Canad. J. Math. Vol. **76** (2), 2024 pp. 394–457
http://dx.doi.org/10.4153/S0008414X2300007X

Residual torsion-free nilpotence, bi-orderability, and two-bridge links

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Abstract. Residual torsion-free nilpotence has proved to be an important property for knot groups with applications to bi-orderability and ribbon concordance. Mayland proposed a strategy to show that a two-bridge knot group has a commutator subgroup which is a union of an ascending chain of para-free groups. This paper proves Mayland's assertion and expands the result to the subgroups of two-bridge link groups that correspond to the kernels of maps to $\mathbb{Z}$. We call these kernels the *Alexander subgroups* of the links. As a result, we show the bi-orderability of a large family of two-bridge link groups. This proof makes use of a modified version of a graph-theoretic construction of Hirasawa and Murasugi in order to understand the structure of the Alexander subgroup for a two-bridge link group.

1 Introduction

Given an oriented smooth link L in S^3, the *link group* of L, denoted $\pi(L)$, is the fundamental group of the complement of L in S^3. Also, let $\Delta_L(t)$ denote the Alexander polynomial of L (see [23, Chapter 6] for details).

Let $h : \pi(L) \to H_1(S^3 - L)$ be the Hurewicz map, and let $\varphi : H_1(S^3 - L) \to \mathbb{Z}$ be the map defined by identifying the oriented meridians of each component of L with each other. The group $\pi(L)$ is canonically an extension of $\mathbb{Z}$ by $\ker(\varphi \circ h)$ as follows:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \ker(\varphi\circ h) & \longrightarrow & \pi(L) & \xrightarrow{\varphi\circ h} & \mathbb{Z} & \longrightarrow & 1 \\
 & & & & {\scriptstyle h}\big\downarrow & \nearrow{\scriptstyle \varphi} & & & \\
 & & & & H_1(S^3 - L) & & & &
\end{array}
\tag{1.1}
$$

We call the subgroup $\ker(\varphi \circ h)$ the *Alexander subgroup* of the oriented link L. When L is a knot, the Alexander subgroup is the commutator subgroup of $\pi(L)$.

A group G is *residually torsion-free nilpotent* if for every nontrivial element $x \in G$, there is a normal subgroup $N \triangleleft G$ such that $x \notin N$ and G/N is a torsion-free nilpotent group. The residual torsion-free nilpotence of the Alexander subgroup of a link group has applications to bi-orderability [13] and ribbon concordance [10]. Several knots are known to have groups with residually torsion-free nilpotent commutator subgroups including fibered knots (since free groups are residually torsion-free nilpotent [17] and

Received by the editors July 26, 2021; revised January 5, 2023; accepted January 9, 2023.
Published online on Cambridge Core January 25, 2023.
The work was supported by NSF grants DMS-1937215 and DMS-2213213.
AMS subject classification: **57K10**, **57M05**, 06F15.
Keywords: Two-bridge knots, residual nilpotence, bi-orderable groups, Alexander subgroups.

https://doi.org/10.4153/S0008414X2300007X Published online by Cambridge University Press

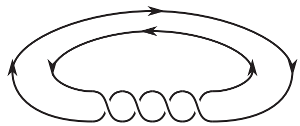

Figure 1: The $(4,2)$-torus link.

the commutator subgroup of a fibered knot group is a finitely generated free group), twist knots [18], all knots in Reidemeister's knot table (see [26]) except 8_{13}, 9_{25}, 9_{35}, 9_{38}, 9_{41}, and 9_{49} [18], and pseudo-alternating links whose Alexander polynomials have prime power leading coefficients [20]. This paper confirms that many two-bridge links, including all two-bridge knots, have groups with residually torsion-free nilpotent Alexander subgroups.

Theorem 1.1 *If L is an oriented two-bridge link whose Alexander polynomial has relatively prime coefficients (collectively, not pairwise), then the Alexander subgroup of $\pi(L)$ is residually torsion-free nilpotent.*

Remark 1.2 The condition on the coefficients of the Alexander polynomial cannot be removed. For example, if L is the $(4,2)$-torus link, as shown in Figure 1, then L has Alexander subgroup isomorphic to

$$\langle \{S_i\}_{i\in\mathbb{Z}} \mid S_i^2 = S_{i+1}^2, i \in \mathbb{Z}\rangle,$$

which is not residually nilpotent. (For details on computing the Alexander subgroup, see Section 3.) The Alexander polynomial of L is $\Delta_L(t) = 2t - 2$.

