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ON TIDY SUBGROUPS OF LOCALLY COMPACT TOTALLY DISCONNECTED GROUPS

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Willis has established a structure theory of locally compact totally disconnected groups. An important feature of this theory is the notion of a tidy subgroup. In this paper we provide several results regarding these subgroups.

In recent years, Willis (see [**3, 4**]) has established several structure theorems for locally compact totally disconnected groups. An important feature of this structure theory is the notion of a tidy subgroup, whose definition is recalled:

DEFINITION: (Willis) Let G be a totally disconnected locally compact group and let f be a bicontinuous automorphism of G. Let U be a compact open subgroup of G and set

$$U_+ = \bigcap_{n=0}^{\infty} f^n(U) \text{ and } U_- = \bigcap_{n=0}^{\infty} f^{-n}(U).$$

Then U is said to be *tidy* for f if it satisfies:

T1 $U = U_+U_- = U_-U_+$, and

T2 $\bigcup_{n=0}^{\infty} f^n(U_+)$ and $\bigcup_{n=0}^{\infty} f^{-n}(U_-)$ are closed in G.

Willis has proved the existence of tidy subgroups and has described properties that are possessed by these subgroups. Using Willis' results, we provide some additional observations regarding tidy subgroups.

Throughout this paper, G will denote a locally compact totally disconnected group. Let f be a bicontinuous automorphism of G. We begin by recalling the following result of Willis [**3**, Lemma 1].

LEMMA 1. *Let U be a compact open subgroup of G. Then there is an integer k such that for all $l \geqslant k$, $U(l) = \bigcap_{n=0}^{l} f^n(U)$ satisfies* T1.

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REMARK. Note that $U(l)_+ = \bigcap_{n=0}^{\infty} f^n(U(l)) = U_+$, while $U(l)_- = \bigcap_{n=0}^{\infty} f^{-n}(U(l)) = U(l) \cap U_-$. Therefore, for $l \geqslant k$, $U(l)_+ = U(k)_+$ and $U(l)_- \subseteq U(k)_-$.

Let U be a compact open subgroup of G, and let $U_{++} = \bigcup_{n=0}^{\infty} f^n(U_+)$. The following proposition gives necessary and sufficient conditions for U_{++} to be closed.

PROPOSITION 2. *Let U be a compact open subgroup of G. Then U_{++} is closed if and only if $U_{++} \cap U \subseteq f^l(U_+)$ for some $l \geqslant 0$.*

PROOF: Suppose U_{++} is closed. Since U_{++} is a countable union of compact subgroups, by the Baire Category Theorem, $f^k(U_+)$ must be open for some k. Thus U_+ is open. Since $\{f^n(U_+) \cap U\}_{n=0}^{\infty}$ is an open cover of $U_{++} \cap U$ and is monotonically increasing, $U_{++} \cap U \subseteq \bigcup_{n=0}^{l} f^n(U_+) \cap U \subseteq f^l(U_+) \cap U \subseteq f^l(U_+)$.

Conversely, if $U_{++} \cap U \subseteq f^l(U_+)$, then $U_{++} \cap U = f^l(U_+) \cap U$, which is closed. Therefore U_{++} is closed by [**2**, Chapter III, Proposition 2.4]. ∎

REMARK. Willis has shown ([**3**, Lemma 3]) that if U satisfies T1, then U_{++} is closed if and only if $U_{++} \cap U = U_+$ (that is, $l = 0$ in Proposition 2). Moreover, [**3**, Lemma 3] also shows that U is tidy if and only if U_{++} is closed.

Let V be a compact open subgroup of G satisfying T1. Define $\mathcal{L}_V = \{g \in G : f^n(g) \in V$ for all but finitely many integers $n\}$ and let L_V be the closure of $\mathcal{L}_V$. Then $\mathcal{L}_V \subseteq V_{++}$ and L_V is a compact f-invariant subgroup of G ([**3**, Lemma 6]).

PROPOSITION 3. *Let U be a compact open subgroup of G. If U satisfies* T1, *then U is tidy if and only if $L_U = \overline{\mathcal{L}_U} \subseteq U_{++}$.*

PROOF: If U_{++} is closed, then $L_U = \overline{\mathcal{L}_U} \subseteq U_{++}$.

