*Euro. Jnl of Applied Mathematics* (2023), *vol.* 34, *pp.* 617–641  
doi:10.1017/S0956792522000195

# Homogenisation of a two-phase problem with nonlinear dynamic Wentzell-interface condition for connected–disconnected porous media

M. GAHN

*Interdisciplinary Center for Scientific Computing, University of Heidelberg, Im Neuenheimer Feld 205, Heidelberg 69120, Germany*
*email: markus.gahn@iwr.uni-heidelberg.de*

(*Received 26 February 2021; revised 4 May 2022; accepted 20 May 2022; first published online 21 June 2022*)

We investigate a reaction–diffusion problem in a two-component porous medium with a nonlinear interface condition between the different components. One component is connected and the other one is disconnected. The ratio between the microscopic pore scale and the size of the whole domain is described by the small parameter $\epsilon$. On the interface between the components, we consider a dynamic Wentzell-boundary condition, where the normal fluxes from the bulk domains are given by a reaction–diffusion equation for the traces of the bulk solutions, including nonlinear reaction kinetics depending on the solutions on both sides of the interface. Using two-scale techniques, we pass to the limit $\epsilon \to 0$ and derive macroscopic models, where we need homogenisation results for surface diffusion. To cope with the nonlinear terms, we derive strong two-scale convergence results.

**Keywords:** Homogenisation, two-scale convergence, reaction–diffusion equations, nonlinear interface conditions, surface diffusion

**2020 Mathematics Subject Classification:** 35B27 (Primary); 35K57 (Secondary)

## 1 Introduction

In this paper, we derive homogenised models for nonlinear reaction–diffusion problems with dynamic Wentzell-boundary conditions in multi-component porous media. The domain consists of two components $\Omega_\epsilon^1$ and $\Omega_\epsilon^2$, where $\Omega_\epsilon^1$ is connected, and $\Omega_\epsilon^2$ is disconnected and consists of periodically distributed inclusions. The small scaling parameter $\epsilon$ represents the ratio between the length of an inclusion and the size of the whole domain. At the interface $\Gamma_\epsilon$ between the two components, we assume a dynamic Wentzell-boundary condition, that is, the normal flux at the surface is given by a reaction–diffusion equation on $\Gamma_\epsilon$. More precisely, this boundary/interface condition describes processes like reactions, adsorption, desorption and diffusion at the interface $\Gamma_\epsilon$. Further, it takes into account exchange of species between the different compartments, what can be modelled by nonlinear reaction kinetics depending on the solutions on both sides of $\Gamma_\epsilon$. The aim is the derivation of macroscopic models with homogenised diffusion coefficients for $\epsilon \to 0$, the solution of which is an approximation of the microscopic solution. An additional focus of the paper is to provide general strong two-scale compactness results, which are based on *a priori* estimates for the microscopic solution.

Reaction–diffusion processes play an important role in many applications, and our model is motivated by metabolic and regulatory processes in living cells. Here, an important example is the carbohydrate metabolism in plant cells, where biochemical species are diffusing and reacting within the (connected) cytoplasm and the (disconnected) organelles (like chloroplasts and mitochondria) and are exchanged between different cellular compartments. At the outer mitochondrial membrane takes place the process of metabolic channelling, where intermediates in metabolic pathways are passed directly from enzyme to enzyme without equilibrating in the bulk-solution phase of the cell [32]. This effect can be modelled by the dynamic Wentzell-boundary condition, see [16, Chapter 4] for more details about the modelling and the derivation of these boundary conditions, which can be derived by asymptotic analysis.

To pass to the limit $\epsilon \to 0$ in the variational equation for the microscopic problem, we have to cope with several difficulties. The main challenges are the coupled bulk-surface diffusion in the perforated domains, as well as the treatment of the nonlinear terms, especially on the oscillating surface $\Gamma_\epsilon$. To overcome these problems, we make use of the two-scale method in perforated domains and on oscillating surfaces, where we need two-scale compactness results for diffusion processes on surfaces. To pass to the limit in the nonlinear terms, we need strong convergence results. Such results are quite standard for the connected domain, but the usual methods fail for the disconnected domain. Here we make use of the unfolding method, which gives us a characterisation for the two-scale convergence via functions defined on fixed domains, and a Kolmogorov–Simon-type compactness result for Banach-valued function spaces. Additionally, due to the nonlinear structure of the problem and the weak assumptions on the data, we have to deal with low regularity for the time derivative.

There exists a large amount of papers dealing with homogenisation problems for parabolic equations in multi-component porous media. However, results for the connected–disconnected case for nonlinear problems, especially for nonlinear interface conditions, seem to be rare. In [20] and [19], systems of reaction–diffusion problems are considered with nonlinear interface conditions. In [20], surface concentration is included and an additional focus lies on the modelling part of the carbohydrate metabolism and the specific structure of the nonlinear reaction kinetics. In the present paper, we extend those models to problems including an additional surface diffusion for the traces of the bulk solutions in $\Omega_\epsilon^1$ and $\Omega_\epsilon^2$. The stationary case for different scalings with a continuous normal flux condition at the interface, given by a nonlinear monotone function depending on the jump of the solutions on both sides, is treated in [14]. There, the nonlinear terms in the disconnected domain only occur for particular scalings and it is not straightforward to generalise those results to systems.

A double porosity model, where the diffusion inside the disconnected domain is of order $\epsilon^2$, is considered in [9, 29] for continuous transmission conditions at the interface for the solutions and the normal fluxes. In [9], a nonlinear diffusion coefficient is considered, and the convergence of the nonlinear term is obtained by using the Kirchhoff transformation and comparing the microscopic and the macroscopic equation, where the last one was obtained by a formal asymptotic expansion. Nonlinear reaction kinetics in the bulk domains and an additional ordinary differential equation on the interface are considered in [29], where the strong convergence is proved by showing that the unfolded sequence of the microscopic solution is a Cauchy sequence. A similar model with different kind of interface conditions is considered in [25], where the method of two-scale convergence is used and a variational principle to identify the limits of the nonlinear terms.

To pass to the limit in the diffusive terms on the interface $\Gamma_\epsilon$ arising from the Wentzell-boundary condition, compactness results for the surface gradient on an oscillating manifolds are needed. For such kind of problems in [4, 22], two-scale compactness results are derived for connected surfaces, where in [22] the method of unfolding is used. Compactness results for a coupled bulk-surface problem when the evolution of the trace of the bulk solution on the surface $\Gamma_\epsilon$ is described by a diffusion equation are treated in [6, 17]. In [6], continuity of the traces across the interface is assumed, where in [17] also jumps across the interface are allowed and also compactness results for the disconnected domain $\Omega_\epsilon^2$ are derived. In [6], the convergence results are applied to a linear problem with a dynamic Wentzell-interface condition. A reaction–diffusion problem including dynamic Wentzell-boundary conditions and nonlinear reaction rates in the bulk domain and on the surface is considered in [7] for a connected perforated domain.

In this paper, we start with the microscopic model and establish existence and uniqueness of a weak solution. The appropriate function space for a weak solution is the space of Sobolev functions of first order with $H^1$-traces on the interface $\Gamma_\epsilon$, which we denote by $\mathbb{H}_{j,\epsilon}$ for $j = 1, 2$. To pass to the limit $\epsilon \to 0$, we make use of the method of two-scale convergence for domains and surfaces, see [2, 3, 26, 28]. For the treatment of the diffusive terms on the oscillating surface, we use the methods developed in [17] for the spaces $\mathbb{H}_{j,\epsilon}$. Those two-scale compactness results are based on *a priori* estimates for the microscopic solution depending explicitly on $\epsilon$. However, to pass to the limit in the nonlinear terms, the usual (weak) two-scale convergence is not enough and we need strong two-scale convergence, what leads to difficulties especially in the disconnected domain $\Omega_\epsilon^2$. The strong convergence is obtained by applying the unfolding operator, see [12] for an overview of the unfolding method, to the microscopic solution and uses a Kolmogorov–Simon-type compactness result for the unfolded sequence. We derive a general strong two-scale compactness result that is based only on *a priori* estimates and estimates for the difference between the solution and discrete shifts (with respect to the microscopic cells) of the solution. Since we only take into account linear shifts, which are not well defined for general surfaces, we use a Banach-valued Kolmogorov–Simon-compactness result, see [18]. Further, for our microscopic model we only obtain low regularity results for the time derivative (which is a functional on $\mathbb{H}_{j,\epsilon}$), what leads to additional difficulties in the control of the time variable in the proof of the strong convergence.

This paper is organised as follows: In Section 2, we introduce the geometrical setting and the microscopic model. In Section 3, we show existence and uniqueness of a microscopic solution and derive *a priori* estimates depending explicitly on $\epsilon$. In Section 4, we prove general strong two-scale compactness results for the connected and disconnected domain. In Section 5, we state the convergence results for the microscopic solution, formulate the macroscopic model and show that the limit of the micro-solutions solves the macro-model. In the Appendix A, we repeat the definition of the two-scale convergence and the unfolding operator and summarise some well-known results from the literature.

## 2 The microscopic model

In this section, we introduce the microscopic problem. We start with the definition of the microscopic domains $\Omega_\epsilon^1$ and $\Omega_\epsilon^2$, as well as the interface $\Gamma_\epsilon$, and explain some geometrical properties. Then we state the microscopic equation for given $\epsilon$ and give the assumptions on the data.

### 2.1 The microscopic geometry

Let $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary and $\epsilon > 0$ a sequence with $\epsilon^{-1} \in \mathbb{N}$. We define the unit cube $Y := (0,1)^n$ and $Y_2 \subset Y$ such that $\overline{Y_2} \subset Y$, so $Y_2$ strictly included in $Y$. Further, we define $Y_1 := Y \setminus \overline{Y_2}$ and $\Gamma := \partial Y_2$, and we suppose that $\Gamma \in C^{1,1}$. We assume that $Y_1$ is connected and for the sake of simplicity we also assume that $Y_2$ is connected. The general case of disconnected $Y_2$ is easily obtained by considering the connected components of $Y_2$, see also Remark 2. Now, the microscopic domains $\Omega_\epsilon^1$ and $\Omega_\epsilon^2$ are defined by scaled and shifted reference elements $Y_j$ for $j = 1, 2$. Let $K_\epsilon := \{k \in \mathbb{Z}^n : \epsilon(k+Y) \subset \Omega\}$ and define

$$\Omega_\epsilon^2 := \bigcup_{k \in K_\epsilon} \epsilon(Y_2 + k), \qquad \Omega_\epsilon^1 := \Omega \setminus \overline{\Omega_\epsilon^2}, \qquad \Gamma_\epsilon := \partial \Omega_\epsilon^2.$$

Hence, $\Gamma_\epsilon$ denotes the oscillating interface between $\Omega_\epsilon^1$ and $\Omega_\epsilon^2$. Due to the assumptions on $Y_1$ and $Y_2$ it holds that $\Omega_\epsilon^1$ is connected and $\Omega_\epsilon^2$ is disconnected, and $\Gamma_\epsilon \in C^{1,1}$ is not touching the outer boundary $\partial\Omega$.

### 2.2 The microscopic model

We are looking for a solution $\left(u_\epsilon^1, u_\epsilon^2\right)$ with $u_\epsilon^j : (0,T) \times \Omega_\epsilon^j \to \mathbb{R}$ for $j = 1, 2$, such that it holds that

$$\begin{aligned} \partial_t u_\epsilon^j - \nabla \cdot \left(D_\epsilon^j \nabla u_\epsilon^j\right) &= f_\epsilon^j\left(u_\epsilon^j\right) && \text{in } (0,T) \times \Omega_\epsilon^j, \\ \epsilon\left(\partial_t u_\epsilon^j - \nabla_{\Gamma_\epsilon} \cdot \left(D_{\Gamma_\epsilon}^j \nabla_{\Gamma_\epsilon} u_\epsilon^j\right) - h_\epsilon^j\left(u_\epsilon^1, u_\epsilon^2\right)\right) &= -D_\epsilon^j \nabla u_\epsilon^j \cdot \nu && \text{on } (0,T) \times \Gamma_\epsilon, \\ -D_\epsilon^1 \nabla u_\epsilon^1 \cdot \nu &= 0 && \text{on } (0,T) \times \partial\Omega, \\ u_\epsilon^j(0) &= u_{\epsilon,i}^j && \text{in } \Omega_\epsilon^j, \\ u_\epsilon^j|_{\Gamma_\epsilon}(0) &= u_{\epsilon,i,\Gamma_\epsilon}^j && \text{on } \Gamma_\epsilon, \end{aligned} \tag{2.1}$$

where $\nu$ denotes the outer unit normal (we neglect a subscript for the underlying domain, since this should be clear from the context), and $u_\epsilon^j|_{\Gamma_\epsilon}$ denotes the trace of $u_\epsilon^j$ on $\Gamma_\epsilon$. If it is clear from the context, we use the same notation for a function and its trace, for example, we just write $u_\epsilon^j$ for $u_\epsilon^j|_{\Gamma_\epsilon}$. The precise weak formulation of the micro-model above is stated in Section 3, see (3.2), after introducing the necessary function spaces.