It is a well-known fact that $\Delta_K(1) = \pm 1$ for every knot K [2]. It follows that the coefficients of the Alexander polynomial of K are relatively prime, so we have the following corollary.

Corollary 1.3 *The commutator subgroup of a two-bridge knot group is residually torsion-free nilpotent.*

The following conjecture is an analog of a question by Mayland in [18].

Conjecture 1.4 *The Alexander subgroup of a link group of an alternating link is residually torsion-free nilpotent whenever the link's Alexander polynomial has relatively prime coefficients.*

1.1 Summary of the techniques used

The proof of Theorem 1.1 relies on Baumslag's work on para-free groups [3, 4]. Let G be a group. Define $\gamma_1 G := G$, and for each positive integer n, define $\gamma_{n+1}G := [G, \gamma_n G]$. A group G is *para-free of rank r* if:

(1) for some free group F of rank r, $G/\gamma_n G \cong F/\gamma_n F$ for each n, and
(2) G is residually nilpotent.

Baumslag provides a sufficient condition for a group to be residually torsion-free nilpotent.

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Proposition 1.5 (Baumslag [4, Proposition 2.1(i)]) *Suppose G is a group which is the union of an ascending chain of subgroups as follows:*

$$G_0 < G_1 < G_2 < \cdots < G_n < \cdots < G = \bigcup_{n=1}^{\infty} G_n.$$

Suppose each G_n is para-free of the same rank. If for each nonnegative integer n, $|G_{n+1} : G_n[G_{n+1}, G_{n+1}]|$ is finite, then G is residually torsion-free nilpotent.

Thus, Theorem 1.1 follows from the following lemma.

Lemma 1.6 *Suppose L is an oriented two-bridge link, and let Y be the Alexander subgroup of L. If the Alexander polynomial of L has relatively prime coefficients, then Y can be written as a union of an ascending chain of subgroups $Y_0 < Y_1 < Y_2 < \cdots < Y$ such that:*

(a) *each Y_n is para-free of the same rank and*
(b) *$|Y_{n+1} : Y_n[Y_{n+1}, Y_{n+1}]|$ is finite for each n.*

Let H be a para-free group of rank r. An element $h \in G$ is *homologically primitive* if the class of h in $H/[H, H] \cong \mathbb{Z}^r$ can be extended to a basis.

Proposition 1.7 (Baumslag [3, Proposition 3]) *Let H be a para-free group of rank r, and let $\langle x \rangle$ be an infinite cyclic group generated by x. Let h be an element in H, and let n be a positive prime integer. If h generates its own centralizer and h is homologically primitive in H, then the group*

$$H \underset{h=x^n}{*} \langle x \rangle$$

is para-free of rank r.

Proposition 1.7 can be strengthened to the following statement.

Proposition 1.8 *Let H be a para-free group of rank r, and let $\langle x \rangle$ be an infinite cyclic group generated by x. Let h be an element in H, and let n be any positive integer. If h is homologically primitive in H, then*

$$H \underset{h=x^n}{*} \langle x \rangle$$

is para-free of rank r.

Proof Let H be a para-free group of rank r, and let h be an element in H which is homologically primitive. Suppose an element $g \in H$ commutes with h, and consider, $\langle g, h \rangle$, the subgroup of H generated by g and h. A theorem of Baumslag [4, Theorem 4.2] states that any two-generator subgroup of a para-free group is free. Since g and h commute, $\langle g, h \rangle$ cannot be a rank-2 free group, so $\langle g, h \rangle$ is an infinite cyclic group. Since h is homologically primitive, it must be a generator of $\langle g, h \rangle$, so $g = h^l$ for some integer l. Therefore, h generates its own centralizer.