Conversely, if U_{++} is not closed, then the proof of [**3**, Lemma 3] shows that there exists a sequence $z_1, z_2, \ldots$ in $U_{++} \cap U$ such that $\lim_{n\to\infty} z_n = w \notin U_{++}$. In fact, the z_n are contained in $\mathcal{L}_U$. Thus $L_U = \overline{\mathcal{L}_U} \not\subseteq U_{++}$. ∎

COROLLARY 4. *Let U be a compact open subgroup of G satisfying* T1 *but not* T2. *Let $V = U \cap L_U$. Then V_{++} is not closed and V_{++} is dense in L_U.*

PROOF: Now $V_{++} = \bigcup_{n=0}^{\infty} f^n(V_+) = \bigcup_{n=0}^{\infty} f^n((U \cap L_U)_+) = \bigcup_{n=0}^{\infty} f^n(U_+ \cap L_U) = \bigcup_{n=0}^{\infty} (f^n(U_+) \cap L_U) = U_{++} \cap L_U \supseteq \mathcal{L}_U$. Thus V_{++} is dense in L_U. Since $L_U \not\subseteq U_{++}$ (Proposition 3), $V_{++} \neq L_U$. Therefore, V_{++} is not closed. ∎

REMARK. $V = U \cap L_U$ is a compact open subgroup of L_U. By Lemma 1, $V(l) = \bigcap_{n=0}^{l} f^n(V)$ satisfies T1 for some integer l. However, since $V(l)_+ = V_+$, $V(l)$ is not tidy in L_U.

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In [4] Willis shows how to construct tidy subgroups in three steps. Given an arbitrary compact open subgroup U of G, the first step is to find a compact open subgroup $V \subseteq U$ such that V satisfies T1. This step is precisely Lemma 1. The second step is to identify a particular compact f-invariant subgroup L of G. Here we take $L = L_V$. The third step uses L_V and V to produce a tidy subgroup W. Define $V' = \{v \in V : lvl^{-1} \in VL \text{ for all } l \in L\}$. Then V' is an open subgroup of V and $W = V'L$ is a tidy subgroup of G ([**4**, Theorem 3.1]). Due to this construction we have the next proposition.

PROPOSITION 5. *Let V be a compact open subgroup of G satisfying* T1. *Then there exists a tidy subgroup $V'L$ such that $\overline{V_{++}} = (V'L)_{++}$.*

PROOF: Let $L = L_V$ and let V' be as above. Then $V'L$ is a tidy subgroup of G. By [**4**, Lemma 3.4], there exists an integer p such that $f^{-p}(V_+) \subseteq V'_+$. Then $V_{++} = \bigcup_{n=0}^{\infty} f^n(V_+) \subseteq \bigcup_{n=0}^{\infty} f^n(f^{-p}(V_+)) \subseteq \bigcup_{n=0}^{\infty} f^n(V'_+) = V'_{++} \subseteq V'_{++}L$. Thus $V_{++} \subseteq V'_{++}L = (V'L)_{++}$ by [**4**, Lemma 3.7]. Since $(V'L)_{++}$ is closed, $\overline{V_{++}} \subseteq (V'L)_{++}$. On the other hand, since $V'_{++} \subseteq V_{++}$ and $L \subseteq \overline{V_{++}}$, $V'_{++}L = (V'L)_{++} \subseteq \overline{V_{++}}$. Therefore, $\overline{V_{++}} = (V'L)_{++}$. ∎

REMARK. Given any compact open subgroup U of G, there exists an integer l such that $U(l) = \bigcap_{n=0}^{l} f^n(U)$ satisfies T1. Then $U(l)_+ = U_+$ so $U(l)_{++} = U_{++}$. Thus, by Proposition 5, there exists a tidy subgroup U' such that $U'_{++} = \overline{U_{++}}$.

LEMMA 6. *Let U be a compact open subgroup of G and let $z \in G$.*

(a) *$z \notin U_{++}$ if and only if there is a sequence $0 < n_1 < n_2 < \cdots < n_k < \cdots$ such that $z \notin \bigcup_{k=1}^{\infty} f^{n_k}(U)$.*

(b) *If U is tidy, then $z \notin U_{++}$ if and only if there exists $l \geqslant 0$ such that $z \notin \bigcup_{n=l}^{\infty} f^n(U)$.*

PROOF: (a) If $z \notin f^l(U_+)$, then there exists $l' \geqslant l$ such that $z \notin f^{l'}(U)$. Thus if $z \notin U_{++}$, then there exists $0 < n_1 < n_2 < \cdots < n_k < \cdots$ such that $z \notin \bigcup_{k=1}^{\infty} f^{n_k}(U)$.