In the following, with $T_y\Gamma$ and $T_x\Gamma_\epsilon$ for $y \in \Gamma$ and $x \in \Gamma_\epsilon$ we denote the tangent spaces of $\Gamma$ at $y$ and $\Gamma_\epsilon$ at $x$, respectively. The orthogonal projection $P_\Gamma(y) : \mathbb{R}^n \to T_y\Gamma$ for $y \in \Gamma$ is given by

$$P_\Gamma(y)\xi = \xi - (\xi \cdot \nu(y))\nu(y) \qquad \text{for } \xi \in \mathbb{R}^n,$$

where $\nu(y)$ denotes the outer unit normal at $y \in \Gamma$. Let us extend the unit normal $Y$-periodically. Then, the orthogonal projection $P_{\Gamma_\epsilon}(x) : \mathbb{R}^n \to T_x\Gamma_\epsilon$ for $x \in \Gamma_\epsilon$ is given by

$$P_{\Gamma_\epsilon}(x)\xi = \xi - \left(\xi \cdot \nu\left(\frac{x}{\epsilon}\right)\right)\nu\left(\frac{x}{\epsilon}\right) \qquad \text{for } \xi \in \mathbb{R}^n.$$

**Assumptions on the data:**
In the following let $j \in \{1, 2\}$.

(A1) For the bulk diffusion, we have $D^j_\epsilon(x) := D^j\left(\frac{x}{\epsilon}\right)$ with $D^j \in L^\infty_{\text{per}}\left(Y_j\right)^{n\times n}$ symmetric and coercive, that is, there exits $c_0 > 0$ such that for almost every $y \in Y_j$

$$D^j(y)\xi \cdot \xi \geqslant c_0|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^n.$$

(A2) For the surface diffusion, we suppose $D^j_{\Gamma_\epsilon}(x) := D^j_\Gamma\left(\frac{x}{\epsilon}\right)$ with $D^j_\Gamma \in L^\infty_{\text{per}}(\Gamma)^{n\times n}$ symmetric and $D^j_\Gamma(y)|_{T_y\Gamma} : T_y\Gamma \to T_y\Gamma$ for almost every $y \in \Gamma$. Further, we assume that $D^j_\Gamma$ is coercive, that is, there exists $c_0 > 0$ such that for almost every $y \in \Gamma$

$$D^j_\Gamma(y)\xi \cdot \xi \geqslant c_0|\xi|^2 \qquad \text{for all } \xi \in T_y\Gamma.$$

(A3) For the reaction rates in the bulk domains, we suppose that $f^j_\epsilon(t, x, z) := f^j\left(t, \frac{x}{\epsilon}, z\right)$ with $f^j \in L^\infty\big((0, T) \times Y_j \times \mathbb{R}\big)$ is $Y$-periodic with respect to the second variable and uniformly Lipschitz continuous with respect to the last variable, that is, there exists $C > 0$ such that for all $z, w \in \mathbb{R}$ and almost every $(t, y) \in (0, T) \times Y_j$ it holds that

$$|f^j(t, y, z) - f^j(t, y, w)| \leqslant C|z - w|.$$

(A4) For the reaction rates on the surface, we suppose that $h^j_\epsilon(t, x, z_1, z_2) := h^j\left(t, \frac{x}{\epsilon}, z_1, z_2\right)$ with $h^j \in L^\infty((0, T) \times \Gamma \times \mathbb{R}^2)$ is $Y$-periodic with respect to the second variable and uniformly Lipschitz continuous with respect to the last variable, that is, there exists $C > 0$ such that for all $z_1, z_2, w_1, w_2 \in \mathbb{R}$ and almost every $(t, y) \in (0, T) \times \Gamma$ it holds that

$$|h^j(t, y, z_1, z_2) - h^j(t, y, w_1, w_2)| \leqslant C\big(|z_1 - w_1| + |z_2 - w_2|\big).$$

(A5) For the initial conditions, we assume $u^j_{\epsilon,i} \in L^2\left(\Omega^j_\epsilon\right)$ and $u^j_{\epsilon,i,\Gamma_\epsilon} \in L^2(\Gamma_\epsilon)$ with

$$\|u^j_{\epsilon,i}\|_{L^2\left(\Omega^j_\epsilon\right)} + \sqrt{\epsilon}\|u^j_{\epsilon,i,\Gamma_\epsilon}\|_{L^2(\Gamma_\epsilon)} \leqslant C.$$

Further, there exist $u^j_{0,i} \in L^2(\Omega)$ and $u^j_{0,i,\Gamma} \in L^2(\Omega)$ such that

$$\begin{aligned} u^j_{\epsilon,i} &\to u^j_{0,i} && \text{in the two-scale sense,} \\ u^j_{\epsilon,i,\Gamma_\epsilon} &\to u^j_{0,i,\Gamma} && \text{in the two-scale sense on } \Gamma_\epsilon. \end{aligned}$$

Additionally, we assume that the sequences $u^2_{\epsilon,i}$ and $u^2_{\epsilon,i,\Gamma_\epsilon}$ converge strongly in the two-scale sense. For the definition of the two-scale convergence see Section 4.

## 3 Existence of a weak solution and *a priori* estimates

The aim of this section is the investigation of the microscopic problem (2.1). We introduce appropriate function spaces and show existence and uniqueness of a microscopic solution. Further, we derive *a priori* estimates for the solution depending explicitly on $\epsilon$. These estimates form the basis for the derivation of the macroscopic problem (5.7) by using the compactness results from Section 4.

https://doi.org/10.1017/S0956792522000195 Published online by Cambridge University Press

### 3.1 Function spaces

Due to the Laplace–Beltrami operator in the boundary condition in (2.1), it is not enough to consider the usual Sobolev space $H^1\left(\Omega_\epsilon^j\right)$ as a solution space, because we need more regular traces. This gives rise to deal with the following function spaces for $j = 1, 2$:

$$\begin{aligned} \mathbb{H}_{j,\epsilon} &:= \left\{\phi_\epsilon^j \in H^1\left(\Omega_\epsilon^j\right) \,:\, \phi_\epsilon^j|_{\Gamma_\epsilon} \in H^1(\Gamma_\epsilon)\right\}, \\ \mathbb{H}_j &:= \left\{\phi \in H^1\left(Y_j\right) \,:\, \phi|_\Gamma \in H^1(\Gamma)\right\}, \end{aligned} \tag{3.1}$$

with the inner products

$$\begin{aligned} \left(\phi_\epsilon^j, \psi_\epsilon^j\right)_{\mathbb{H}_{j,\epsilon}} &:= \left(\phi_\epsilon^j, \psi_\epsilon^j\right)_{H^1\left(\Omega_\epsilon^j\right)} + \epsilon\left(\phi_\epsilon^j, \psi_\epsilon^j\right)_{H^1(\Gamma_\epsilon)}, \\ (\phi, \psi)_{\mathbb{H}_j} &:= (\phi, \psi)_{H^1(Y_j)} + (\phi, \psi)_{H^1(\Gamma)}. \end{aligned}$$

The associated norms are denoted by $\|\cdot\|_{\mathbb{H}_{j,\epsilon}}$ and $\|\cdot\|_{\mathbb{H}_j}$. Obviously, the spaces $\mathbb{H}_{j,\epsilon}$ and $\mathbb{H}_j$ are separable Hilbert spaces and we have the dense embeddings $C^\infty\left(\overline{\Omega_\epsilon^j}\right) \subset \mathbb{H}_{j,\epsilon}$ and $C^\infty\left(\overline{Y_j}\right) \subset \mathbb{H}_j$, see [17, Proposition 5]. We also define the space

$$\mathbb{L}_{j,\epsilon} := L^2\left(\Omega_\epsilon^j\right) \times L^2(\Gamma_\epsilon), \qquad \mathbb{L}_j := L^2\left(Y_j\right) \times L^2(\Gamma)$$

with inner products

$$\begin{aligned} \left(\phi_\epsilon^j, \psi_\epsilon^j\right)_{\mathbb{L}_{j,\epsilon}} &:= \left(\phi_\epsilon^j, \psi_\epsilon^j\right)_{L^2\left(\Omega_\epsilon^j\right)} + \epsilon\left(\phi_\epsilon^j, \psi_\epsilon^j\right)_{L^2(\Gamma_\epsilon)}, \\ (\phi, \psi)_{\mathbb{L}_j} &:= (\phi, \psi)_{L^2(Y_j)} + (\phi, \psi)_{L^2(\Gamma)}, \end{aligned}$$

and again denote the associated norms by $\|\cdot\|_{\mathbb{L}_{j,\epsilon}}$ and $\|\cdot\|_{\mathbb{L}_j}$. Obviously, we have

$$\mathbb{H}_{j,\epsilon} \cong \left\{(u_\epsilon, v_\epsilon) \in H^1\left(\Omega_\epsilon^j\right) \times H^1(\Gamma_\epsilon) \,:\, u_\epsilon|_{\Gamma_\epsilon} = v_\epsilon\right\},$$

and a similar result for $\mathbb{H}_j$. Therefore, we have the following Gelfand triples:

$$\mathbb{H}_{j,\epsilon} \hookrightarrow \mathbb{L}_{j,\epsilon} \hookrightarrow \mathbb{H}_{j,\epsilon}', \qquad \mathbb{H}_j \hookrightarrow \mathbb{L}_j \hookrightarrow \mathbb{H}_j'.$$

We will also make use for $\alpha \in \left(\frac{1}{2}, 1\right]$ of the function space

$$\mathbb{H}_j^\alpha := \left\{\phi \in H^\alpha\left(Y_j\right) \,:\, \phi|_\Gamma \in H^\alpha(\Gamma)\right\}$$

with inner product

$$(\phi, \psi)_{\mathbb{H}_j^\alpha} := (\phi, \psi)_{H^\alpha(Y_j)} + (\phi, \psi)_{H^\alpha(\Gamma)}.$$

By definition, we have $\mathbb{H}_j = \mathbb{H}_j^1$. Obviously, we have the compact embedding $\mathbb{H}_j \hookrightarrow \mathbb{H}_j^\alpha$ for $\alpha \in \left(\frac{1}{2}, 1\right)$.

### 3.2 Existence and uniqueness of a weak solution

A weak solution of Problem (2.1) is defined in the following way: The tuple $\left(u_\epsilon^1, u_\epsilon^2\right)$ is a weak solution of (2.1) if for $j = 1, 2$

$$u_\epsilon^j \in L^2\left((0, T), \mathbb{H}_{j,\epsilon}\right) \cap H^1\left((0, T), \mathbb{H}_{j,\epsilon}'\right),$$

https://doi.org/10.1017/S0956792522000195 Published online by Cambridge University Press

$u_\epsilon^j$ and $u_\epsilon^j|_{\Gamma_\epsilon}$ fulfil the initial condition $u_\epsilon^j(0) = u_{\epsilon,i}^j$ and $u_\epsilon^j|_{\Gamma_\epsilon}(0) = u_{\epsilon,i,\Gamma_\epsilon}^j$, and for all $\phi_\epsilon^j \in \mathbb{H}_{j,\epsilon}$ it holds almost everywhere in $(0,T)$

$$\begin{aligned}\langle \partial_t u_\epsilon^j, \phi_\epsilon^j \rangle_{\mathbb{H}'_{j,\epsilon},\mathbb{H}_{j,\epsilon}} &+ \left(D_\epsilon^j \nabla u_\epsilon^j, \nabla \phi_\epsilon^j\right)_{\Omega_\epsilon^j} + \epsilon \left(D_{\Gamma_\epsilon}^j \nabla_{\Gamma_\epsilon} u_\epsilon^j, \nabla_{\Gamma_\epsilon} \phi_\epsilon^j\right)_{\Gamma_\epsilon} \\ &= \left(f_\epsilon^j(u_\epsilon^j), \phi_\epsilon^j\right)_{\Omega_\epsilon^j} + \epsilon \left(h_\epsilon^j\left(u_\epsilon^1, u_\epsilon^2\right), \phi_\epsilon^j\right)_{\Gamma_\epsilon}.\end{aligned} \tag{3.2}$$

Here $(\cdot,\cdot)_U$ stands for the inner product on $L^2(U)$, for a suitable set $U \subset \mathbb{R}^n$ and for a Banach space $X$ and its dual $X'$ we write $\langle \cdot,\cdot \rangle_{X',X}$ for the duality pairing between $X'$ and $X$. The scaling factor $\epsilon$ for the time derivative in (2.1) is included in the duality pairing $\langle \cdot,\cdot \rangle_{\mathbb{H}'_{j,\epsilon},\mathbb{H}_{j,\epsilon}}$. In fact, if additionally it holds that $\partial_t u_\epsilon^j \in L^2\left((0,T), H^1\left(\Omega_\epsilon^j\right)'\right)$ and $\partial_t u_\epsilon^j|_{\Gamma_\epsilon} \in L^2((0,T), H^1(\Gamma_\epsilon)')$ with respect to the Gelfand triples $H^1\left(\Omega_\epsilon^j\right) \hookrightarrow L^2\left(\Omega_\epsilon^j\right) \hookrightarrow H^1\left(\Omega_\epsilon^j\right)'$ and $H^1(\Gamma_\epsilon) \hookrightarrow L^2(\Gamma_\epsilon) \hookrightarrow H^1(\Gamma_\epsilon)'$, we get for all $\phi_\epsilon^j \in \mathbb{H}_{j,\epsilon}$

$$\langle \partial_t u_\epsilon^j, \phi_\epsilon^j \rangle_{\mathbb{H}'_{j,\epsilon},\mathbb{H}_{j,\epsilon}} = \langle \partial_t u_\epsilon^j, \phi_\epsilon^j \rangle_{H^1\left(\Omega_\epsilon^j\right)',H^1\left(\Omega_\epsilon^j\right)} + \epsilon \langle \partial_t u_\epsilon^j|_{\Gamma_\epsilon}, \phi_\epsilon^j|_{\Gamma_\epsilon} \rangle_{H^1(\Gamma_\epsilon)',H^1(\Gamma_\epsilon)}.$$

**Proposition 1** *There exists a unique weak solution* $u_\epsilon = \left(u_\epsilon^1, u_\epsilon^2\right)$ *of the microscopic problem* (2.1).

**Proof** This is an easy consequence of the Galerkin method and the Leray–Schauder principle, where we have to use similar estimates as in Proposition 2 below. The uniqueness follows from standard energy estimates. For more details see [16]. □

### 3.3 *A priori* estimates

We derive *a priori* estimates for the microscopic solution depending explicitly on $\epsilon$. These estimates are necessary for the application of the two-scale compactness results from Section 4 to derive the macroscopic model. In a first step, we give estimates in the spaces $L^2\left((0,T), \mathbb{H}_{j,\epsilon}\right)$ and $H^1\left((0,T), \mathbb{H}'_{j,\epsilon}\right)$. Such kind of estimates is also needed to establish the existence of a weak solution via the Galerkin method. In a second step, we derive estimates for the difference of shifted microscopic solution with respect to the macroscopic variable. These estimates are necessary for strong two-scale compactness results in the disconnected domain.

