Bull. Aust. Math. Soc. **98** (2018), 372–375
doi:10.1017/S0004972718000503

SOME HYPERGEOMETRIC INTEGRALS FOR LINEAR FORMS IN ZETA VALUES

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(Received 23 April 2018; accepted 12 May 2018; first published online 19 July 2018)

Dedicated to Carlo Viola, whose creativity in and love of integrals
for linear forms in zeta values are boundless, on the occasion of his 75th birthday

Abstract

We prove new integral representations of the approximation forms in zeta values.

2010 *Mathematics subject classification*: primary 11M06; secondary 11J72, 33C20.

Keywords and phrases: Hurwitz zeta function, hypergeometric integrals, integral representation.

In the exposition below, s and D are positive integers such that $s \ge 3D-1$, while the parameter n is assumed to be a positive *even* integer. The notation

$$\zeta(s,\alpha)=\sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^s}$$

is used for the Hurwitz zeta function, so that $\zeta(s)=\zeta(s,1)$, and $d_n=\operatorname{lcm}(1,2,\dots,n)$.

In [1] the following approximations are constructed: for any $j\in\{1,\dots,D\}$, take

$$r_{n,j}=\sum_{m=1}^{\infty}R_n\Big(m+\frac{j}{D}\Big),\quad \text{where } R_n(t)=D^{3Dn}n!^{s+1-3D}\frac{\prod_{l=0}^{3Dn}(t-n+l/D)}{\prod_{l=0}^{n}(t+l)^{s+1}}.$$

It is shown that

$$r_{n,j}=a_{0,j}+\sum_{\substack{2\le i\le s\\ i\equiv s \ (\mathrm{mod}\ 2)}}a_i\zeta\Big(i,\frac{j}{D}\Big), \tag{1}$$

with

$$d_n^{s+1-i}a_i\in\mathbb{Z}\quad \text{for } i=2,3,4,\dots,s \text{ and } i\equiv s \pmod 2,$$
$$d_{n+1}^{s+1}a_{0,j}\in\mathbb{Z}\quad \text{for } j\in\{1,\dots,D\}$$

(see [1, Lemmas 1 and 2]), and some further information is provided for the asymptotic growth of the *positive* quantities $r_{n,j}$ as $n \to \infty$. (We note that choosing n even implies that $3Dn + 1 + (s+1)(n+1) \equiv s \pmod 2$ and hence $R_n(-n-t) = (-1)^s R_n(t)$. This reflects on the parity in the summation in (1)—consideration in [1] is restricted to the case of s odd.) The approximations are building blocks for linear forms in zeta values $\zeta(i)$ with i of the same parity as s, with the help of the elementary formula

$$\sum_{j=1}^{d} \zeta\Big(i, \frac{j(D/d)}{D}\Big) = \sum_{j=1}^{d} \zeta\Big(i, \frac{j}{d}\Big) = d^i \zeta(i)$$

valid for any divisor d of D.

The principal goal of this note is to establish the following integral representation of the approximations $r_{n,j}$ for $j \in \{1, \ldots, D\}$.

Theorem 1. *The linear forms (1) admit the integral representation*

$$r_{n,j} = \frac{D^{s-1}(3Dn+1)!}{n!^{3D}} \sum_{m=1}^{D} \xi^{-mj} r^*_{n,m},$$

where

$$r^*_{n,m} = \xi^m \underset{[0,1]^{s+1}}{\int \cdots \int} \frac{\prod_{i=0}^{s} x_i^{Dn}(1-x_i^D)^n \, dx_i}{(1-\xi^m x_0 \cdots x_s)^{3Dn+2}} = \int_0^{\xi^m} \underset{[0,1]^{s}}{\int \cdots \int} \frac{\prod_{i=0}^{s} x_i^{Dn}(1-x_i^D)^n \, dx_i}{(1-x_0 \cdots x_s)^{3Dn+2}}$$

and $\xi = \xi_D$ denotes a primitive root of unity of degree D.

