Can. J. Math., Vol. XXVI, No. 4, 1974, pp. 917–919

THE NAKAYAMA MAP AND RAMIFICATION FOR MAXIMALLY COMPLETE FIELDS

MURRAY A. MARSHALL

Let K be a maximally complete valued field and let L be a totally ramified Galois extension of K with Galois group G. Assume (i) the value group quotient of $L|K$ is cyclic and (ii) there exists an unramified cyclic extension of K of the same degree as L. Then there is an isomorphism of G^a onto a subgroup $A/N(L^\times)$ of $K^\times/N(L^\times)$ which maps the ramification group G^i onto $A^iN(L^\times)/N(L^\times)$ for all $i > 0$ where $A^i = \{x \in A | v(x-1) \geqq i\}$. This generalizes certain results of Local Class Field Theory.

1. The Nakayama map. Throughout, $L|K$ denotes a totally ramified Galois extension of valued fields, K is maximally complete [**4**], and the value group quotient Γ_L/Γ_K is cyclic. Assume also that K has an unramified cyclic extension $K'|K$ of the same degree as $L|K$. Let $L' = LK'$ denote the composition of L and K'. Identify the Galois groups $G_{L|K} = G_{L'|K'} = G$, $G_{K'|K} = G_{L'|L} = G'$, and the norm mappings $N_{K'|K} = N_{L'|L} = N'$, $N_{L|K} = N_{L'|K'} = N$. P_K, $P_K{}^i$, $i > 0$ will denote respectively

$$\{x \in K | v(x) > 0\}, \quad \{x \in K | v(x) \geqq i\}$$

where v is the valuation (written additively).

LEMMA 1. *Let $i \in \Gamma_K$, $i > 0$, and let ρ be a generator of G'. Then*
(i) $N'(1 + P_{K'}{}^i) = 1 + P_K{}^i$;
(ii) *if $x \in 1 + P_{K'}{}^i$, satisfies $N'(x) = 1$, then there exists $y \in 1 + P_{K'}{}^i$ such that $x = y^{\rho-1}$.*

Proof. The lemma says, in effect, that the G'-module $1 + P_{K'}{}^i$ has trivial cohomology. It is well-known that the additive group of the residue field $\bar{K}'$ has trivial cohomology as a G'-module. The result follows from this together with maximal completeness. (For a detailed proof of (i), see [**1**, Theorem 3].)

LEMMA 2. *There is an element $z \in L'$ satisfying*
(i) *$v(z)$ generates $\Gamma_{L'} = \Gamma_L$ modulo Γ_K;*
(ii) *$N'(z) \in K$.*

Proof. Let $y \in L$ be such that $v(y)$ generates Γ_L modulo Γ_K. If we can find $z \in L'$ such that $\pm N(y) = N'(z)$, we are finished. To this end write $N(y) = y^n u$ where $u = \Pi_{\sigma \in G} y^{\sigma-1}$. The map $G \to U_L/1 + P_L$ (where $U_L =$

Received February 14, 1973 and in revised form, February 6, 1974.

$\{x \in L | v(x) = 0\})$ given by $\sigma \to y^{\sigma-1}$ is a group homomorphism. If $\sigma^2 \neq 1$, then the terms $y^{\sigma-1}$ and $y^{\sigma^{-1}-1}$ in u cancel modulo $1 + P_L$. Hence $u^2 \equiv 1 \bmod (1 + P_L)$ so $u \equiv \pm 1 \bmod (1 + P_L)$. Thus $\pm u \in 1 + P_L = N'(1 + P_{L'})$ by Lemma 1 applied to $L'|L$.

THEOREM 1. *There is an injective homomorphism*

$$\alpha : G^a \to K^\times / N(L^\times).$$

(Here G^a denotes the maximal abelian factor group of G, and $K^\times$ the multiplicative group of K.)

Proof. Fix a generator ρ of G' and an element $z \in L'$ as in Lemma 2. If $\sigma \in G$, $N'(z^{\sigma-1}) = (N'(z))^{\sigma-1} = 1$ by property (ii) of z. Hence by Hilbert's Satz 90, there exists $y \in L'$ such that $z^{\sigma-1} = y^{\rho-1}$. $N(y)^{\rho-1} = N(y^{\rho-1}) = N(z^{\sigma-1}) = 1$ so $N(y) \in K^\times$. Also if $y^{\rho-1} = y'^{\rho-1}$, $y' \in L'$, then $y' = yc$, $c \in L$ so $N(y') \equiv N(y) \bmod N(L^\times)$, i.e. the class of $N(y)$ modulo $N(L^\times)$ depends only on σ. If $\sigma_1, \sigma_2 \in G$, and if $y_1, y_2 \in L'$ satisfy $y_i^{\rho-1} = z^{\sigma_i-1}$, $i = 1, 2$, then

$$z^{\sigma_1\sigma_2-1} = z^{(\sigma_2-1)\sigma_1} z^{\sigma_1-1} = (y_2^{\rho-1})^{\sigma_1} y_1^{\rho-1} = (y_2^{\sigma_1} y_1)^{\rho-1},$$