The following trace inequality for perforated domains will be used frequently throughout the paper and follows easily by a standard decomposition argument and the trace inequality on the reference element $Y_j$, see also [21, Theorem II.4.1 and Exercise II.4.1]: For every $\theta > 0$, there exists a $C(\theta) > 0$ such that for every $v_\epsilon \in H^1\left(\Omega_\epsilon^j\right)$ it holds that

$$\sqrt{\epsilon}\|v_\epsilon\|_{L^2(\Gamma_\epsilon)} \leqslant C(\theta)\|v_\epsilon\|_{L^2\left(\Omega_\epsilon^j\right)} + \theta\epsilon\|\nabla v_\epsilon\|_{L^2\left(\Omega_\epsilon^j\right)}. \tag{3.3}$$

**Proposition 2** *The weak solution* $u_\epsilon = \left(u_\epsilon^1, u_\epsilon^2\right)$ *of the microscopic problem* (2.1) *fulfils the following a priori estimate*

$$\left\|\partial_t u_\epsilon^j\right\|_{L^2\left((0,T),\mathbb{H}'_{j,\epsilon}\right)} + \left\|u_\epsilon^j\right\|_{L^2\left((0,T),\mathbb{H}_{j,\epsilon}\right)} \leqslant C,$$

*for a constant* $C > 0$ *independent of* $\epsilon$.

https://doi.org/10.1017/S0956792522000195 Published online by Cambridge University Press

**Proof** We choose $u_\epsilon^j$ as a test function in (3) (for $j=1,2$) to obtain with the Assumptions (A3) and (A4) on $f^j$ and $h^j$

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_\epsilon^j\|^2_{\mathbb{L}_{j,\epsilon}} + \left(D_\epsilon^j\nabla u_\epsilon^j, \nabla u_\epsilon^j\right)_{\Omega_\epsilon^j} + \epsilon\left(D_{\Gamma_\epsilon}^j\nabla_{\Gamma_\epsilon}u_\epsilon^j, \nabla_{\Gamma_\epsilon}u_\epsilon^j\right)_{\Gamma_\epsilon} \\
= \left(f_\epsilon^j(u_\epsilon^j), u_\epsilon^j\right)_{\Omega_\epsilon^j} + \epsilon\left(h_\epsilon^j\left(u_\epsilon^1, u_\epsilon^2\right), u_\epsilon^j\right)_{\Gamma_\epsilon} \\
\leqslant C\left(1 + \|u_\epsilon^j\|^2_{L^2\left(\Omega_\epsilon^j\right)} + \epsilon\|u_\epsilon^1\|^2_{L^2(\Gamma_\epsilon)} + \epsilon\|u_\epsilon^2\|^2_{L^2(\Gamma_\epsilon)}\right) \\
\leqslant C\left(1 + \|u_\epsilon^1\|^2_{\mathbb{L}_{1,\epsilon}} + \|u_\epsilon^2\|^2_{\mathbb{L}_{2,\epsilon}}\right).
\end{aligned}
$$

Using the coercivity of $D_\epsilon^j$ and $D_{\Gamma_\epsilon}^j$ from the Assumptions (A1) and (A2), we obtain for $j=1,2$

$$
\frac{d}{dt}\|u_\epsilon^j\|^2_{\mathbb{L}_{j,\epsilon}} + \|\nabla u_\epsilon^j\|^2_{L^2\left(\Omega_\epsilon^j\right)} + \epsilon\|\nabla_{\Gamma_\epsilon}u_\epsilon^j\|^2_{L^2(\Gamma_\epsilon)} \leqslant C\left(1 + \|u_\epsilon^1\|^2_{\mathbb{L}_{1,\epsilon}} + \|u_\epsilon^2\|^2_{\mathbb{L}_{2,\epsilon}}\right).
$$

Summing over $j=1,2$, integrating with respect to time, Assumption (A5) and the Gronwall inequality implies the boundedness of $\|u_\epsilon^j\|_{L^2\left((0,T),\mathbb{H}_{j,\epsilon}\right)}$ uniformly with respect to $\epsilon$.

It remains to check the bound for the time derivative $\partial_t u_\epsilon^j$. As a test function in (3.2) we choose $\phi_\epsilon^j \in \mathbb{H}_{j,\epsilon}$ with $\|\phi_\epsilon^j\|_{\mathbb{H}_{j,\epsilon}} \leqslant 1$ to obtain (using the boundedness of the diffusion tensors and again the growth condition for $h^j$ and $f^j$):

$$
\left\langle \partial_t u_\epsilon^j, \phi_\epsilon^j\right\rangle_{\mathbb{H}'_{j,\epsilon},\mathbb{H}_{j,\epsilon}} \leqslant C\left(\|u_\epsilon^1\|_{\mathbb{H}_{1,\epsilon}} + \|u_\epsilon^2\|_{\mathbb{H}_{2,\epsilon}}\right)\|\phi_\epsilon^j\|_{\mathbb{H}_{j,\epsilon}} \leqslant C\left(\|u_\epsilon^1\|_{\mathbb{H}_{1,\epsilon}} + \|u_\epsilon^2\|_{\mathbb{H}_{2,\epsilon}}\right).
$$

Squaring, integrating with respect to time and the boundedness of $u_\epsilon^j$ for $j=1,2$ already obtained above imply the desired result. □

Next, we derive estimates for the difference of the shifted functions. First of all, we introduce some additional notations. For $h>0$ let us define

$$
\begin{aligned}
\Omega_h &:= \{x\in\Omega \,:\, \mathrm{dist}(x,\partial\Omega) > h\}, \\
K_{\epsilon,h} &:= \{k\in\mathbb{Z}^n \,:\, \epsilon(Y+k)\subset\Omega_h\}, \\
\Omega_{\epsilon,h} &:= \mathrm{int}\bigcup_{k\in K_{\epsilon,h}} \epsilon\left(\overline{Y}+k\right),
\end{aligned}
$$

and the related perforated domains and the related surface

$$
\Omega^2_{\epsilon,h} := \bigcup_{k\in K_{\epsilon,h}} \epsilon\left(Y_2+k\right), \quad \Omega^1_{\epsilon,h} := \Omega_{\epsilon,h}\setminus\overline{\Omega^2_{\epsilon,h}}, \quad \Gamma_{\epsilon,h} := \partial\Omega^2_{\epsilon,h}.
$$

For $l\in\mathbb{Z}^n$ with $|l\epsilon|<h$ and $G_{\epsilon,h}\in\left\{\Omega_{\epsilon,h}, \Omega^1_{\epsilon,h}, \Omega^2_{\epsilon,h}\right\}$, we define for an arbitrary function $v_\epsilon : G_{\epsilon,h}\to\mathbb{R}$ the shifted function

$$
v_\epsilon^l(x) := v_\epsilon(x+l\epsilon),
$$

and the difference between the shifted function and the function itself

$$
\delta_l v_\epsilon(x) := \delta v_\epsilon(x) := v_\epsilon^l(x) - v_\epsilon(x) = v_\epsilon(x+l\epsilon) - v_\epsilon(x). \tag{3.4}
$$

https://doi.org/10.1017/S0956792522000195 Published online by Cambridge University Press

Here, in the writing $\delta v_\epsilon$ we neglect the dependence on $l$ if it is clear from the context. Further, we define $\mathbb{H}_{j,\epsilon,h}$ in the same way as $\mathbb{H}_{j,\epsilon}$ in (3.1) by replacing $\Omega_\epsilon^j$ and $\Gamma_\epsilon$ with $\Omega_{\epsilon,h}^j$ and $\Gamma_{\epsilon,h}$. In the same way, we define $\mathbb{L}_{j,\epsilon,h}$. Further, for any function $\phi_{\epsilon,h} \in \mathbb{H}_{2,\epsilon,h}$ we write $\overline{\phi}_{\epsilon,h}$ for the zero extension to $\Omega_\epsilon^2$. Especially it holds that $\overline{\phi}_{\epsilon,h} \in \mathbb{H}_{2,\epsilon}$, since $\Omega_\epsilon^2$ is disconnected.

**Proposition 3** *Let* $0 < h \ll 1$*, then for all* $l \in \mathbb{Z}^n$ *with* $|\epsilon l| < h$*, it holds that*

$$\|\delta u_\epsilon^2\|_{L^\infty((0,T),\mathbb{L}_{2,\epsilon,h})} + \|\nabla \delta u_\epsilon^2\|_{L^2((0,T),\mathbb{L}_{2,\epsilon,h})} \leqslant C\left(\|\delta u_\epsilon^1\|_{L^2\left((0,T),L^2\left(\Omega_{\epsilon,h}^1\right)\right)} + \left\|\delta\left(u_{\epsilon,i}^2, u_{\epsilon,i,\Gamma_\epsilon}^2\right)\right\|_{\mathbb{L}_{2,\epsilon,h}} + \epsilon\right),$$

*for a constant* $C > 0$ *independent of* $h$, $\epsilon$, *and* $l$.

**Proof** Let $0 < h \ll 1$ and $l \in \mathbb{Z}^n$ with $|\epsilon l| < h$, and we shortly write $u_\epsilon^{2,l} := \left(u_\epsilon^2|_{\Omega_{\epsilon,h}^2}\right)^l$, that is, the shifts with respect to $l\epsilon$ of the restriction $u_\epsilon^2|_{\Omega_{\epsilon,h}^2}$ (we neglect the index $h$). In the same way, we define $u_\epsilon^{1,l}$. Let $\phi_{\epsilon,h} \in \mathbb{H}_{2,\epsilon,h}$. Then, for $x \in \Omega_\epsilon^2 \setminus \left(\Omega_{\epsilon,h}^2 + \epsilon l\right)$ it holds that $x - l\epsilon \notin \Omega_{\epsilon,h}^2$ and therefore $\overline{\phi}_{\epsilon,h}^{-l}(x) = 0$ and similar from $x \in \Gamma_\epsilon \setminus \left(\Gamma_{\epsilon,h} + \epsilon l\right)$ it follows $\overline{\phi}_{\epsilon,h}^{-l}(x) = 0$. This implies for all $\psi \in C_0^\infty(0,T)$

$$\begin{aligned}
&\int_0^T \left(u_\epsilon^{2,l}, \phi_{\epsilon,h}\right)_{\mathbb{L}_{2,\epsilon,h}} \psi'(t)dt \\
&= \int_0^T \left[\int_{\Omega_{\epsilon,h}^2} u_\epsilon^2(t, x + l\epsilon)\phi_{\epsilon,h}(x)dx + \epsilon \int_{\Gamma_{\epsilon,h}} u_\epsilon^2(t, x + l\epsilon)\phi_{\epsilon,h}(x)d\sigma\right]\psi'(t)dt \\
&= \int_0^T \left[\int_{\Omega_{\epsilon,h}^2 + l\epsilon} u_\epsilon^2(t,x)\phi_{\epsilon,h}^{-l}(x)dx + \epsilon \int_{\Gamma_{\epsilon,h}+l\epsilon} u_\epsilon^2(t,x)\phi_{\epsilon,h}^{-l}(x)d\sigma\right]\psi'(t)dt \\
&= \int_0^T \left[\int_{\Omega_\epsilon^2} u_\epsilon^2(t,x)\phi_{\epsilon,h}^{-l}(x)dx + \epsilon \int_{\Gamma_\epsilon} u_\epsilon^2(t,x)\phi_{\epsilon,h}^{-l}(x)d\sigma\right]\psi'(t)dt \\
&= \int_0^T \left(u_\epsilon^2, \overline{\phi}_{\epsilon,h}^{-l}\right)_{\mathbb{L}_{2,\epsilon}} \psi'(t)dt = -\int_0^T \left\langle \partial_t u_\epsilon^2, \overline{\phi}_{\epsilon,h}^{-l}\right\rangle_{\mathbb{H}_{2,\epsilon}',\mathbb{H}_{2,\epsilon}} \psi(t)dt.
\end{aligned}$$

Hence, we have $\partial_t u_\epsilon^{2,l} \in L^2\left((0,T), \mathbb{H}_{2,\epsilon,h}'\right)$ with

$$\left\langle \partial_t u_\epsilon^{2,l}, \phi_{\epsilon,h}\right\rangle_{\mathbb{H}_{2,\epsilon,h}',\mathbb{H}_{2,\epsilon,h}} = \left\langle \partial_t u_\epsilon^2, \overline{\phi}_{\epsilon,h}^{-l}\right\rangle_{\mathbb{H}_{2,\epsilon}',\mathbb{H}_{2,\epsilon}}$$

almost everywhere in $(0,T)$. Using $\overline{\phi}_{\epsilon,h}^{-l} \in \mathbb{H}_{2,\epsilon}$ as a test function in (3.2), we obtain using the periodicity of $D^2$, $D_\Gamma^2$, $f^2$ and $h^2$, by an elemental calculation

$$\begin{aligned}
\left\langle \partial_t u_\epsilon^2, \overline{\phi}_{\epsilon,h}^{-l}\right\rangle_{\mathbb{H}_{2,\epsilon}',\mathbb{H}_{2,\epsilon}} &= -\left(D_\epsilon^2 \nabla u_\epsilon^{2,l}, \nabla \phi_{\epsilon,h}\right)_{\Omega_{\epsilon,h}^2} - \epsilon\left(D_{\Gamma_\epsilon}^2 \nabla_{\Gamma_\epsilon} u_\epsilon^{2,l}, \nabla_{\Gamma_\epsilon}\phi_{\epsilon,h}\right)_{\Gamma_{\epsilon,h}} \\
&\quad + \left(f_\epsilon^2\left(u_\epsilon^{2,l}\right), \phi_{\epsilon,h}\right)_{\Omega_{\epsilon,h}^2} + \epsilon\left(h_\epsilon^2\left(u_\epsilon^{1,l}, u_\epsilon^{2,l}\right), \phi_{\epsilon,h}\right)_{\Gamma_{\epsilon,h}}.
\end{aligned}$$

https://doi.org/10.1017/S0956792522000195 Published online by Cambridge University Press

Subtracting the above equation for $l=0$ and arbitrary $l\in\mathbb{Z}^n$ with $|\epsilon l|<h$ we obtain

$$\left\langle \partial_t \delta u_\epsilon^2, \phi_{\epsilon,h}\right\rangle_{\mathbb{H}'_{2,\epsilon,h},\mathbb{H}_{2,\epsilon,h}} + \left(D_\epsilon^2 \nabla \delta u_\epsilon^2, \nabla \phi_{\epsilon,h}\right)_{\Omega^2_{\epsilon,h}} + \epsilon \left(D^2_{\Gamma_\epsilon}\nabla_{\Gamma_\epsilon}\delta u_\epsilon^2, \nabla_{\Gamma_\epsilon}\phi_{\epsilon,h}\right)_{\Gamma_{\epsilon,h}}$$
$$= \left(f_\epsilon^2\left(\left(u_\epsilon^2\right)^l\right) - f_\epsilon^2\left(u_\epsilon^2\right), \phi_{\epsilon,h}\right)_{\Omega^2_{\epsilon,h}} + \epsilon\left(h_\epsilon^2\left(\left(u_\epsilon^1\right)^l, \left(u_\epsilon^2\right)^l\right) - h_\epsilon^2\left(u_\epsilon^1, u_\epsilon^2\right), \phi_{\epsilon,h}\right)_{\Gamma_{\epsilon,h}}.$$