Conversely, suppose $z \notin \bigcup_{k=1}^{\infty} f^{n_k}(U)$. Since $z \notin f^{n_k}(U)$, $f^{-n_k}(z) \notin U$, which implies that $f^{-n_k}(z) \notin U_+$. Thus $z \notin f^{n_k}(U_+)$ for all n_k, where $n_k \to \infty$. Since $\{f^n(U_+)\}_{n=0}^{\infty}$ is monotonically increasing, $z \notin \bigcup_{n=0}^{\infty} f^n(U_+) = U_{++}$.

(b) If no such l exists, then there exists $n_1 < n_2 < \cdots < n_k < \cdots$ such that $z \in f^{n_k}(U)$ for all n_k. By [**3**, Lemma 6], $f^{-n_1}(z) \in L_U U_+$. Since U is tidy,

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$L_U = \mathcal{L}_U = U_+ \cap U_- \subseteq U_+$ ([**3**, Corollary to Lemma 3]). Therefore, $L_U U_+ = U_+$, and $z \in f^{n_1}(U_+) \subseteq U_{++}$.

The converse follows from part (a). ∎

Proposition 7. *Suppose U is a compact open subgroup of G satisfying* T1. *Then U is tidy if and only if the following two conditions hold.*

(1) *Let $z \in G$. Then $z \notin U_{++}$ if and only if there exists $l \geqslant 0$ such that $z \notin \bigcup_{n=l}^{\infty} f^n(U)$.*

(2) *$\bigcup_{n=l}^{\infty} f^n(U)$ is closed for each $l \geqslant 0$.*

Proof: If U is tidy, then condition (1) is satisfied by Lemma 6(b). Since U is tidy, $f^l(U)$ is tidy for each $l \geqslant 0$, so $\bigcup_{n=l}^{\infty} f^n(U)$ is closed by [**3**, Proposition 1].

Conversely, conditions (1) and (2) together imply that U is tidy because from (1) we have that $G \backslash U_{++} = \bigcup_{l=0}^{\infty} \left(G \backslash \bigcup_{n=l}^{\infty} f^n(U) \right)$ and from (2) we have $G \backslash \bigcup_{n=l}^{\infty} f^n(U)$ is open for each l. Therefore $G \backslash U_{++}$ is an open subset, so U_{++} is closed. ∎

Proposition 8. *Let U be a compact open subgroup of G satisfying* T1. *Then U is tidy if and only if $\bigcup_{n \in F} f^n(U)$ is closed, where F is any subset of non-negative integers.*

Proof: By Lemma 6(a), we have that $G \backslash U_{++} = \bigcup_{F \in \mathfrak{F}} \left(G \backslash \bigcup_{n \in F} f^n(U) \right)$, where $\mathfrak{F}$ is the family of all infinite subsets of non-negative integers. Thus, if we assume that $\bigcup_{n \in F} f^n(U)$ is closed for each F, then U_{++} is also closed.

Conversely, suppose U is tidy. To show that $\bigcup_{n \in F} f^n(U)$ is closed for any $F \in \mathfrak{F}$, we can use the same proof that Willis used to show that $\bigcup_{n=0}^{\infty} f^n(U)$ is closed ([**3**, Proposition 1]). ∎

Recall that for every locally compact group, there is a topology on the space of closed subgroups (see [**1**, Chapter VIII, Section 5]). In this topology, a sequence of closed subgroups $\{S_n\}$ converges to S if for every compact set K and for every neighbourhood V of e, there exists an integer N such that for all $n > N$, $S \cap K \subseteq VS_n$ and $S_n \cap K \subseteq VS$.

Proposition 9. *Let U be a tidy subgroup of G. Then $\{f^n(U)\}$ converges to U_{++}.*

Proof: Let K be a compact set and let V be a neighbourhood of e.

Since U_{++} is closed, U_+ is open and $U_{++} \cap K$ is compact. Thus $U_{++} \cap K \subseteq f^{N_1}(U_+)$ for some N_1. Therefore, if $z \in U_{++} \cap K$, then $z \in f^{N_1}(U_+)$, which implies that $z \in f^n(U)$ for all $n > N_1$. Thus $U_{++} \cap K \subseteq Vf^n(U)$ for all $n > N_1$.