Proof. As the rational function $R_n(t)$ has zeros at $t = m - (D-j)/D$ for $m = 1, \ldots, n$ and $j \in \{1, \ldots, D\}$, we can write

$$\begin{aligned}
r_{n,j} &= \sum_{m=n}^{\infty} R_n\Big(m + \frac{j}{D}\Big) = D^{3Dn} n!^{s+1-3D} \sum_{k=0}^{\infty} \frac{\prod_{l=0}^{3Dn}(k+(l+j)/D)}{\prod_{l=0}^{n}(k+n+l+j/D)^{s+1}} \\
&= \frac{n!^{s+1-3D} \prod_{l=0}^{3Dn}(l+j)}{D \prod_{l=0}^{n}(n+l+j/D)^{s+1}} \\
&\qquad \times {}_{s+D+1}F_{s+D}\left(\left.\begin{matrix} \Big\{3n + \frac{j+l}{D} : l = 1, \ldots, D\Big\}, \Big\{n + \frac{j}{D}\Big\}^{s+1} \\ \Big\{1 + \frac{j-l}{D} : l = 1, \ldots, D, j \neq l\Big\}, \Big\{2n + 1 + \frac{j}{D}\Big\}^{s+1} \end{matrix} \right| 1\right) \\
&= \frac{(3Dn+j)!}{Dn!^{3D}(j-1)!} \underset{[0,1]^{s+1}}{\int \cdots \int} f_j(t_0 \cdots t_s) \prod_{i=0}^{s} t_i^{n+j/D-1}(1-t_i)^n \, dt_i, \qquad (2)
\end{aligned}$$

https://doi.org/10.1017/S0004972718000503 Published online by Cambridge University Press

where

$$f_j(t) = {}_DF_{D-1}\left(\begin{matrix}\left\{3n + \dfrac{j+l}{D} : l = 1, \dots, D\right\} \\ \left\{1 + \dfrac{j-l}{D} : l = 1, \dots, D,\ j \ne l\right\}\end{matrix}\middle|\, t\right)$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{l=1}^{D}\left(3n + \dfrac{j+l}{D}\right)_k}{\prod_{l=1}^{D}\left(1 + \dfrac{j-l}{D}\right)_k} t^k = \sum_{k=0}^{\infty} \frac{(3Dn + j + 1)_{Dk}}{(j)_{Dk}} t^k \quad \text{for } j \in \{1, \dots, D\}.$$

Recall that

$$\sum_{l=0}^{\infty} \frac{(a)_l}{l!} x^l = \frac{1}{(1-x)^a}$$

and observe that

$$\frac{(3Dn+2)_{j-1}}{(j-1)!} x^{j-1} f_j(x^D) = \sum_{k=0}^{\infty} \frac{(3Dn+2)_{Dk+j-1}}{(Dk+j-1)!} x^{Dk+j-1}$$

$$= \sum_{\substack{l=0 \\ l \equiv j-1 \pmod D}}^{\infty} \frac{(3Dn+2)_l}{l!} x^l = \frac{1}{D} \sum_{m=1}^{D} \frac{\xi^{-m(j-1)}}{(1-\xi^m x)^{3Dn+2}}.$$

Taking $t_i = x_i^D$ for $i = 0, 1, \dots, s$ in the integrals (2), we thus obtain

$$r_{n,j} = \frac{D^{s-1}(3Dn+1)!}{n!^{3D}} \sum_{m=1}^{D} \xi^{-m(j-1)} \int\cdots\int_{[0,1]^{s+1}} \frac{\prod_{i=0}^{s} x_i^{Dn}(1-x_i^D)^n\, dx_i}{(1-\xi^m x_0 \cdots x_s)^{3Dn+2}}$$

for each $j \in \{1, \dots, D\}$. □

Choosing $D = 2$ and $s \geq 5$ odd, we obtain the linear forms

$$7r_{n,2} - r_{n,1} = \frac{2^s(6n+1)!}{n!^6} \int\cdots\int_{[0,1]^{s+1}} \left(\frac{3}{(1-x_0x_1\cdots x_s)^{6n+2}} - \frac{4}{(1+x_0x_1\cdots x_s)^{6n+2}}\right) \prod_{i=0}^{s} x_i^{2n}(1-x_i^2)^n\, dx_i$$

$$= \frac{2^s(6n+1)!}{n!^6} \int\cdots\int_{\gamma\times[0,1]^s} \frac{\prod_{i=0}^{s} x_i^{2n}(1-x_i^2)^n\, dx_i}{(1-x_0x_1\cdots x_s)^{6n+2}}$$

in $\mathbb{Q} + \mathbb{Q}\zeta(5) + \cdots + \mathbb{Q}\zeta(s)$ considered previously in [2]. Here the path $\gamma \subset \mathbb{R}$ for integrating with respect to x_0 is given by $\gamma = 3[0,1] + 4[0,-1]$ and the parity assumption on n can be dropped.

Acknowledgements

This note was produced during the trimester on *Periods in Number Theory, Algebraic Geometry and Physics* at the Hausdorff Research Institute for Mathematics (Bonn, Germany). I thank Clément Dupont for his encouragement to write the integrals for the hypergeometric approximations used in [1, 2].

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https://doi.org/10.1017/S0004972718000503 Published online by Cambridge University Press