Choosing $\phi_{\epsilon,h} := \delta u_\epsilon^2$ (more precisely we take the restriction of $u_\epsilon^2$ to $\Omega^2_{\epsilon,h}$) we obtain with the coercivity of $D^2$ and $D^2_\Gamma$, as well as the Lipschitz continuity of $f^2$ and $h^2$

$$\frac{1}{2}\frac{d}{dt}\left\|\delta u_\epsilon^2\right\|^2_{\mathbb{L}_{2,\epsilon,h}} + c_0\left\|\nabla\delta u_\epsilon^2\right\|^2_{\mathbb{L}_{2,\epsilon,h}} \leqslant C\left(\left\|\delta u_\epsilon^2\right\|^2_{L^2\left(\Omega^2_{\epsilon,h}\right)} + \epsilon\sum_{j=1}^2 \left\|\delta u_\epsilon^j\right\|^2_{L^2\left(\Gamma_{\epsilon,h}\right)}\right)$$
$$\leqslant \sum_{j=1}^2\left(C(\theta)\left\|\delta u_\epsilon^j\right\|^2_{L^2\left(\Omega^j_{\epsilon,h}\right)} + \epsilon\theta\left\|\nabla u_\epsilon^j\right\|^2_{L^2\left(\Omega^j_{\epsilon,h}\right)}\right),$$

for arbitrary $\theta>0$, where we used the trace inequality (3.3). Choosing $\theta$ small enough the gradient term for $j=2$ can be absorbed from the left-hand side. Integrating with respect to time, using the *a priori* estimates from Proposition 2 for the gradients of $u_\epsilon^1$, as well as the Gronwall inequality, we obtain the desired result. □

## 4 Two-scale compactness results

In this section, we prove general strong two-scale compactness results for functions in the space $L^2\left((0,T),\mathbb{H}_{j,\epsilon}\right)\cap H^1\left((0,T),\mathbb{H}'_{j,\epsilon}\right)$ for $j\in\{1,2\}$ based on suitable *a priori* estimates. These estimates are fulfilled by the microscopic solution $u_\epsilon = \left(u_\epsilon^1, u_\epsilon^2\right)$ which fulfils Propositions 2 and 3, but are not restricted to them. The connected and disconnected cases are completely different and are therefore treated separately. These strong compactness results are enough to pass to the limit in the nonlinear terms in the microscopic equation (3.2), in fact we have

**Lemma 1** *Let* $p\in(1,\infty)$.

*(i) For* $j\in\{1,2\}$, *let* $\left(u_\epsilon^j\right)\subset L^p\left((0,T)\times\Omega_\epsilon^j\right)$ *be a sequence converging strongly in the two-scale sense to* $u_0^j\in L^p((0,T)\times\Omega\times Y_j)$. *Further* $f:[0,T]\times Y_j\times\mathbb{R}\to\mathbb{R}$ *is continuous, Y-periodic with respect to the second variable, and fulfils the growth condition*

$$|f(t,y,z)|\leqslant C(1+|z|)\quad \text{for all } (t,y,z)\in[0,T]\times Y_j\times\mathbb{R}.$$

*Then it holds up to a subsequence*

$$f\left(\cdot_t, \frac{\cdot_x}{\epsilon}, u_\epsilon^j\right)\to f\left(\cdot_t,\cdot_y,u_0^j\right)\qquad \text{in the two-scale sense in } L^p.$$

*(ii) Let* $(u_\epsilon)\subset L^p\left((0,T)\times\Gamma_\epsilon\right)$ *be a sequence converging strongly in the two-scale sense on* $\Gamma_\epsilon$ *to* $u_0\in L^p((0,T)\times\Omega\times\Gamma)$. *Further,* $h:[0,T]\times\Gamma\times\mathbb{R}\to\mathbb{R}$ *is continuous, Y-periodic with respect to the second variable, and fulfils the growth condition*

$$|h(t,y,z)|\leqslant C(1+|z|)\quad \text{for all } (t,y,z)\in[0,T]\times\Gamma\times\mathbb{R}.$$

https://doi.org/10.1017/S0956792522000195 Published online by Cambridge University Press

*Then it holds up to a subsequence*

$$h\left(\cdot_t, \frac{\cdot_x}{\epsilon}, u_\epsilon\right) \to h(\cdot_t, \cdot_y, u_0) \qquad \text{in the two-scale sense on } \Gamma_\epsilon \text{ in } L^p.$$

*We emphasise that for functions f and h uniformly Lipschitz continuous with respect to the last variable, the growth conditions are fulfilled. For such Lipschitz-continuous functions we also easily obtain the strong two-scale convergence of the whole sequence.*

**Proof** We only prove (ii). The other statement follows the same way. Due to Lemma 6 in the Appendix A, the sequence $\mathcal{T}_\epsilon u_\epsilon$ converges in $L^p((0, T) \times \Omega \times \Gamma)$ to $u_0$. Hence, up to a subsequence, $\mathcal{T}_\epsilon u_\epsilon \to u_0$ almost everywhere in $(0, T) \times \Omega \times \Gamma$. Further, we have

$$\mathcal{T}_\epsilon\left(h\left(\cdot_t, \frac{\cdot_x}{\epsilon}, u_\epsilon\right)\right) = h\left(\cdot_t, y, \mathcal{T}_\epsilon u_\epsilon\right) \to h(\cdot_t, \cdot_y, u_0) \quad \text{a.e. in } (0, T) \times \Omega \times \Gamma.$$

The growth condition on $h$ implies $\mathcal{T}_\epsilon\left(h\left(\cdot_t, \frac{\cdot_x}{\epsilon}, u_\epsilon\right)\right)$ bounded in $L^p((0, T) \times \Omega \times \Gamma)$. Egorov's theorem (see also [24, Theorem 13.44]) implies

$$\mathcal{T}_\epsilon\left(h\left(\cdot_t, \frac{\cdot_x}{\epsilon}, u_\epsilon\right)\right) \rightharpoonup h(\cdot_t, \cdot_y, u_0) \quad \text{weakly in } L^p((0, T) \times \Omega \times \Gamma).$$

Using again Lemma 6, we obtain the desired result. □

### 4.1 The connected domain $\Omega_\epsilon^1$

Here we give a strong compactness result for a sequence in the connected domain $\Omega_\epsilon^1$ under suitable *a priori* estimates. The case of a connected perforated domain can be treated more easily than a disconnected domain, because we can extend a bounded sequence in $H^1(\Omega_\epsilon)$ to a bounded sequence in $H^1(\Omega)$, due to [1, 13]. Hence, we can work in fixed Bochner spaces (not depending on $\epsilon$) and use standard methods from functional analysis. For this we need control for the time variable, which can be obtained from the uniform bound of the time derivative $\partial_t u_\epsilon^1$. However, since $\partial_t u_\epsilon^1$ is pointwise only an element in the space $\mathbb{H}'_{1,\epsilon}$ it is not clear if the time derivative of the extension of $u_\epsilon^1$ exists and if it is bounded uniformly with respect to $\epsilon$. The following lemma gives us an estimate for the difference of the shifts with respect to time for functions with generalised time derivative. It is just an easy generalisation of [19, Lemma 9].

**Lemma 2** *Let V and H be Hilbert spaces and we assume that $(V, H, V')$ is a Gelfand triple. Let $v \in L^2((0, T), V) \cap H^1((0, T), V')$. Then, for every $\phi \in V$ and almost every $t \in (0, T)$, $s \in (-T, T)$, such that $t + s \in (0, T)$, we have*

$$\left|(v(t+s) - v(t), \phi)_H\right| \leqslant \sqrt{|s|}\, \|\phi\|_V \|\partial_t v\|_{L^2((t,t+s),V')}.$$

*Especially, it holds that*

$$\left\|v(t+s) - v(t)\right\|_H^2 \leqslant \sqrt{|s|}\left\|v(t+s) - v(t)\right\|_V \|\partial_t v\|_{L^2((t,t+s),V')}.$$

**Proof** The proof follows the same lines as the proof of [19, Lemma 9], if we replace the Gelfand triple $\left(H^1\left(\Omega_j^\epsilon\right), L^2\left(\Omega_j^\epsilon\right), H^1\left(\Omega_j^\epsilon\right)'\right)$ by the Gelfand triple $(V, H, V')$. □

https://doi.org/10.1017/S0956792522000195 Published online by Cambridge University Press

In the following, for $v_\epsilon \in H^1(\Omega_\epsilon^1)$ we denote by $\tilde{v}_\epsilon \in H^1(\Omega)$ the extension from [1, 13] with

$$\|\tilde{v}_\epsilon\|_{L^2(\Omega)} \leqslant C\|v_\epsilon\|_{L^2(\Omega_\epsilon^1)}, \quad \|\nabla\tilde{v}_\epsilon\|_{L^2(\Omega)} \leqslant C\|\nabla v_\epsilon\|_{L^2(\Omega_\epsilon^1)},$$

with a constant $C > 0$ independent of $\epsilon$.

**Proposition 4** *Let* $(v_\epsilon) \subset L^2((0,T), \mathbb{H}_{1,\epsilon}) \cap H^1((0,T), \mathbb{H}'_{1,\epsilon})$ *be a sequence with*

$$\|\partial_t v_\epsilon\|_{L^2((0,T),\mathbb{H}'_{1,\epsilon})} + \|v_\epsilon\|_{L^2((0,T),\mathbb{H}_{1,\epsilon})} \leqslant C. \tag{4.1}$$

*There exists* $v_0 \in L^2((0,T), H^1(\Omega))$ *such that for all* $\beta \in \left(\frac{1}{2}, 1\right)$ *up to a subsequence it holds that*

$$\tilde{v}_\epsilon \to v_0 \quad in\ L^2((0,T), H^\beta(\Omega)).$$

*Further, it holds that (up to a subsequence)*

$$\mathcal{T}_\epsilon v_\epsilon \to v_0 \quad in\ L^2((0,T)\times\Omega, \mathbb{H}_1).$$

**Proof** Since $\tilde{v}_\epsilon$ is bounded in $L^2((0,T), H^1(\Omega))$ there exists $v_0 \in L^2((0,T), H^1(\Omega))$, such that up to a subsequence $v_\epsilon$ converges weakly to $v_0$ in $L^2((0,T), H^1(\Omega))$. Lemma 2 and inequality (4.1) imply for $0 < h \ll 0$

$$\begin{aligned}\int_0^{T-h} \|v_\epsilon(t+h) - v_\epsilon\|_{\mathbb{L}_{1,\epsilon}}^2 dt &\leqslant C\sqrt{h}\|\partial_t v_\epsilon\|_{L^2((0,T),\mathbb{H}'_{1,\epsilon})} \int_0^{T-h} \|v_\epsilon(t+h) - v_\epsilon\|_{\mathbb{H}_{1,\epsilon}} dt \\ &\leqslant C\sqrt{h}.\end{aligned} \tag{4.2}$$

Now, from the properties of the extension $\tilde{v}_\epsilon$ we obtain

$$\int_0^{T-h} \|\tilde{v}_\epsilon(t+h) - \tilde{v}_\epsilon\|_{L^2(\Omega)}^2 dt \leqslant C\int_0^{T-h} \|v_\epsilon(t+h) - v_\epsilon\|_{L^2(\Omega_\epsilon^1)}^2 dt \leqslant C\sqrt{h}.$$

Since $H^1(\Omega) \hookrightarrow H^\beta(\Omega)$ is compact for $\beta \in \left(\frac{1}{2}, 1\right)$ we can apply [30, Theorem 1] to $(\tilde{v}_\epsilon)$ as a sequence in $L^2((0,T), H^\beta(\Omega))$ and obtain the strong convergence of $\tilde{v}_\epsilon$ to $v_0$ in $L^2((0,T), H^\beta(\Omega))$.

Now we prove the convergence of $\mathcal{T}_\epsilon v_\epsilon$. It holds that

$$\|\mathcal{T}_\epsilon v_\epsilon - v_0\|_{L^2((0,T)\times\Omega,\mathbb{H}_1)}^2 = \|\mathcal{T}_\epsilon v_\epsilon - v_0\|_{L^2((0,T)\times\Omega,H^1(Y_1))}^2 + \|\mathcal{T}_\epsilon v_\epsilon - v_0\|_{L^2((0,T)\times\Omega,H^1(\Gamma))}^2.$$

We only prove the convergence to zero for the second term, since the first one can be treated in a similar way. We obtain from the properties of the unfolding operator from Lemma 5, the trace inequality and the inequality (4.1)

$$\begin{aligned}\|\mathcal{T}_\epsilon v_\epsilon - v_0\|_{L^2((0,T)\times\Omega,H^1(\Gamma))} &\leqslant C\|\mathcal{T}_\epsilon v_\epsilon - v_0\|_{L^2((0,T)\times\Omega\times\Gamma)} + C\|\nabla_{\Gamma,y}\mathcal{T}_\epsilon v_\epsilon\|_{L^2((0,T)\times\Omega\times\Gamma)} \\ &\leqslant C\left(\|\mathcal{T}_\epsilon v_\epsilon - v_0\|_{L^2((0,T)\times\Omega\times Y_1)} + \epsilon\|\nabla v_\epsilon\|_{L^2((0,T)\times\Omega_\epsilon^1)} + \epsilon^{\frac{3}{2}}\|\nabla_{\Gamma_\epsilon} v_\epsilon\|_{L^2((0,T)\times\Gamma_\epsilon)}\right) \\ &\leqslant C\left(\|v_\epsilon - v_0\|_{L^2((0,T)\times\Omega_\epsilon^1)} + \|\mathcal{T}_\epsilon v_0 - v_0\|_{L^2((0,T)\times\Omega\times Y_1)} + \epsilon\right).\end{aligned}$$

https://doi.org/10.1017/S0956792522000195 Published online by Cambridge University Press

The first term converges to zero for $\epsilon \to 0$, due to the strong convergence of $\tilde{v}_\epsilon$ to $v_0$, and the second term because of [12, Proposition 4.4]. This gives the desired result. □

### 4.2 The disconnected domain $\Omega_\epsilon^2$

In this section, we give a strong two-scale compactness result for the disconnected domain $\Omega_\epsilon^2$ of Kolmogorov–Simon-type, that is, it is based on *a priori* estimates for the difference of discrete shifts, see condition (ii) in Theorem 4.1. As already mentioned above, it is in general not possible to find an extension for a function in $H^1\left(\Omega_\epsilon^2\right)$ to the whole domain $\Omega$ which preserves the *a priori* estimates. Hence, the method from Section 4.1 for the connected domain fails. Therefore, we consider the unfolded sequence in the Bochner space $L^p\left(\Omega, L^2\left((0,T), \mathbb{H}_2^\beta\right)\right)$ with $\beta \in \left(\frac{1}{2}, 1\right)$ and $p \in (1,2)$ and apply the Kolmogorov–Simon-compactness result from [18], which gives an extension of [30, Theorem 1] to higher-dimensional domains of definition. Here, a crucial point is the estimate for the shifts. An important reason to work here with general Bochner spaces, that is, Banach-valued functions spaces, is that we are dealing with manifolds and therefore linear shifts with respect to the space variable are not well defined.

