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ITÔ'S THEOREM AND MONOMIAL BRAUER CHARACTERS

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Abstract

Let G be a finite solvable group and let p be a prime. In this note, we prove that p does not divide $\varphi(1)$ for every irreducible monomial p-Brauer character φ of G if and only if G has a normal Sylow p-subgroup.

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1. Introduction

Throughout this paper, all groups are finite. In [4], Pang and Lu show that the properties of solvable groups coming from the degrees of the irreducible characters can be determined using only the degrees of the monomial irreducible characters. In other words, for solvable groups, the monomial irreducible characters are plentiful enough to be used in place of the irreducible characters.

In particular, Pang and Lu prove in [4, Theorem 1.3] that if G is solvable and p is a prime, then p does not divide $\chi(1)$ for every monomial character $\chi \in \mathrm{Irr}(G)$ if and only if G has a normal Sylow p-subgroup. Hence, Pang and Lu are able to generalise the normal Sylow subgroup portion of Itô's theorem [1, Corollary 12.34]. Note that the degrees of irreducible monomial characters for the group $\mathrm{SL}(2,3)$ are one and three, and the normal Sylow 2-subgroup of $\mathrm{SL}(2,3)$ is not abelian, so they are not able to recover the full strength of Itô's theorem in this situation.

Itô also proved his theorem for Brauer characters of p-solvable groups. In this case also, only the normality of Sylow subgroups is recovered (see [2, Theorem 13.1(b) and (c)]. Our main theorem shows that, for solvable groups, we only need the monomial Brauer characters to prove this result.

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THEOREM 1.1. *Let G be a solvable group and let p be a prime. Then G has a normal Sylow p-subgroup if and only if p does not divide $\varphi(1)$ for every monomial Brauer character $\varphi \in \mathrm{IBr}(G)$.*

Observe that the hypothesis that G is solvable cannot be entirely dropped. For example, let $G = S_5$ be the symmetric group of degree five and $p = 2$. It is easy to see that S_5 has no subgroup of order 30, so none of the nonlinear irreducible 2-Brauer characters of G are monomial since they have degree four. Obviously, G has no normal Sylow 2-subgroup. At this time, we have not determined if we can weaken the hypothesis of solvability to p-solvability. Our proof is motivated by the proof of [4, Theorem 1.3].

2. Proof of Theorem 1.1

If a Sylow p-subgroup P is normal in G, then $P = \mathbf{O}_p(G)$ is contained in the kernel of every irreducible p-Brauer character of G. Thus $\mathrm{IBr}(G) = \mathrm{IBr}(G/P) = \mathrm{Irr}(G/P)$. Since G/P is a p'-group, for every Brauer character $\varphi \in \mathrm{IBr}(G)$, p does not divide $\varphi(1)$.

Conversely, suppose that p does not divide $\varphi(1)$ for every monomial Brauer character $\varphi \in \mathrm{IBr}(G)$. Let N be a minimal normal subgroup of G and let P be a Sylow p-subgroup of G. By induction, G/N has a normal Sylow p-subgroup PN/N, and so PN is normal in G. If N is a p-group, then $PN = P$. Thus P is normal in G, as desired. Thus, we may assume that N is an elementary abelian q-group for some prime $q \neq p$. By the Frattini argument, it follows that $G = NP\mathbf{N}_G(P) = N\mathbf{N}_G(P)$. Since $N \cap \mathbf{N}_G(P)$ is normal in $\mathbf{N}_G(P)$ and N is abelian, $N \cap \mathbf{N}_G(P)$ will be normal in $N\mathbf{N}_G(P) = G$. The minimality of N implies that either $N \leq \mathbf{N}_G(P)$ or $N \cap \mathbf{N}_G(P) = 1$. If $N \leq \mathbf{N}_G(P)$, then $G = \mathbf{N}_G(P)$ and P is normal in G, as desired.

We assume that $N \cap \mathbf{N}_G(P) = 1$. Let $1_N \neq \lambda \in \mathrm{IBr}(N) = \mathrm{Irr}(N)$ and take T to be the inertia group of λ in G. Since N is complemented in G, it follows that N is complemented in T. Using [1, Problem 6.18], we see that λ extends to $\nu \in \mathrm{Irr}(T)$. Taking μ to be the restriction of ν to the p-regular elements of T, we see that $\mu \in \mathrm{IBr}(T)$ and $\mu_N = \lambda$. Applying the Clifford correspondence for Brauer characters [3, Theorem 8.9], $\varphi = \mu^G \in \mathrm{IBr}(G)$. This implies that φ is monomial with degree $|G : T|$. By hypothesis, p does not divide $\varphi(1) = |G : T|$. It follows that T contains some Sylow p-subgroup of G and, without loss of generality, we may assume that $P \leq T$. Now, for all elements $x \in P$ and $n \in N$, $\lambda(n) = \lambda^x(n) = \lambda(xnx^{-1})$. Since λ is linear, we obtain $\lambda(xnx^{-1}n^{-1}) = 1$. Because λ is arbitrary, it follows that $[P, N] \leq \bigcap_{\lambda \in \mathrm{IBr}(N)} \ker \lambda = 1$. Therefore N normalises P. This implies that P is a characteristic subgroup of NP, and thus P is a normal subgroup of G.

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