and $N(y_2^{\sigma_1} y_1) = N(y_1)N(y_2)$. Thus $\sigma \to N(y)$ where $y \in L'$ satisfies $y^{\rho-1} = z^{\sigma-1}$ defines a homomorphism from G^a into $K^\times/N(L^\times)$. To show this is injective assume $N(y) = N(y')$ for some $y' \in L$. It is enough to show that the restriction $\sigma|_M$ of σ to M is the identity for all cyclic extensions M of K in L. If M is any such extension, let $\sigma|_M = \tau^s$ where τ is a generator of $G_{M|K} = G_{M'|K'}$ $(M' = MK')$. By assumption $N_{L'|M'}(y)$ and $N_{L'|M'}(y')$ have the same norm relative to $M'|K'$. Thus by Hilbert's Satz 90, $N_{L'|M'}(y) = N_{L'|M'}(y')w^{\tau-1}$ for some $w \in M'$. Applying $\rho - 1$ to this and referring to the definition of y we have

$$\begin{aligned} N_{L'|M'}(z)^{\tau^s-1} &= N_{L'|M'}(z)^{\sigma-1} = N_{L'|M'}(z^{\sigma-1}) \\ &= N_{L'|M'}(y^{\rho-1}) = (N_{L'|M'}(y))^{\rho-1} = N_{L'|M'}(y')^{\rho-1}. w^{(\tau-1)(\rho-1)} \\ &= w^{(\rho-1)(\tau-1)}. \end{aligned}$$

since $y' \in L$. Factoring $\tau^s - 1$ we obtain

$$N_{L'|M'}(z)^{1+\tau+\cdots+\tau^{s-1}} = w^{\rho-1}. c, \quad c \in K'.$$

Comparing values in this last equation we get $s[L : M] \cdot v(z) \equiv 0 \bmod \Gamma_K$. But by property (i) of z, $v(z)$ has order $[L : K]$ modulo Γ_K. Thus $[M : K]$ divides s, so $\sigma|_M = \tau^s = 1$.

Remark. For each $\sigma \in G$ fix an element $y_\sigma \in L'^\times$ such that $y_\sigma^{\rho-1} = z^{\sigma-1}$; then $f : G \times G \to L^\times$ defined by $f(\tau, \sigma) = y_\sigma^\tau \cdot y_{\tau\sigma}^{-1}. y_\tau$ is a 2-cocycle and

$$N(y_\sigma) = \prod_{\tau \in G} f(\tau, \sigma).$$

Thus α as defined in Theorem 1 is the Nakayama Map determined by f [**3**].

Remark. As defined, α depends on the choice of an unramified extension K', a generator ρ of G', and the class of $v(z)$ in Γ_L/Γ_K. The image $A/N(L^\times)$ of α in $K^\times/N(L^\times)$ depends only on the choice of K' (and on L). In the classical case where K is discrete and $\bar{K}$ is finite, there is a unique choice for K' and G' and Γ_L/Γ_K have canonical generators so there is a canonical choice for α. The statement $A = K^\times$ is the "second fundamental inequality". This holds in the classical case and more generally in the case discussed in [**1**].

2. Ramification. Define the Herbrand function ϕ and the ramification groups $G_j = G^{\phi(j)}$, $j \geqq 0$ of the extension $L|K$ as in [**2**]. As Γ_L/Γ_K is cyclic, Theorem 2 of [**2**] holds.

LEMMA 3. *For all* $j > 0$, $N(1 + P_L{}^j) \subseteq 1 + P_K{}^{\phi(j)}$ *with equality if* $G_j = 1$.

Proof. By solvability of G together with transitivity of N, ϕ we can assume $[L : K]$ is a prime. The case is dealt with in [**1**, pp. 422, 426].

Defining A as in the remark, section 1, let

$$A^i = A \cap (1 + P_K{}^i) \quad i > 0.$$

THEOREM 2. *The Nakayama map* α *carries* G^i *onto* $A^iN(L^\times)/N(L^\times)$ *for all* $i \in \Gamma_L$, $i > 0$.

Proof. Suppose $\sigma \in G_j$. Thus $z^{\sigma-1} \in 1 + P_{L'}{}^j$. By Lemma 1 applied to $L'|L$, we can choose $y \in 1 + P_{L'}{}^j$ so that $y^{\rho-1} = z^{\sigma-1}$. Thus $N(y) \in N(1 + P_{L'}{}^j) \subseteq 1 + P_{K'}{}^{\phi(j)}$ by Lemma 3 applied to $L'|K'$. Conversely suppose $\sigma \in G$, $y^{\rho-1} = z^{\sigma-1}$, $N(y) \in 1 + P_K{}^i$. Let M be any cyclic extension of K in L fixed by G^i. If we can show $\sigma|_M = 1$ we are finished (since the fixed field of $G^i(G, G)$ is generated by such cyclic extensions). Since G^i fixes M, $G_{M|K}{}^i = 1$. Hence $N(y) \in N_{M|K}(M^\times)$ by Lemma 3 applied to $M|K$. Thus there exists $y' \in M$ such that $N_{L'|M'}(y)$ and y' have the same norm in $M'|K'$. From this point, the proof that $\sigma|_M = 1$ parallels the latter part of the proof of Theorem 1.

REFERENCES

1. L. Geissinger, *A reciprocity law for maximal fields*, Trans. Amer. Math. Soc. *125* (1966), 422—431.
2. M. Marshall, *Ramification theory for valuations of arbitrary rank*, Can. J. Math. *26* (1974), 908–916.
3. T. Nakayama, *Über die Beziehungen zwischen den Faktorensystemen und der Normklassengruppe eines galoisschen Erweiterungskörpers*, Math. Ann. *112* (1936), 85–91.
4. O. F. G. Shilling, *The theory of valuations*, Math. Surveys, No. IV (Amer. Math. Soc. publ., Providence, 1950).

University of Saskatchewan,
Saskatoon, Saskatchewan