In the following lemma, we estimate the shifts of the unfolded sequence with respect to the macroscopic variable by the shifts of the function itself, see again Section 3.3 for the notations.

**Lemma 3** *Let* $v_\epsilon \in L^2\left((0,T)\times\Omega_\epsilon^j\right)$ *for* $j \in \{1,2\}$ *and* $w_\epsilon \in L^2\left((0,T)\times\Gamma_\epsilon\right)$*. Then, for* $0 < h \ll 1$*,* $|z| < h$*, and* $\epsilon$ *small enough it holds that*

$$\left\|\mathcal{T}_\epsilon v_\epsilon(t,x+z,y) - \mathcal{T}_\epsilon v_\epsilon\right\|^2_{L^2(0,T)\times\Omega_{2h}\times Y_j)} \leqslant \sum_{k\in\{0,1\}^n} \|\delta v_\epsilon\|^2_{L^2\left((0,T)\times\Omega^j_{\epsilon,h}\right)},$$

$$\left\|\mathcal{T}_\epsilon w_\epsilon(t,x+z,y) - \mathcal{T}_\epsilon w_\epsilon\right\|^2_{L^2(0,T)\times\Omega_{2h}\times\Gamma)} \leqslant \epsilon \sum_{k\in\{0,1\}^n} \|\delta w_\epsilon\|^2_{L^2\left((0,T)\times\Gamma_{\epsilon,h}\right)},$$

*with* $l = l(\epsilon,z,k) = k + \left[\frac{z}{\epsilon}\right]$.

**Proof** The proof for a thin layer can be found in [27, p. 709] and can be easily extended to our setting. See also [19] for more details. □

**Theorem 4.1** *Let* $v_\epsilon \in L^2\left((0,T),\mathbb{H}_{2,\epsilon}\right) \cap H^1\left((0,T),\mathbb{H}'_{2,\epsilon}\right)$ *with:*

*(i) It holds the estimate*

$$\|v_\epsilon\|_{L^2((0,T),\mathbb{H}_{2,\epsilon})} + \|\partial_t v_\epsilon\|_{L^2\left((0,T),\mathbb{H}'_{2,\epsilon}\right)} \leqslant C.$$

*(ii) For* $0 < h \ll 1$ *and* $l \in \mathbb{Z}^n$ *with* $|l\epsilon| < h$ *it holds that*

$$\|\delta v_\epsilon\|_{L^2\left((0,T),L^2\left(\Omega^2_{\epsilon,h}\right)\right)} \xrightarrow{\epsilon l \to 0} 0.$$

*Then, there exists* $v_0 \in L^2\left((0,T),L^2(\Omega)\right)$*, such that for* $\beta \in \left(\frac{1}{2},1\right)$ *and* $p \in [1,2)$ *it holds up to a subsequence that*

$$\mathcal{T}_\epsilon v_\epsilon \to v_0 \quad \text{in } L^p\left(\Omega, L^2\left((0,T),\mathbb{H}_2^\beta\right)\right).$$

*Especially,* $v_\epsilon$ *and* $v_\epsilon|_{\Gamma_\epsilon}$ *converge strongly in the two-scale sense to* $v_0$ *(with respect to* $L^p$*).*

https://doi.org/10.1017/S0956792522000195 Published online by Cambridge University Press

**Proof** Our aim is to apply [18, Corollary 2.5] to $(\mathcal{T}_\epsilon v_\epsilon)$ as a sequence in $L^p\Big(\Omega, L^2\big((0,T), \mathbb{H}_2^\beta\big)\Big)$ for $p \in [1,2)$ and $\beta \in \left(\frac{1}{2}, 1\right)$. Hence, we have to check the following three conditions:

(K1) For every measurable set $A \subset \Omega$, the sequence $\left\{\int_A v_\epsilon dx\right\}$ is relatively compact in $L^2\big((0,T), \mathbb{H}_2^\beta\big)$,

(K2) for all $0 < h \ll 1$ and $|z| < h$ it holds that

$$\sup_\epsilon \|v_\epsilon(\cdot + z) - v_\epsilon\|_{L^p\left(\Omega_h, L^2\left((0,T), \mathbb{H}_2^\beta\right)\right)} \to 0 \quad \text{for } z \to 0,$$

(K3) for $h > 0$ it holds that $\sup_\epsilon \int_{\Omega \setminus \Omega_h} |v_\epsilon(x)|^p dx \to 0$ for $h \to 0$.

We start with the condition (K1). Let $A \subset \Omega$ be measurable and we define $V_\epsilon(t,y) := \int_A \mathcal{T}_\epsilon v_\epsilon(t,x,y) dx$. To show the relative compactness of $(V_\epsilon)$, we use again [30, Theorem 1] as in the proof of Proposition 4. First of all, due to our assumptions on $v_\epsilon$ and the properties of the unfolding operator, for $t_1, t_2 \in (0,T)$ it holds that

$$\left\| \int_{t_1}^{t_2} V_\epsilon dt \right\|_{\mathbb{H}_2} \leqslant \|\mathcal{T}_\epsilon v_\epsilon\|_{L^2((0,T)\times\Omega, \mathbb{H}_2)} \leqslant C.$$

Due to the compact embedding $\mathbb{H}_2 \hookrightarrow \mathbb{H}_2^\beta$ we obtain that $\int_{t_1}^{t_2} V_\epsilon dt$ is relatively compact in $\mathbb{H}_2^\beta$. Further, for $0 < s \ll 1$ we obtain with the estimates for $v_\epsilon$ and the trace inequality (3.3)

$$\begin{aligned}
&\|V_\epsilon(t+s,y) - V_\epsilon\|_{L^2((0,T-s),\mathbb{H}_2)} \leqslant \|\mathcal{T}_\epsilon v_\epsilon(t+s,x,y) - \mathcal{T}_\epsilon v_\epsilon\|_{L^2((0,T-s)\times\Omega,\mathbb{H}_2)} \\
&\leqslant C\|v_\epsilon(t+s,x) - v_\epsilon\|_{L^2\left((0,T-s)\times\Omega_\epsilon^2\right)} + C\epsilon\|\nabla v_\epsilon(t+s,x) - \nabla v_\epsilon\|_{L^2\left((0,T-s)\times\Omega_\epsilon^2\right)} \\
&\quad + C\epsilon^{\frac{3}{2}}\|\nabla_{\Gamma_\epsilon} v_\epsilon(t+s,x) - \nabla_{\Gamma_\epsilon} v_\epsilon\|_{L^2((0,T-s)\times\Gamma_\epsilon)} \\
&\leqslant C\left(s^{\frac{1}{4}} + \epsilon\right),
\end{aligned} \tag{4.3}$$

where for the last inequality we used Lemma 2 to estimate the first term in the line before by using the same arguments as for the inequality (4.2) in the proof of Proposition 4. We show that inequality (4.3) implies the convergence of the difference of the shifts to zero for $s \to 0$ uniformly with respect to $\epsilon$, see also [27, pp. 710–711] or [15, pp. 1476–1477] for the same argument. First of all, from (4.3) we obtain for every $0 < \rho$ the existence of $0 < \epsilon_0$, $\delta_0$, such that for all $\epsilon \leqslant \epsilon_0$ and $s \leqslant \delta_0$ it holds that

$$\|V_\epsilon(t+s,y) - V_\epsilon\|_{L^2((0,T-s),\mathbb{H}_2)} \leqslant \rho. \tag{4.4}$$

Since $\epsilon^{-1} \in \mathbb{N}$, there are only finitely many elements $\epsilon$, denoted by $\epsilon_i$ with $i = 1, \ldots, N$, such that $\epsilon_0 < \epsilon_i$. For every $\epsilon_i$ there exists a $0 < \delta_i$, such that (4.4) is valid for $\epsilon = \epsilon_i$ and all $s \leqslant \delta_i$. In fact, this follows directly from $V_{\epsilon_i} \in L^2((0,T), \mathbb{H}_2)$, see for example [5, Theorem 4.15] or use the Kolmogorov–Simon-compactness theorem from [30, Theorem 1] applied to the function $V_{\epsilon_i}$. Choosing $\delta := \max_{i=0,\ldots,N}\{\delta_i\}$, we obtain for all $s \leqslant \delta$

$$\sup_\epsilon \|V_\epsilon(t+s,y) - V_\epsilon\|_{L^2((0,T-s),\mathbb{H}_2)} \leqslant \rho.$$

Hence, [30, Theorem 1] implies that $(V_\epsilon)$ is relatively compact in $L^2\big((0,T), \mathbb{H}_2^\beta\big)$, that is, condition (K1).

https://doi.org/10.1017/S0956792522000195 Published online by Cambridge University Press

For (K2) we fix $0 < h \ll 1$ and choose $|z| < h$. Lemma 3 with $l = k + \left[\frac{z}{\epsilon}\right]$ (see the definition of the difference $\delta$ in (3.4)), the conditions (i) and (ii), as well as the trace inequality (3.3) imply

$$
\begin{aligned}
&\left\|\mathcal{T}_\epsilon v_\epsilon(t,x+z,y) - \mathcal{T}_\epsilon v_\epsilon\right\|_{L^2(\Omega_{2h},L^2((0,T),\mathbb{H}_2))} \\
&\leqslant C \sum_{k\in\{0,1\}^n} \left( \|\delta v_\epsilon\|_{L^2\left((0,T)\times\Omega^2_{\epsilon,h}\right)} + \epsilon \|\delta \nabla v_\epsilon\|_{L^2\left((0,T)\times\Omega^2_{\epsilon,h}\right)} + \epsilon^{\frac{3}{2}} \|\delta \nabla_{\Gamma_\epsilon} v_\epsilon\|_{L^2((0,T)\times\Gamma_{\epsilon,h})} \right) \\
&\leqslant C \left( \sum_{k\in\{0,1\}^n} \|\delta v_\epsilon\|_{L^2\left((0,T)\times\Omega^2_{\epsilon,h}\right)} + \epsilon \right) \xrightarrow{\epsilon,z\to 0} 0.
\end{aligned}
$$

Again we obtain for every $0 < \rho$ the existence of $0 < \epsilon_0,\ \delta_0$, such that for all $\epsilon \leqslant \epsilon_0$ and $|z| \leqslant \delta_0$ it holds that

$$
\left\|\mathcal{T}_\epsilon v_\epsilon(t,x+z,y) - \mathcal{T}_\epsilon v_\epsilon\right\|_{L^2(\Omega_{2h},L^2((0,T),\mathbb{H}_2))} \leqslant \rho. \tag{4.5}
$$

Arguing as above, this inequality is also valid for all (finitely many) $\epsilon > \epsilon_0$ and $|z|$ small enough. This implies (K2). For the last condition (K3) we use the Hölder inequality to obtain for $p \in [1,2)$ and $0 < h \ll 1$

$$
\left\|\mathcal{T}_\epsilon v_\epsilon\right\|_{L^p(\Omega\setminus\Omega_h,L^2((0,T),\mathbb{H}_2))} \leqslant |\Omega \setminus \Omega_h|^{\frac{2-p}{2p}} \left\|\mathcal{T}_\epsilon v_\epsilon\right\|_{L^2((0,T)\times\Omega\setminus\Omega_h,\mathbb{H}_2)} \leqslant C h^{\frac{2-p}{2p}} \xrightarrow{h\to 0} 0,
$$

where we used again estimate (i). Now, [18, Corollary 2.5] implies the the strong convergence of $\mathcal{T}_\epsilon v_\epsilon$ up to a subsequence in $L^p\left(\Omega, L^2\left((0,T),\mathbb{H}_2^\beta\right)\right)$ to a limit function $v_0$. Lemma 6 (see Appendix A) implies the strong two-scale convergence of $v_\epsilon$ to the same limit. The fact $v_0 \in L^2\left((0,T),L^2(\Omega)\right)$ follows from standard two-scale compactness results, see [2], based on the estimate (i). □

**Remark 1** *Theorem 4.1 and its proof remain valid if we replace* $\mathbb{H}_{2,\epsilon}$ *and* $\mathbb{H}^\beta$ *by* $H^1\left(\Omega_\epsilon^2\right)$ *and* $H^\beta(Y_2)$.

## 5 Derivation of the macroscopic model

The aim of this section is the derivation of the macroscopic model (5.7) from Theorem 5.1 for $\epsilon \to 0$. Therefore, we make use of compactness results from Section 4 and the *a priori* estimates from Section 3. In the following proposition, we collect the convergence results for the microscopic solution $u_\epsilon = \left(u_\epsilon^1, u_\epsilon^2\right)$. Again we use the notation $\widetilde{u}_\epsilon^1$ for the extension of $u_\epsilon^1$ from [1, 13] used in Section 4.1.

