimensional Hilbert space. Suppose that $\varphi, \psi : A \to Q(H)$ are two unital injective homomorphisms, where $A = \mathcal{O}_\infty$ or E_n. Then there is a unitary $u \in B(H)$ such that $\varphi(x) = \pi(u)^* \psi(x) \pi(u)$ for all $x \in A$.*

PROOF. Now, φ and ψ are two unital essential extensions of A by $\mathcal{K}$, so the result follows immediately from the fact that $\mathbf{Ext}_s^{\mathrm{u}}(A) = 0$. □

COROLLARY 2.10. *Let H be a separable infinite-dimensional Hilbert space.*

(1) *Let $v_1, v_2, \ldots, v_n$ ($n \geq 2$) be isometries in the Calkin algebra $Q(H)$ with mutually orthogonal range projections such that $\sum_{i=1}^n t_i t_i^* < 1$. Then there are isometries $V_1, V_2, \ldots, V_n$ in $B(H)$ with mutually orthogonal range projections such that $\pi(V_i) = v_i$ for $i = 1, 2, \ldots, n$.*
(2) *Let $\{v_i\}$ be a sequence of isometries in $Q(H)$ with mutually orthogonal range projections. Then there is a sequence of isometries $\{V_i\}$ in $B(H)$ with mutually orthogonal range projections such that $\pi(V_i) = v_i$ for $i = 1, 2, \ldots$.*

PROOF. For part (1), the map $\tau : E_n \to Q(H)$ defined by $\tau(t_i) = v_i$ is a unital essential extension of E_n by $\mathcal{K}$. It is trivial since $\mathbf{Ext}_s^{\mathrm{u}}(E_n) = 0$; thus τ lifts to a unital injective homomorphism $\sigma : E_n \to B(H)$. Now put $V_i = \sigma(t_i)$. The proof of (2) is similar, since $\mathbf{Ext}_s^{\mathrm{u}}(\mathcal{O}_\infty) = 0$. □

3. Homomorphisms from E_∞

DEFINITION 3.1. Let A and B be C^*-algebras, let F be a finite subset of A, and let φ and ψ be two homomorphisms from A into B. Let $\varepsilon > 0$. We say that φ and ψ are approximately unitarily equivalent within ε, with respect to F, if there is a unitary $u \in \tilde{B}$ such that

$$\|\varphi(x) - u\psi(x)u^*\| < \varepsilon$$

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for all $x \in F$, where $\tilde{B}$ is the unitization of B. We abbreviate this as $\varphi \sim_\varepsilon \psi$ with respect to F. When the set F is understood, we shall omit mention of it.

We further say that φ and ψ are approximately unitarily equivalent if for every finite $F \subset A$ and $\varepsilon > 0$, we have $\varphi \sim_\varepsilon \psi$ with respect to F.

THEOREM 3.2. *Let A be a unital purely infinite simple C^*-algebra, and let $\varphi, \psi : E_\infty \to A$ be two unital monomorphisms. Then the following statements are equivalent:*

(i) *φ and ψ are approximately unitarily equivalent;*
(ii) *$[\varphi] = [\psi]$ in $KK(E_\infty, A)$;*
(iii) *$\varphi_* = \psi_* : K_0(E_\infty) \to K_0(A)$;*
(iv) *$\varphi(p) \sim \psi(p)$ in A, where $p = 1 - t_1^* t_1$;*
(v) *$\varphi(t_1^* t_1) \sim \psi(t_1^* t_1)$ in A.*

PROOF. (i) ⇔ (ii). Since $K_*(E_\infty)$ is finitely generated, $KK(E_\infty, A) = KL(E_\infty, A)$. The result then follows immediately from [10, Theorem 2.9], since every unital purely infinite simple C^*-algebra has properties (P_1), (P_2) and (P_3) and every monomorphism into a simple C^*-algebra is full in the sense of [10].

(ii) ⇔ (iii). Since $K_0(E_\infty) = \mathbb{Z} \oplus \mathbb{Z}$ is a free abelian group and $K_1(E_\infty) = 0$, by the *UCT* we have $KK(E_\infty, A) \cong \mathrm{Hom}(K_0(E_\infty), K_0(A))$. Note that $\gamma(\varphi) = \varphi_*$. Therefore, $[\varphi] = [\psi]$ in $KK(E_\infty, A)$ if and only if $\varphi_* = \psi_* : K_0(E_\infty) \to K_0(A)$.

(iii) ⇔ (iv). Since $K_0(E_\infty) = \mathbb{Z} \oplus \mathbb{Z}$ is an abelian group generated by $[1]_0$ and $[p]_0$, $\varphi_* = \psi_* : K_0(E_\infty) \to K_0(A)$ if and only if $[\varphi(1)]_0 = [\psi(1)]_0$ and $[\varphi(p)]_0 = [\psi(p)]_0$ in $K_0(A)$; this occurs if and only if $[\varphi(p)]_0 = [\psi(p)]_0$, since φ and ψ are unital. But A is a purely infinite simple C^*-algebra, so $[\varphi(p)]_0 = [\psi(p)]_0$ in $K_0(A)$ if and only if $\varphi(p) \sim \psi(p)$ in A.

(iv) ⇔ (v). This is obvious since $\varphi(p)$, $\psi(p)$, $\varphi(t_1^* t_1) = 1 - \varphi(p)$ and $\psi(t_1^* t_1) = 1 - \psi(p)$ are nonzero projections in a purely infinite simple C^*-algebra. □

Similar results hold for F_n, with the same proof.

PROPOSITION 3.3. *Let A be a unital purely infinite simple C^*-algebra, and let $\varphi, \psi : F_n \to A$ be two unital monomorphisms. Then the following statements are equivalent:*

(i) *φ and ψ are approximately unitarily equivalent;*
(ii) *$[\varphi] = [\psi]$ in $KK(F_n, A)$;*
(iii) *$\varphi_* = \psi_* : K_0(F_n) \to K_0(A)$;*
(iv) *$\varphi(p) \sim \psi(p)$ in A, where $p = 1 - t_1^* t_1$;*
(v) *$\varphi(t_1^* t_1) \sim \psi(t_1^* t_1)$ in A.*

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