Now we want to show that there exists an integer N_2 such that for all $n > N_2, f^n(U) \cap K \subseteq VU_{++}$. Suppose not. Then there exists a sequence $0 < n_1 < n_2 < \cdots < n_i < \cdots$ such that $f^{n_i}(U) \cap K \not\subseteq VU_{++}$. That is, there exists $z_i \in f^{n_i}(U) \cap K$ such that $z_i \notin VU_{++}$. Since $\bigcup_{i=1}^{\infty} f^{n_i}(U)$ is closed (Proposition 8), $\bigcup_{i=1}^{\infty} f^{n_i}(U) \cap K$ is compact. Thus, by passing to a subsequence if necessary, we may assume that the sequence $\{z_i\}$ converges to z. Since $z_i \notin U_{++}$ and U_{++} is open, $z \notin U_{++}$. By Proposition 7 there exists an l such that $z \notin \bigcup_{n=l}^{\infty} f^n(U)$. Now $\bigcup_{n=l}^{\infty} f^n(U)$ is closed (Proposition 7), so there exists a neighbourhood W of e such that $zW \cap \bigcup_{n=l}^{\infty} f^n(U) = \emptyset$. Given W, there exists N_3 such that for all $i > N_3$, $z_i \in zW$. Choose $m > N_3$ such that $n_m > l$. Then $z_m \in zW \cap \bigcup_{n=l}^{\infty} f^n(U)$, which is a contradicton. Thus, there exists N_2 such that for all $n > N_2$, $f^n(U) \cap K \subseteq VU_{++}$. ∎

If U is a compact open subgroup of G, then $U_+ \cap U_-$ is a compact f-invariant subgroup of G. In particular, if U is tidy, the next proposition shows that $U_+ \cap U_-$ is a maximal compact f-invariant subgroup of U_{++}.

PROPOSITION 10. *Let U be a tidy subgroup of G. Then $U_+ \cap U_-$ is a maximal compact f-invariant subgroup of U_{++}.*

PROOF: Suppose M is a compact f-invariant subgroup of U_{++} which contains $U_+ \cap U_-$. Since $\{f^n(U_+)\}_{n=0}^{\infty}$ is an open cover of U_{++} and is monotonically increasing, $M \subseteq f^l(U_+)$ for some $l \geqslant 0$. If $l = 0$, then $M \subseteq U_+ \subseteq U$. Thus $M = U_+ \cap U_-$. If $l > 0$, then $f^{-l}(M) \subseteq U_+ \subseteq U$. Since $f^{-l}(M)$ is f-invariant and $U_+ \cap U_- \subseteq f^{-l}(M) \subseteq U_+$, $f^{-l}(M) = U_+ \cap U_-$. Thus $M = f^l(U_+ \cap U_-) = U_+ \cap U_-$. ∎

COROLLARY 11. *Let U be a compact open subgroup of G. If U satisfies* T1, *then U is tidy if and only if $U_+ \cap U_-$ is a maximal compact f-invariant subgroup of $\overline{U_{++}}$.*

PROOF: If U is tidy, then U_{++} is closed. By Proposition 10, $U_+ \cap U_-$ is a maximal compact f-invariant subgroup of $U_{++} = \overline{U_{++}}$.

Conversely, suppose $U_+ \cap U_-$ is a maximal compact f-invariant subgroup of $\overline{U_{++}}$. Since $U_+ \cap U_- \subseteq \mathcal{L}_U \subseteq L_U \subseteq \overline{U_{++}}$ and L_U is a compact f-invariant subgroup, $L_U = \mathcal{L}_U = U_+ \cap U_-$. Thus U is tidy by [**3**, Corollary to Lemma 3]. ∎

We conclude this paper by considering compact totally disconnected groups.

PROPOSITION 12. *Suppose G is compact. Let U be a compact open subgroup of G.*

(a) *U_{++} is closed if and only if $f(U_+) = U_+$.*

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(b) *U is tidy if and only if $f(U) = U = U_+ = U_-$.*

PROOF:

(a) Suppose $U_{++} = \bigcup_{n=0}^{\infty} f^n(U_+)$ is closed. Then U_{++} is compact. Thus there exists an l such that $U_{++} = f^l(U_+) = f^n(U_+)$ for all $n \geqslant l$. Hence $f(U_+) = U_+$. Conversely, if $f(U_+) = U_+$, then $U_{++} = U_+$ is closed.

(b) If U is tidy, then U satisfies T1, so $[f(U) : f(U) \cap U] = [f(U_+) : U_+]$ (see [**3**, p. 354]). Thus $f(U) \subseteq U$. But U is also tidy for f^{-1}, so $f^{-1}(U) \subseteq U$. Hence $f(U) = U$. $\square$

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