**Proposition 5** *Let* $u_\epsilon = \left(u_\epsilon^1, u_\epsilon^2\right)$ *be the microscopic solution of the problem* (2.1)*. There exist*

$$
u_0^1 \in L^2\left((0,T),H^1(\Omega)\right), \quad u_1^1 \in L^2((0,T),\mathbb{H}_1/\mathbb{R}), \quad u_0^2 \in L^2((0,T)\times\Omega),
$$

*such that up to a subsequence it holds for* $p \in [1,2)$

https://doi.org/10.1017/S0956792522000195 Published online by Cambridge University Press

$$\widetilde{u}^1_\epsilon \to u^1_0 \qquad \textit{strongly in the two-scale sense,} \tag{5.1a}$$
$$\nabla \widetilde{u}^1_\epsilon \to \nabla u^1_0 + \nabla_y u^1_1 \qquad \textit{in the two-scale sense,} \tag{5.1b}$$
$$u^1_\epsilon|_{\Gamma_\epsilon} \to u^1_0 \qquad \textit{strongly in the two-scale sense on } \Gamma_\epsilon, \tag{5.1c}$$
$$\nabla_{\Gamma_\epsilon} u^1_\epsilon \to P_\Gamma \nabla u^1_0 + \nabla_{\Gamma,y} u^1_1 \qquad \textit{in the two-scale sense on } \Gamma_\epsilon, \tag{5.1d}$$
$$\chi_{\Omega^2_\epsilon} u^2_\epsilon \to \chi_{Y_2} u^2_0 \qquad \textit{strongly in the two-scale sense in } L^p, \tag{5.1e}$$
$$\chi_{\Omega^2_\epsilon} \nabla u^2_\epsilon \to 0 \qquad \textit{in the two-scale sense,} \tag{5.1f}$$
$$u^2_\epsilon|_{\Gamma_\epsilon} \to u^2_0 \qquad \textit{strongly in the two-scale sense in } L^p \textit{ on } \Gamma_\epsilon, \tag{5.1g}$$
$$\nabla_{\Gamma_\epsilon} u^2_\epsilon \to 0 \qquad \textit{in the two-scale sense on } \Gamma_\epsilon. \tag{5.1h}$$

**Proof** The convergence results (5.1a)–(5.1d) follow immediately from Proposition 4, Lemma 4 (see Appendix A) and the *a priori* estimates in Proposition 2.

For (5.1e)–(5.1h) we first notice that due to Lemma 4 there exists $u^2_0 \in L^2((0,T)\times\Omega)$ and $u^2_1 \in L^2((0,T)\times\Omega, \mathbb{H}_2/\mathbb{R})$, such that up to a subsequence

$$\begin{aligned}
\chi_{\Omega^2_\epsilon} u^2_\epsilon &\to \chi_{Y_2} u^2_0 && \text{in the two-scale sense,}\\
\chi_{\Omega^2_\epsilon} \nabla u^2_\epsilon &\to \chi_{Y_2} \nabla_y u^2_1 && \text{in the two-scale sense,}\\
u^2_\epsilon|_{\Gamma_\epsilon} &\to u^2_0 && \text{in the two-scale sense on } \Gamma_\epsilon,\\
\nabla_{\Gamma_\epsilon} u^2_\epsilon|_{\Gamma_\epsilon} &\to \nabla_\Gamma u^2_1 && \text{in the two-scale sense on } \Gamma_\epsilon.
\end{aligned}$$

For the strong two-scale convergence of $u^2_\epsilon$ and $u^2_\epsilon|_{\Gamma_\epsilon}$ we make use of Theorem 4.1, where we have to check the conditions (i) and (ii). The first one is just the *a priori* estimate from Proposition 2. For (ii), we use Proposition 3 to obtain for fixed $0 < h \ll 1$ and $l \in \mathbb{Z}^n$ with $\epsilon|l| < h$

$$\left\|\delta u^2_\epsilon\right\|_{L^2\left((0,T),L^2\left(\Omega^2_{\epsilon,h}\right)\right)} \leqslant C\left(\left\|\delta u^1_\epsilon\right\|_{L^2\left((0,T),L^2\left(\Omega^1_{\epsilon,h}\right)\right)} + \left\|\delta\left(u^2_{\epsilon,i}, u^2_{\epsilon,i,\Gamma_\epsilon}\right)\right\|_{\mathbb{L}_{2,\epsilon,h}} + \epsilon\right). \tag{5.2}$$

For the first term on the right-hand side, we have

$$\left\|\delta u^1_\epsilon\right\|_{L^2\left((0,T),L^2\left(\Omega^1_{\epsilon,h}\right)\right)} \leqslant \left\|\mathcal{T}_\epsilon u^1_\epsilon(t, x+l\epsilon, y) - \mathcal{T}_\epsilon u^1_\epsilon\right\|_{L^2\left((0,T)\times\Omega_h\times Y_1\right)}.$$

The right-hand side converges to zero for $\epsilon l \to 0$, due to the strong two-scale convergence of $u^1_\epsilon$, that is, the strong convergence of $\mathcal{T}_\epsilon u^1_\epsilon$ in $L^2\left((0,T)\times\Omega\times Y_1\right)$ (see again Lemma 6 in the Appendix A), and the standard Kolmogorov-compactness theorem. For the term including the shifts of the initial values in (5.2) we argue in a similar way: We have

$$\sqrt{\epsilon}\left\|\delta u^2_{\epsilon,i,\Gamma_\epsilon}\right\|_{L^2(\Gamma_\epsilon)} \leqslant \left\|\mathcal{T}_\epsilon u^2_{\epsilon,i,\Gamma_\epsilon}(x+l\epsilon, y) - \mathcal{T}_\epsilon u^2_{\epsilon,i,\Gamma_\epsilon}\right\|_{L^2(\Omega_h, L^2(\Gamma))}. \tag{5.3}$$

Due to Assumption (A5), the sequence $u^2_{\epsilon,i,\Gamma_\epsilon}$ converges strongly in the two-scale sense on $\Gamma_\epsilon$, which implies the strong convergence of $\mathcal{T}_\epsilon u_{\epsilon,i,\Gamma_\epsilon}$ in $L^2\left(\Omega, L^2(\Gamma)\right)$. Especially, the sequence $\mathcal{T}_\epsilon u_{\epsilon,i,\Gamma_\epsilon}$ is relatively compact in $L^2\left(\Omega, L^2(\Gamma)\right)$. Hence, we can apply the Kolmogorov–Simon-compactness result from [18, Corollary 2.5] and obtain that the right-hand side in (5.3) converges to zero for $\epsilon l \to 0$. In the same way, we can treat the term in (5.2) including $\delta u^2_{\epsilon,i}$. Hence, the condition (ii) of Theorem 4.1 is proved and we obtain the strong two-scale convergence of $u^2_\epsilon$ and its trace.

https://doi.org/10.1017/S0956792522000195 Published online by Cambridge University Press

To prove (5.1f) and (5.1h), we have to show that $u_1^2$ is constant with respect to $y$. Therefore, we choose $\phi_\epsilon(t,x) := \epsilon\phi\left(t,x,\frac{x}{\epsilon}\right)$ with $\phi \in C_0^\infty\left((0,T)\times\Omega\times\overline{Y_2}\right)$ (periodically extended in the last variable) as a test function in (3.2) for $j=2$ and integrate with respect to time to obtain

$$\begin{aligned}\int_0^T \left\langle \partial_t u_\epsilon^2, \phi_\epsilon \right\rangle_{\mathbb{H}_{2,\epsilon}',\mathbb{H}_{2,\epsilon}} dt + \int_0^T \int_{\Omega_\epsilon^2} D_\epsilon^2 \nabla u_\epsilon^2 \cdot \left(\epsilon \nabla_x \phi\left(t,x,\frac{x}{\epsilon}\right) + \nabla_y \phi\left(t,x,\frac{x}{\epsilon}\right)\right) dxdt \\ + \epsilon \int_0^T \int_{\Gamma_\epsilon} D_{\Gamma_\epsilon}^2 \nabla_{\Gamma_\epsilon} u_\epsilon^2 \cdot \left(\epsilon P_{\Gamma_\epsilon} \nabla_x \phi\left(t,x,\frac{x}{\epsilon}\right) + P_{\Gamma_\epsilon} \nabla_y \phi\left(t,x,\frac{x}{\epsilon}\right)\right) d\sigma dt \\ = \epsilon \int_0^T \int_{\Omega_\epsilon^2} f_\epsilon^2(u_\epsilon^2)\phi\left(t,x,\frac{x}{\epsilon}\right) dxdt + \epsilon^2 \int_0^T \int_{\Gamma_\epsilon} h_\epsilon^2\left(u_\epsilon^1, u_\epsilon^2\right)\phi\left(t,x,\frac{x}{\epsilon}\right) dxdt.\end{aligned} \tag{5.4}$$

For the first term on the left-hand side including the time derivative, we get by integration by parts in time

$$\begin{aligned}\int_0^T \left\langle \partial_t u_\epsilon^2, \phi_\epsilon \right\rangle_{\mathbb{H}_{2,\epsilon}',\mathbb{H}_{2,\epsilon}} dt = -\epsilon \int_0^T \int_{\Omega_\epsilon^2} \partial_t \phi\left(t,x,\frac{x}{\epsilon}\right) u_\epsilon^2 dxdt \\ - \epsilon^2 \int_0^T \int_{\Gamma_\epsilon} \partial_t \phi\left(t,x,\frac{x}{\epsilon}\right) u_\epsilon^2 d\sigma dt.\end{aligned}$$

The right-hand side is of order $\epsilon$, due to the estimates in Proposition 2. Hence, all terms in (5.4) except the terms including the $\nabla_y$ are of order $\epsilon$ (again because of Proposition 2) and we obtain for $\epsilon \to 0$

$$\begin{aligned}\int_0^T \int_\Omega \int_{Y_2} D^2(y) \nabla_y u_1^2(t,x,y) \cdot \nabla_y \phi(t,x,y) dydxdt \\ + \int_0^T \int_\Omega \int_\Gamma D_\Gamma^2(y) \nabla_{\Gamma,y} u_1^2(t,x,y) \cdot \nabla_{\Gamma,y}\phi(t,x,y) d\sigma_y dxdt = 0.\end{aligned}$$

Due to the density of $C^\infty(\overline{Y_2})$ in $\mathbb{H}_2$, see [17, Lemma 2.1], the equation above holds for all $\phi \in L^2((0,T)\times\Omega, \mathbb{H}_2)$. This implies $u_1^2 = 0$. The proposition is proved. □

We have the following representation of $u_1^1$:

**Corollary 1** *Almost everywhere in* $(0,T)\times\Omega\times Y_1$ *it holds that*

$$u_1^1(t,x,y) = \sum_{i=1}^n \partial_{x_i} u_0^1(t,x) w_i^1(y), \tag{5.5}$$

*where* $w_i^1 \in \mathbb{H}_1/\mathbb{R}$ *with Y-periodic boundary conditions is the unique weak solution of the following cell problem* $(i = 1, \ldots, n)$

$$\begin{aligned} -\nabla_y \cdot \left(D^1\left(\nabla_y w_i^1 + e_i\right)\right) = 0 & \quad \text{in } Y_1, \\ -D^1\left(\nabla_y w_i^1 + e_i\right)\cdot\nu = -\nabla_{\Gamma,y}\cdot\left(D_\Gamma^1\left(\nabla_{\Gamma,y} w_i^1 + \nabla_{\Gamma,y} y_i\right)\right) & \quad \text{on } \Gamma, \\ w_i^1 \text{ is } Y\text{-periodic and } \int_\Gamma w_i^1 d\sigma = 0. & \end{aligned} \tag{5.6}$$

**Proof** The procedure is quite standard, see, for example, [2], but for the sake of completeness we give the main steps. We choose $\phi_\epsilon(t,x) = \epsilon\phi\left(t,x,\frac{x}{\epsilon}\right)$ with $\phi \in C_0^\infty\left((0,T)\times\Omega, C_{\text{per}}^\infty\left(\overline{Y_1}\right)\right)$ as

https://doi.org/10.1017/S0956792522000195 Published online by Cambridge University Press

a test function in (3.2) and integrate with respect to time to obtain (5.4) if we replace $j=2$ by $j=1$. From Proposition 5, we get for $\epsilon \to 0$

$$0 = \int_0^T \int_\Omega \int_{Y_1} D^1(y) \left[ \nabla_x u_0^1(t,x) + \nabla_y u_1^1(t,x,y) \right] \cdot \nabla_y \phi(t,x,y) dy dx dt + \int_0^T \int_\Omega \int_\Gamma D_\Gamma^1(y) \left[ P_\Gamma(y) \nabla_x u_0^1(t,x) + \nabla_{\Gamma,y} u_1^1(t,x,y) \right] \cdot \nabla_{\Gamma,y} \phi(t,x,y) d\sigma_y dx dt.$$

Due to the Lax–Milgram lemma, this problem has a unique solution $u_1^1$ and due to its linearity we easily obtain the representation (5.5). □

Now, we are able to formulate the macroscopic model. We show that $u_0 = \left(u_0^1, u_0^2\right)$ from Proposition 5 is the unique weak solution (the definition of a weak solution is given below) of the macro-model

$$\begin{aligned}
(|Y_1| + |\Gamma|)\partial_t u_0^1 - \nabla \cdot \left(\widehat{D}^1 \nabla u_0^1\right) &= \int_{Y_1} f^1\left(t,y,u_0^1\right) dy + \int_\Gamma h^1\left(t,y,u_0^1,u_0^2\right) d\sigma_y && \text{in } (0,T)\times\Omega, \\
(|Y_2| + |\Gamma|)\partial_t u_0^2 &= \int_{Y_2} f^2\left(t,y,u_0^2\right) dy + \int_\Gamma h^2\left(t,y,u_0^1,u_0^2\right) d\sigma_y && \text{in } (0,T)\times\Omega, \\
-\widehat{D}^1 \nabla u_0^1 \cdot \nu &= 0 && \text{on } (0,T)\times\partial\Omega, \\
u_0^j(0) &= \frac{|Y_j| u_{0,i}^j + |\Gamma| u_{0,i,\Gamma}^j}{|Y_j| + |\Gamma|} && \text{in } \Omega,
\end{aligned} \tag{5.7}$$

where the homogenised diffusion coefficient $\widehat{D}^1 \in \mathbb{R}^{n\times n}$ is defined by $(i,l=1,\ldots,n)$

$$\left(\widehat{D}^1\right)_{il} := \int_{Y_1} D^1\left(\nabla_y w_i^1 + e_i\right) \cdot \left(\nabla_y w_l^1 + e_l\right) dy + \int_\Gamma D_\Gamma^1 \left(\nabla_{\Gamma,y} w_i^1 + \nabla_{\Gamma,y} y_i\right) \cdot \left(\nabla_{\Gamma,y} w_l^1 + \nabla_{\Gamma,y} y_l\right) d\sigma,$$

and $w_i^1 \in \mathbb{H}_1/\mathbb{R}$ (see Section 3.1 for the definition of this space) for $i=1,\ldots,n$ are the solutions of the cell problems (5.6).