Let $n = p_1 \cdots p_k$ be the prime decomposition of n. Let $\langle x_1 \rangle, \ldots, \langle x_k \rangle$ be infinite cyclic groups. Define $G_0 = H$ and $x_0 = h$. Then, for $j = 1, \ldots, k$, define

$$G_j := \frac{H * \langle x_1 \rangle * \cdots * \langle x_j \rangle}{N(x_0^{-1} x_1^{p_1}, x_1^{-1} x_2^{p_2}, \ldots, x_{j-1}^{-1} x_j^{p_j})},$$

where N means the normal closure of the indicated elements. Thus,

$$G_j = G_{j-1} \underset{x_{j-1} = x_j^{p_j}}{*} \langle x_j \rangle \tag{1.2}$$

for each j. We can substitute $x_i^{p_i}$ for x_{i-1} for $i = 1, \ldots, j$ so that

$$G_j \cong H \underset{h = x_j^{n_j}}{*} \langle x_j \rangle,$$

where $n_j = p_1 \cdots p_j$.

Since h is homologically primitive in H, the class of h in H', the abelianization of H, extends to a basis $\mathcal{B}$ of $H' \cong \mathbb{Z}^r$. After adjoining a root of h to obtain G_j, H' embeds into G_j', the abelianization of G_j.

The elements in $\mathcal{B}$ remain linearly independent in G_k'. Removing the class of h from $\mathcal{B}$ and replacing it with the class of x_j produces a basis of G_j'. Therefore, x_j is homologically primitive in G_j.

Since $G_0 = H$ is para-free of rank r, inductively, each G_j is para-free of rank r by (1.2) and Proposition 1.7. Thus,

$$G_k \cong H \underset{h = x^n}{*} \langle x \rangle$$

is para-free of rank r. ∎

Mayland [19] proposes a strategy that uses the Reidemeister–Schreier rewriting process to describe the commutator subgroup of a two-bridge knot group as the union of an ascending chain of subgroups satisfying the conditions of Lemma 1.6. The first term Y_0 is a free group, and ideally, for each $n \geq 1$, Y_n is isomorphic to Y_{n-1} after adjoining roots of homologically primitive elements, in the manner of Proposition 1.8, a finite number of times. Mayland attempts to show that, for a given two-bridge knot, each Y_n is obtained by adjoining roots to Y_{n-1} using a recursive argument. However, it is not at all obvious that Mayland's recursive argument is valid. While it is straightforward to verify Mayland's argument on a case-by-case basis, proving his recursive argument works in general is quite difficult. Furthermore, there are errors in Mayland's argument that the elements, whose roots are adjoined, are homologically primitive. Unfortunately, Mayland never published a proof of his assertion. In a later paper by Mayland and Murasugi [20], it is stated that Mayland plans to present a proof using a different strategy. This paper has not appeared.

Here, we use a slightly different approach. In this paper, we use a graph-theoretic construction similar to one used by Hirasawa and Murasugi [11] to relate the Alexander subgroups of more complicated two-bridge link groups to those of simpler two-bridge link groups. Then, it is proved inductively that the Alexander subgroups of all two-bridge links can be described by adjoining roots to a free group, and we show that when two-bridge links have Alexander polynomials with relatively prime coefficients, their Alexander subgroups satisfy Lemma 1.6 via Mayland's strategy.

1.2 Application to bi-orderability

Residually torsion-free nilpotence is useful for determining when a link group is *bi-orderable*, i.e., admits a total order invariant under both left and right multiplication [7, 25, 30]. Let L be a smooth link in S^3. The link group $\pi(L)$ is an extension of $\langle t\rangle$ (an infinite cyclic group generated by t) by the Alexander subgroup Y. Let Y^{ab} denote the abelianization of Y, and let L_t be the linear map induced on $\mathbb{Q}\otimes Y^{\mathrm{ab}}$ by conjugating Y by t. The following result is shown by Linnell, Rhemtulla, and Rolfsen [13] and is stated more explicitly by Chiswell, Glass, and Wilson [6].

Theorem 1.9 (Chiswell–Glass–Wilson [6, Theorem B]) *Suppose Y is residually torsion-free nilpotent. If the dimension of $\mathbb{Q}\otimes Y^{ab}$ is finite and all the eigenvalues of L_t are real and positive, then $\pi(L)$ is bi-orderable.*

The Alexander polynomial of L, $\Delta_L(t)$, is a scalar multiple of the characteristic polynomial of L_t, and the dimension of $\mathbb{Q}\otimes Y^{\mathrm{ab}}$ is the degree of $\Delta_L(t)$ (for details, see [27, Chapter VIII]), which implies the following corollary.

Corollary 1.10 *Let L be a link in S^3. If the Alexander subgroup of L is residually torsion-free nilpotent and $\Delta_L(t)$ has all real positive roots, then $\pi(L)$ is bi-orderable.*

Remark 1.11 Linnell, Rhemtulla, and Rolfsen actually show that a weaker condition on the Alexander polynomial is sufficient for bi-orderability. However, since two bridge links are alternating, the coefficients of their Alexander polynomials alternate sign [8], so the signs of the even degree terms are all opposite to the signs of the odd degree terms. It follows that the Alexander polynomials of two-bridge links cannot have negative roots. Therefore, for a two-bridge link, having an Alexander polynomial which is "special" in the sense of Linnell, Rhemtulla, and Rolfsen [13] is equivalent to the Alexander polynomial having all real and positive roots.

By combining Theorem 1.1 with Corollary 1.10, we have the following result.

Theorem 1.12 *Let L be an oriented two-bridge link with Alexander polynomial $\Delta_L(t)$. If all the roots of $\Delta_L(t)$ are real and positive and the coefficients of $\Delta_L(t)$ are relatively prime, then the link group of L is bi-orderable. In particular, if K is a two-bridge knot and all the roots of $\Delta_K(t)$ are real and positive, then the knot group of K is bi-orderable.*

Remark 1.13 Theorem 1.12 is not true if either condition on the Alexander polynomial is removed. The link group of the $(4,2)$-torus link has presentation

$$\langle x, y | x^{-1}y^{-2}xy^2\rangle.$$

Since x and y do not commute but x commutes with y^2, the $(4,2)$-torus link does not have bi-orderable link group [24, Lemma 1.1]. As stated in Remark 1.2, the $(4,2)$-torus link, oriented as in Figure 1, has Alexander polynomial $2t-2$, which has only one real positive root but does not have relatively prime coefficients. If we reverse the orientation of one of the components, the Alexander polynomial is t^3-t^2+t-1, which has relatively prime coefficients, but no real roots.

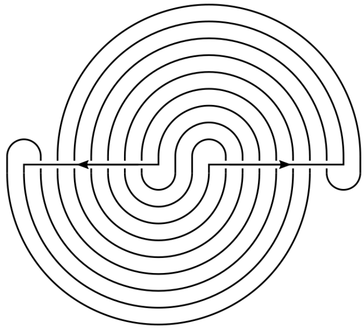

Figure 2: Schubert's projection of $L(8/3)$.

1.3 A family of bi-orderable two-bridge links

Every oriented two-bridge link is the closure of rational tangle. Thus, by Conway's correspondence, we can associate a two-bridge link to a rational fraction p/q with $p > 0$ (see [5, Chapter 12] for details). Let $L(p/q)$ denote the two-bridge link represented by p/q. Choose an orientation of $L(p/q)$ so that the two overstrands of Schubert's projection of $L(p/q)$ are oriented away from each other, as shown in Figure 2. This correspondence satisfies the following properties:

(1) $L(p/q)$ and $L(p'/q')$ are equivalent as unoriented links if and only if:
  (a) $p = p'$ and
  (b) $q \cong q' \pmod{p}$ or $qq' \cong 1 \pmod{p}$.
(2) $L(p/q)$ and $L(p'/q')$ are equivalent as oriented links if and only if:
  (a) $p = p'$ and
  (b) $q \cong q' \pmod{2p}$ or $qq' \cong 1 \pmod{2p}$.
(3) $L(p/q)$ is a knot if and only if p is odd.
(4) $L(p/q)$ and $L(-p/q)$ are mirrors.
(5) If $L(p/q)$ is a link, $L(p/(q \pm p))$ is the oriented link obtained by reversing the orientation of one of the components of $L(p/q)$.