We say that $u_0 = \left(u_0^1, u_0^2\right)$ is a weak solution of the macroscopic model, if

$$\begin{aligned}
&u_0^1 \in L^2\left((0,T), H^1(\Omega)\right) \cap H^1\left((0,T), H^1(\Omega)'\right), \\
&u_0^2 \in H^1\left((0,T), L^2(\Omega)\right),
\end{aligned}$$

the equation for $\partial_t u_0^2$ in (5.7) is valid in $L^2((0,T)\times\Omega)$, and for all $\phi \in H^1(\Omega)$ it holds almost everywhere in $(0,T)$

$$(|Y_1| + |\Gamma|)\left\langle \partial_t u_0^1, \phi \right\rangle_{H^1(\Omega)', H^1(\Omega)} + \int_\Omega \widehat{D}^1 \nabla u_0^1 \cdot \nabla \phi dx = \int_\Omega \int_{Y_1} f^1\left(y, u_0^1\right) \phi dy dx + \int_\Omega \int_\Gamma h^1\left(y, u_0^1, u_0^2\right) \phi d\sigma_y dx,$$

together with the initial conditions from (5.7).

https://doi.org/10.1017/S0956792522000195 Published online by Cambridge University Press

**Theorem 5.1** *The limit function* $u_0 = \left(u_0^1, u_0^2\right)$ *from Proposition 5 is the unique solution of the macroscopic problem* (5.7).

**Proof** We illustrate the procedure for $j = 1$ (the case $j = 2$ follows by similar arguments, where the diffusion terms vanish in the limit). As a test function in (3.2) for $j = 1$ we choose $\phi \in C_0^\infty([0, T) \times \overline{\Omega})$ and integrate with respect to time. By integration by parts in time, we obtain

$$\begin{aligned}
&-\int_0^T \int_{\Omega_\epsilon^1} u_\epsilon^1 \partial_t \phi dxdt - \epsilon \int_0^T \int_{\Gamma_\epsilon} u_\epsilon^1 \partial_t \phi d\sigma dt \\
&\quad + \int_0^T \int_{\Omega_\epsilon^1} D_\epsilon^1 \nabla u_\epsilon^1 \cdot \nabla \phi dxdt + \epsilon \int_0^T \int_{\Gamma_\epsilon} D_{\Gamma_\epsilon}^1 \nabla_{\Gamma_\epsilon} u_\epsilon^1 \cdot \nabla_{\Gamma_\epsilon} \phi d\sigma dt \\
&= \int_0^T \int_{\Omega_\epsilon^1} f_\epsilon^1 \left(u_\epsilon^1\right) \phi dxdt + \epsilon \int_0^T \int_{\Gamma_\epsilon} h_\epsilon^1 \left(u_\epsilon^1, u_\epsilon^2\right) \phi d\sigma dt \\
&\quad + \int_{\Omega_\epsilon^1} u_{\epsilon,i}^1 \phi dx + \epsilon \int_{\Gamma_\epsilon} u_{\epsilon,i,\Gamma_\epsilon}^1 \phi d\sigma .
\end{aligned}$$

Using the convergence results from Proposition 5, Corollary 1, and Lemma 1, as well as the Assumption (A5) on the initial conditions, we obtain for $\epsilon \to 0$

$$\begin{aligned}
-\left(|Y_1| + |\Gamma|\right) &\int_0^T \int_\Omega u_0^1 \partial_t \phi dxdt + \int_0^T \int_\Omega \widehat{D}^1 \nabla u_0^1 \cdot \nabla \phi dxdt \\
&= \int_0^T \int_\Omega \int_{Y_1} f^1\left(u_0^1\right) \phi dydxdt + \int_0^T \int_\Omega \int_\Gamma h^1\left(u_0^1, u_0^2\right) \phi d\sigma_y dxdt \\
&\quad + \int_\Omega |Y_1| u_{0,i}^1 \phi(0) dx + \int_\Omega |\Gamma| u_{0,i,\Gamma}^1 \phi(0) dx.
\end{aligned}$$

Choosing $\phi$ with compact support in $(0,T)$ we get $\partial_t u_0^1 \in L^2\left((0, T), H^1(\Omega)'\right)$ (see also Remark 2) with $u_0^1(0) = \frac{|Y_1| u_{0,i}^1 + |\Gamma| u_{0,i,\Gamma}^1}{|Y_1| + |\Gamma|}$, and by density we obtain that $u_0^1$ is a weak solution of the macroscopic equation for $j = 1$ in (5.7). Following the same arguments as above, we obtain for all $\phi \in C_0^\infty(\Omega)$ and $\psi \in C_0^\infty(0, T)$

$$-\int_0^T \int_\Omega u_0^2 \phi \psi' dxdt = \int_0^T \left\{ \frac{1}{|Y_2| + |\Gamma|} \int_\Omega \left[ \int_{Y_2} f^2\left(t, y, u_0^2\right) dy + \int_\Gamma h^2\left(t, y, u_0^1, u_0^2\right) d\sigma_y \right] \phi dx \right\} \psi dt.$$

By density, this result is valid for all $\phi \in L^2(\Omega)$. Hence, the expression in the curly brackets defines a linear functional on $L^2(\Omega)$. This implies $\partial_t u_0^2 \in L^2((0, T) \times \Omega)$. Uniqueness of the macroscopic problem follows by standard energy estimates. □

**Remark 2**

(i) *We established the regularity for the time derivatives via the variational equations derived in the proof of Theorem 5.1. However, we emphasise that some regularity is also a direct consequence of the a priori estimates in Proposition 2. In fact, define for* $0 < h \ll 1$ *and* $v : (0, T) \to X$ *for a Banach space* $X$ *the difference quotient for* $t \in (0, T - h)$

$$\partial_t^h v(t) := \frac{v(t + h) - v(t)}{h}.$$

https://doi.org/10.1017/S0956792522000195 Published online by Cambridge University Press

*Then for all* $\phi \in C_0^\infty((0,T), C^\infty(\overline{\Omega}))$ *it holds, due to Proposition 5 and the a priori estimates for the time derivative in Proposition 2:*

$$\begin{aligned}
\langle \partial_t^h u_0^1, \phi \rangle_{L^2\left((0,T-h),H^1(\Omega)'\right),L^2\left((0,T-h),H^1(\Omega)\right)} &= \int_0^{T-h} \int_\Omega \partial_t^h u_0^1 \phi dx dt \\
&= \lim_{\epsilon \to 0} \frac{1}{|Y_1| + |\Gamma|} \left( \int_0^{T-h} \int_{\Omega_\epsilon^1} \partial_t^h u_\epsilon^1 \phi dx + \epsilon \int_{\Gamma_\epsilon} \partial_t^h u_\epsilon^1 \phi d\sigma dt \right) \\
&= \lim_{\epsilon \to 0} \frac{1}{|Y_1| + |\Gamma|} \int_0^T \langle \partial_t^h u_\epsilon^1, \phi \rangle_{\mathbb{H}_{1,\epsilon}', \mathbb{H}_{1,\epsilon}} dt. \\
&\leqslant \lim_{\epsilon \to 0} \frac{1}{|Y_1| + |\Gamma|} \| \partial_t^h u_\epsilon^1 \|_{L^2\left((0,T),\mathbb{H}_{1,\epsilon}'\right)} \|\phi\|_{L^2\left((0,T),\mathbb{H}_{1,\epsilon}\right)} \\
&\leqslant C \lim_{\epsilon \to 0} \|\partial_t u_\epsilon^1\|_{L^2\left((0,T),\mathbb{H}_{1,\epsilon}'\right)} \|\phi\|_{L^2\left((0,T),\mathbb{H}_{1,\epsilon}\right)} \\
&\leqslant C \lim_{\epsilon \to 0} \|\phi\|_{L^2\left((0,T),\mathbb{H}_{1,\epsilon}\right)} \leqslant C \|\phi\|_{L^2\left((0,T),H^1(\Omega)\right)},
\end{aligned}$$

*where at the end we used that* $P_\Gamma$ *is an orthogonal projection. By density and the reflexivity of* $L^2\left((0,T-h),H^1(\Omega)\right)$*, we obtain the boundedness*

$$\left\| \partial_t^h u_0^1 \right\|_{L^2\left((0,T-h),H^1(\Omega)'\right)} \leqslant C,$$

*for a constant C independent of h. This implies* $\partial_t u_0^1 \in L^2\left((0,T),H^1(\Omega)'\right)$*. A similar argument implies* $\partial_t u_0^2 \in L^2\left((0,T),H^1(\Omega)'\right)$*. However, the limit equation for* $u_0^2$ *even improves the regularity of* $\partial_t u_0^2$*.*

(ii) *We can also consider the case of a connected–connected porous medium (for* $n \geqslant 3$ *and a domain* $\Omega$ *which can be decomposed in microscopic cells, for example, a rectangle with integer side length, and an additional boundary condition on* $\partial\Gamma_\epsilon$ *is needed). In this case, both macroscopic solutions are described by a reaction–diffusion equation as for* $u_0^1$ *in Theorem 5.1. The derivation of the macroscopic model for the connected–connected case even gets simpler, because we only need the a priori estimates from Proposition 2 and the convergence results for the connected domain in Section 4.1. The estimates for the shifts in Proposition 3 are no longer necessary.*

(iii) *The results can be easily extended to systems, see [16] for more details.*

## 6 Discussion

By the methods of two-scale convergence and the unfolding operator we derived a macroscopic model for a reaction–diffusion equation in a connected–disconnected porous medium with a nonlinear dynamic Wentzell-interface condition across the interface. The crucial point was to pass to the limit in the nonlinear terms, especially on the interface. Therefore, we established strong two-scale compactness results just depending on *a priori* estimates for the sequence of solutions. We emphasise that the strong compactness result in Theorem 4.1 is not restricted to our specific problem, but on the *a priori* estimates and the estimates for the shifts for the sequence. Therefore, it can be easily applied to other problems. Especially, the results above can be extended to systems in an obvious way.

The time derivative in the Wentzell-boundary condition on the interface $\Gamma_\epsilon$ regularises the problem and leads to a simple variational structure with respect to the Gelfand triple

$(\mathbb{H}_{j,\epsilon}, \mathbb{L}_{j,\epsilon}, \mathbb{H}'_{j,\epsilon})$, see (3.2). Hence, the problem seems to be more complex regarding stationary interface conditions (neglecting the time derivative). On the other hand, neglecting the diffusion term on the surface leads to an ordinary differential equation on the surface and we loose spatial regularity on the surface. Hence, we have to replace $\mathbb{H}_{j,\epsilon}$ by $H^1\left(\Omega_\epsilon^j\right)$ and it could be expected that the methods in this paper can be adapted to that case.


## Acknowledgements

The author was supported by the Odysseus program of the Research Foundation – Flanders FWO (Project-Nr. G0G1316N) and the project SCIDATOS (Scientific Computing for Improved Detection and Therapy of Sepsis), which was funded by the Klaus Tschira Foundation, Germany (Grant number 00.0277.2015). Further, the author thanks an anonymous referee for detailed and helpful comments about an earlier version of the manuscript, which helped to improve the structure and thus the readability of the paper.


## References


[1] ACERBI, E., CHIADÒ, V., MASO, G. D. & PERCIVALE, D. (1992) An extension theorem from connected sets, and homogenization in general periodic domains. *Nonlinear Anal. Theory Methods Appl.* **18**(5), 481–496.

[2] ALLAIRE, G. (1992) Homogenization and two-scale convergence. *SIAM J. Math. Anal.* **23**, 1482–1518.

[3] ALLAIRE, G., DAMLAMIAN, A. & HORNUNG, U. (1996) Two-scale convergence on periodic surfaces and applications. In: A. BOURGEAT and C. CARASSO (editors), *Proceedings of the International Conference on Mathematical Modelling of Flow Through Porous Media, World Scientific, Singapore*, pp. 15–25.

[4] ALLAIRE, G. & HUTRIDURGA, H. (2012) Homogenization of reactive flows in porous media and competition between bulk and surface diffusion. *IMA J. Appl. Math.* **77**, 788–215.

[5] ALT, H. W. (2016) *Linear Functional Analysis*, Springer, Berlin Heidelberg.

[6] AMAR, M. & GIANNI, R. (2018) Laplace-Beltrami operator for the heat conduction in polymer coating of electronic devices. *Discrete Cont. Dyn. Syst. B* **23**(4), 1739–1756.

[7] ANGUIANO, M. (2020) Existence, uniqueness and homogenization of nonlinear parabolic problems with dynamical boundary conditions in perforated media. *Mediterr. J. Math.* **17**, 1–22

[8] ARBOGAST, T., DOUGLAS, J. & HORNUNG, U. (1990) Derivation of the double porosity model of single phase flow via homogenization theory. *SIAM J. Math. Anal.* **27**, 823–836.

[9] BOURGEAT, A., LUCKHAUS, S. & MIKELIĆ, A. (1996) Convergence of the homogenization process for a double-porosity model of immiscible two-phase flow. *SIAM J. Math. Anal.* **27**, 1520–1543.

[10] CIORANESCU, D., DAMLAMIAN, A., DONATO, P., GRISO, G. & ZAKI, R. (2012) The periodic unfolding method in domains with holes. *SIAM J. Math. Anal.* **44**(2), 718–760.

[11] CIORANESCU, D., DAMLAMIAN, A. & GRISO, G. (2008) The periodic unfolding method in homogenization. *SIAM J. Math. Anal.* **40**, 1585–1620.

[12] CIORANESCU, D., GRISO, G. & DAMLAMIAN, A. (2018) *The Periodic Unfolding Method*, Springer, Singapore.

[13] CIORANESCU, D. & PAULIN, J. S. J. (1979) Homogenization in open sets with holes. *J. Math. Pures Appl.* **71**, 590–607.

[14] DONATO, P. & NGUYEN, K. H. L. (2015) Homogenization of diffusion problems with a nonlinear interfacial resistance. *NoDEA Nonlinear Differ. Equations Appl.* **22**(5), 1345–1380.

[15] FRIESECKE, G., JAMES, R. D. & MÜLLER, S. (2002) A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. *Commun. Pure Appl. Math. J. Issued Courant Inst. Math. Sci.* **55**(11), 1461–1506.

[16] GAHN, M. (2017) *Derivation of Effective Models for Reaction-Diffusion Processes in Multi-component Media*. PhD thesis, Friedrich-Alexander-Universität Erlangen-Nürnberg, Shaker Verlag.
[17] GAHN, M. (2019) Multi-scale modeling of processes in porous media - coupling reaction-diffusion processes in the solid and the fluid phase and on the separating interfaces. *Discrete Contin. Dyn. Syst. Ser. B* **24**(12), 6511–6531.
[18] GAHN, M. & NEUSS-RADU, M. (2016) A characterization of relatively compact sets in $L^p(\Omega, B)$. *Stud. Univ. Babeş-Bolyai Math.* **61**(3), 279–290.
[19] GAHN, M., NEUSS-RADU, M. & KNABNER, P. (2016) Homogenization of reaction–diffusion processes in a two-component porous medium with nonlinear flux conditions at the interface. *SIAM J. Appl. Math.* **76**(5), 1819–1843.
[20] GAHN, M., NEUSS-RADU, M. & KNABNER, P. (2017) Derivation of an effective model for metabolic processes in living cells including substrate channeling. *Vietnam J. Math.* **45**(1–2), 265–293.
[21] GALDI, G. P. (2011) *An Introduction to the Mathematical Theory of the Navier–Stokes Equations*. *Springer Monographs in Mathematics*, Springer-Verlag, New York, New York.
[22] GRAF, I. & PETER, M. A. (2014) A convergence result for the periodic unfolding method related to fast diffusion on manifolds. *C. R. Acad. Sci. Paris Ser. I* **352**(6), 485–490.
[23] GRAF, I. & PETER, M. A. (2014) Diffusion on surfaces and the boundary periodic unfolding operator with an application to carcinogenesis in human cells. *SIAM J. Math. Anal.* **46**(4), 3025–3049.
[24] HEWITT, E. & STROMBERG, K. (1975) *Real and Abstract Analysis*, Springer-Verlag, New York.
[25] HORNUNG, U., JÄGER, W. & MIKELIĆ, A. (1994) Reactive transport through an array of cells with semi-permeable membranes. *ESAIM Math. Model. Numer. Anal.* **28**, 59–94.
[26] NEUSS-RADU, M. (1996) Some extensions of two-scale convergence. *C. R. Acad. Sci. Paris Sér. I Math.* **322**, 899–904.
[27] NEUSS-RADU, M. & JÄGER, W. (2007) Effective transmission conditions for reaction-diffusion processes in domains separated by an interface. *SIAM J. Math. Anal.* **39**, 687–720.
[28] NGUETSENG, G. (1989) A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal.* **20**, 608–623.
[29] PTASHNYK, M. & ROOSE, T. (2010) Derivation of a macroscopic model for transport of strongly sorbed solutes in the soil using homogenization theory. *SIAM J. Appl. Math.* **70**, 2097–2118.
[30] SIMON, J. (1987) Compact sets in the space $L^p(0, T; B)$. *Ann. Mat. Pura Appl.* **146**, 65–96.
[31] VOGT, C. (1982) *A Homogenization Theorem Leading to a Volterra Integro-Differential Equation for Permeation Chromatography.* SFB 123, University of Heidelberg, Preprint 155 and Diploma-thesis.
[32] WINKEL, B. S. (2004) Metabolic channeling in plants. *Annu. Rev. Plant Biol.* **55**, 85–107.

## Appendix A. Two-scale convergence and unfolding operator

We repeat the definition of the two-scale convergence and the unfolding operator and summarise some well-known properties and compactness results.

### A.1 Two-scale convergence

In the following, unless stated otherwise, we assume that $p \in (1, \infty)$ and $p'$ is the dual exponent of $p$. We start with the definition of the two-scale convergence, see [2, 28].

**Definition A.1** We say the sequence $u_\epsilon \in L^p((0, T) \times \Omega)$ converges in the two-scale sense (in $L^p$) to a limit function $u_0 \in L^p((0, T) \times Y)$, if for all $\phi \in L^{p'}\left((0, T) \times \Omega, C^0_{\text{per}}(Y)\right)$ it holds that

$$\lim_{\epsilon \to 0} \int_0^T \int_\Omega u_\epsilon(t, x)\phi\left(t, x, \frac{x}{\epsilon}\right) dxdt = \int_0^T \int_\Omega \int_Y u_0(t, x, y)\phi(t, x, y)dxdydt.$$

https://doi.org/10.1017/S0956792522000195 Published online by Cambridge University Press

We say the sequence converges strongly in the two-scale sense (in $L^p$), if it holds that

$$\lim_{\epsilon\to 0} \|u_\epsilon\|_{L^p((0,T)\times\Omega)} = \|u_0\|_{L^p((0,T)\times\Omega\times Y)}.$$

**Remark 3**

*(i)* *For sequences in* $L^p\left((0,T)\times\Omega_\epsilon^j\right)$ *on the perforated domain, we also use the designation 'two-scale convergence'. The definition is also valid for such functions by extension by zero (or with the extension operator from [1]), and considering suitable test functions.*

*(ii)* *The two-scale convergence introduced above should actually be referred to as 'weak two-scale convergence'. However, in accordance with the definition in [2] we neglect the word 'weak' and only use 'strong' to highlight the 'strong two-scale convergence'.*

*(iii)* *For the 'two-scale convergence in* $L^2$*' we just write 'two-scale convergence'.*

Next, we give the definition of the two-scale convergence on oscillating surfaces, see [3, 26].

**Definition A.2** We say the sequence $u_\epsilon \in L^p\left((0,T)\times\Gamma_\epsilon\right)$ converges in the two-scale sense (in $L^p$) to a limit function $u_0 \in L^p((0,T)\times\Omega\times\Gamma)$, if for all $\phi \in C^0\left([0,T]\times\overline{\Omega}, C^0_{\text{per}}(\Gamma)\right)$ it holds that

$$\lim_{\epsilon\to 0} \epsilon \int_0^T \int_{\Gamma_\epsilon} u_\epsilon(t,x)\phi\left(t,x,\frac{x}{\epsilon}\right) d\sigma_x dt = \int_0^T \int_\Omega \int_\Gamma u_0(t,x,y)\phi(t,x,y) d\sigma_y dx dt.$$

We say the sequence converges strongly in the two-scale sense, if it holds that

$$\lim_{\epsilon\to 0} \epsilon^{\frac{1}{p}} \|u_\epsilon\|_{L^p((0,T)\times\Gamma_\epsilon)} = \|u_0\|_{L^p((0,T)\times\Omega\times\Gamma)}.$$

In accordance with Remark 3, we proceed analogously for the two-scale convergence on $\Gamma_\epsilon$ and neglect the word 'weak' and the addition '$L^2$'.

To pass to the limit $\epsilon \to 0$ in the diffusion terms in the bulk domain $\Omega_\epsilon^j$ and the surface $\Gamma_\epsilon$ in the microscopic equation (3.2) we need compactness results for the spaces $\mathbb{H}_{j,\epsilon}$. In the following lemma, we summarise some weak two-scale compactness results for such functions, which can be found in [17]:

**Lemma 4** *For* $j \in \{1,2\}$ *let* $u_\epsilon^j \in L^2\left((0,T),\mathbb{H}_{j,\epsilon}\right)$ *be a sequence with*

$$\left\|u_\epsilon^j\right\|_{L^2\left((0,T),\mathbb{H}_{j,\epsilon}\right)} \leqslant C.$$

*Then it holds:*

*(i)* *For* $j=1$ *there exists an extension* $\widetilde{u}_\epsilon^1 \in L^2\left((0,T),H^1(\Omega)\right)$ *of* $u_\epsilon^1$ *(see Section 4.1 for more details), and* $u_0^1 \in L^2\left((0,T),H^1(\Omega)\right)$ *and a Y-periodic function* $u_1^1 \in L^2((0,T)\times\Omega,\mathbb{H}_1/\mathbb{R})$*, such that up to a subsequence*

https://doi.org/10.1017/S0956792522000195 Published online by Cambridge University Press

$$
\begin{aligned}
\widetilde{u}_\epsilon^1 &\to u_0^1 && \text{in the two-scale sense,}\\
\nabla \widetilde{u}_\epsilon^1 &\to \nabla_x u_0^1 + \nabla_y u_1^1 && \text{in the two-scale sense,}\\
u_\epsilon^1|_{\Gamma_\epsilon} &\to u_0^1 && \text{in the two-scale sense on } \Gamma_\epsilon,\\
\nabla_{\Gamma_\epsilon} u_\epsilon^1|_{\Gamma_\epsilon} &\to P_\Gamma \nabla u_0^1 + \nabla_\Gamma u_1^1|_\Gamma && \text{in the two-scale sense on } \Gamma_\epsilon.
\end{aligned}
$$

*(ii) For $j = 2$ there exist $u_0^2 \in L^2((0,T)\times\Omega)$ and $u_1^2 \in L^2((0,T)\times\Omega, \mathbb{H}_2/\mathbb{R})$ such that up to a subsequence*

$$
\begin{aligned}
\chi_{\Omega_\epsilon^2} u_\epsilon^2 &\to \chi_{Y_2} u_0^2 && \text{in the two-scale sense,}\\
\chi_{\Omega_\epsilon^2} \nabla u_\epsilon^2 &\to \chi_{Y_2} \nabla_y u_1^2 && \text{in the two-scale sense,}\\
u_\epsilon^2|_{\Gamma_\epsilon} &\to u_0^2 && \text{in the two-scale sense on } \Gamma_\epsilon,\\
\nabla_{\Gamma_\epsilon} u_\epsilon^2|_{\Gamma_\epsilon} &\to \nabla_\Gamma u_1^2 && \text{in the two-scale sense on } \Gamma_\epsilon.
\end{aligned}
$$

### A.2 The unfolding operator

In the following, we give the definition of the unfolding operator and summarise some well-known properties, see the monograph [12] for an overview about this topic, and also [8–11, 31]. In the following, we consider the tuple $(G_\epsilon, G) \in \{(\Omega, Y), (\Omega_\epsilon^1, Y_1), (\Omega_\epsilon^2, Y_2), (\Gamma_\epsilon, \Gamma)\}$ and we define

$$
\widehat{G}_\epsilon := \operatorname{int} \bigcup_{k\in K_\epsilon} \epsilon\left(\overline{G} + k\right), \qquad \Lambda_\epsilon := \Omega \setminus \overline{\widehat{G}_\epsilon}.
$$

Then, for $p \in (1,\infty)$ we define the unfolding operator

$$
\mathcal{T}_\epsilon : L^p((0,T)\times G_\epsilon) \to L^p((0,T)\times\Omega\times G),
$$

with

$$
\mathcal{T}_\epsilon(\phi_\epsilon)(t,x,y) := \begin{cases} \phi_\epsilon\left(t, \epsilon\left[\frac{x}{\epsilon}\right] + \epsilon y\right) & \text{for } x \in \widehat{G}_\epsilon,\\ 0 & \text{for } x \in \Lambda_\epsilon. \end{cases}
$$

We emphasise that we use the same notation for the unfolding operator for the different choices of the tuple $(G_\epsilon, G)$. It should be clear from the context in which sense it has to be understood. Further, we mention that unfolding operator commutes with the trace operator in the following sense: For $\phi_\epsilon \in L^p\left((0,T), W^{1,p}\left(\Omega_\epsilon^j\right)\right)$ for $j \in \{1,2\}$ it holds that

$$
\mathcal{T}_\epsilon\left(\phi_\epsilon|_{\Gamma_\epsilon}\right) = \left(\mathcal{T}_\epsilon(\phi_\epsilon)\right)|_\Gamma.
$$

**Lemma 5**

*(a) For $(G_\epsilon, G) \in \left\{(\Omega, Y), \left(\Omega_\epsilon^1, Y_1\right), \left(\Omega_\epsilon^2, Y_2\right)\right\}$ we have:*

*(i) For $\phi_\epsilon \in L^p((0,T)\times G_\epsilon)$ it holds that*

$$
\|\mathcal{T}_\epsilon \phi_\epsilon\|_{L^p((0,T)\times\Omega\times G)} = \|\phi_\epsilon\|_{L^p((0,T)\times\widehat{G}_\epsilon)}.
$$

*(ii) For $\phi_\epsilon \in L^p\left((0,T), W^{1,p}(G_\epsilon)\right)$ it holds that*

$$
\nabla_y \mathcal{T}_\epsilon \phi_\epsilon = \epsilon \mathcal{T}_\epsilon \nabla_x \phi_\epsilon.
$$

https://doi.org/10.1017/S0956792522000195 Published online by Cambridge University Press

*(b) For the unfolding operator on the surface, we have*

*(i) For* $\phi_\epsilon \in L^p((0,T)\times\Gamma_\epsilon)$ *it holds that*

$$\|\mathcal{T}_\epsilon\phi_\epsilon\|_{L^p((0,T)\times\Omega\times\Gamma)} = \epsilon^{\frac{1}{p}}\|\phi_\epsilon\|_{L^p((0,T)\times\Gamma_\epsilon)}.$$

*(ii) For* $\phi_\epsilon \in L^p\big((0,T), W^{1,p}(\Gamma_\epsilon)\big)$ *it holds that*

$$\nabla_{\Gamma,y}\mathcal{T}_\epsilon\phi_\epsilon = \epsilon\mathcal{T}_\epsilon\nabla_{\Gamma_\epsilon}\phi_\epsilon.$$

**Proof** For (a) and (b)(i) see [12]. A proof for (b)(ii) can be found in [23]. □

In the following lemma, we give an equivalent relation between the unfolding operator and the two-scale convergence. For a proof see for example [9, 10, 12].

**Lemma 6** *Let* $p \in (1,\infty)$.

*(a) For* $(G_\epsilon, G) \in \big\{(\Omega, Y), \big(\Omega_\epsilon^1, Y_1\big), \big(\Omega_\epsilon^2, Y_2\big)\big\}$ *and a sequence* $u_\epsilon \in L^p((0,T)\times G_\epsilon)$*, the following statements are equivalent:*

(a) $u_\epsilon \to u_0$ *weakly/strongly in the two-scale sense in* $L^p$*,*
(b) $\mathcal{T}_\epsilon u_\epsilon \to u_0$ *weakly/strongly in* $L^p((0,T)\times\Omega\times G)$*.*

*(b) For a sequence* $u_\epsilon \in L^p((0,T)\times\Gamma_\epsilon)$ *with* $\epsilon^{\frac{1}{p}}\|u_\epsilon\|_{L^p((0,T)\times\Gamma_\epsilon)} \leqslant C$*, the following statements are equivalent:*

(a) $u_\epsilon \to u_0$ *weakly/strongly in the two-scale sense on* $\Gamma_\epsilon$ *in* $L^p$*,*
(b) $\mathcal{T}_\epsilon u_\epsilon \to u_0$ *weakly/strongly in* $L^p((0,T)\times\Omega\times\Gamma)$*.*