When q is odd, there are nonzero integers $k_1, \dots, k_n$ such that $p/(p-q) = [2k_1, \dots, 2k_n]$. Here, $[2k_1, \dots, 2k_n]$ denotes the continued fraction expansion

$$[2k_1, \dots, 2k_n] = 2k_1 + \cfrac{1}{2k_2 + \cfrac{1}{2k_3 + \cfrac{1}{\cdots + \cfrac{1}{2k_n}}}}.$$

The integers $2k_1, \dots, 2k_n$ correspond to the number of twist in the rational tangle p/q (see Figure 3). See [23, Chapter 9] for details on fraction expansions and rational tangles. When n is even, $L(p/q)$ is a knot with genus $n/2$. When n is odd, $L(p/q)$ is a two-component link with genus $(n-1)/2$.

Every oriented two-bridge link is associated to a fraction p/q with q odd and $|p/q| > 1$. When $L(p/q)$ is a link, p is always even and q is always odd. Suppose $L(p/q)$ is a knot

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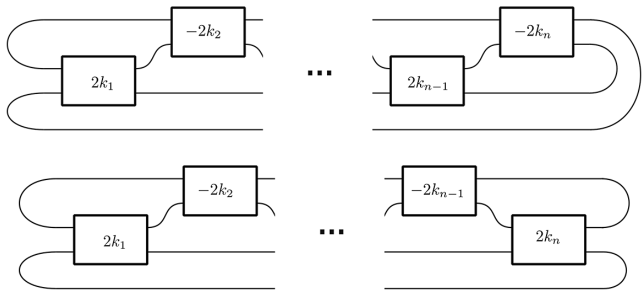

Figure 3: Rational tangle form of a two-bridge knot (top) and link (bottom).

with q even. Let q' be the inverse q modulo $2p$. Since q is even, q' is odd, and $L(p/q)$ is equivalent to $L(p/q')$. Furthermore, since $L(p/q)$ is equivalent to $L(p/(q+2pk))$ for all integers k, q can be chosen such that $-p < q < p$ so $|p/q| > 1$. Therefore, we adopt the convention that $p > |q| > 0$ and q is odd.

Chiswell, Glass, and Wilson showed that groups that admit presentations with two generators and one relator satisfying certain conditions have residually torsion-free nilpotent commutator subgroups [6]. Clay, Desmarais, and Naylor used this to show that twist knots (knots represented by $[2, 2k]$ with $k > 0$) have bi-orderable knot groups in [7]. In [30], Yamada used the same idea to extend this to the family of two-bridge links represented by $[2, 2, \dots, 2, 2k]$, where $k > 0$. Using the following result of Lyubich and Murasugi, this paper extends this family further.

Theorem 1.14 (Lyubich–Murasugi [16, Theorem 2]) *Let p/q be a fraction of co-prime integers p and q with $q \neq 0$, and let L be the two-bridge link $L(p/q)$. If for some positive integer n, $p/q = [2k_1, \dots, 2k_n]$ with $k_i > 0$ for each $i = 1, \dots, n$, then all the roots of $\Delta_L(t)$ are real and positive.*

Combining this theorem with Corollary 1.3 implies the following.

Corollary 1.15 *Let p/q be a fraction of co-prime integers p and q with $q \neq 0$, and $p/(p-q) = [2k_1, \dots, 2k_n]$ with $k_i > 0$ for each $i = 1, \dots, n$.*

If the coefficients of the Alexander polynomial of $L(p/q)$ are relatively prime, then the link group of $L(p/q)$ is bi-orderable. In particular, when $L(p/q)$ is a knot, the knot group of $L(p/q)$ is bi-orderable.

Theorem 1.14 does not characterize all two-bridge links with Alexander polynomial that have all real and positive roots.

Example 1.16 Let $K = L(81/49)$. $81/(81-49) = [2, 2, -8, -2]$,

$$\Delta_K(t) = 4t^4 - 20t^3 + 33t^2 - 20t + 4 = (t-2)^2(2t-1)^2,$$

which has two real roots of multiplicity 2. Thus, the knot group of K is bi-orderable.

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1.4 Genus one two-bridge links

Suppose L is an oriented genus-one two-bridge link $L(p/q)$. When L is a genus-one knot, $p/(p-q) = [2k_1, 2k_2]$ for some nonzero integers k_1 and k_2. The Alexander polynomial of L is

$$\Delta_L(t) = k_1k_2t^2 - (2k_1k_2 + 1)t + k_1k_2.$$

When $k_1k_2 > 0$, $\Delta_L(t)$ has two positive real roots, so $\pi(L)$ is bi-orderable by Theorem 1.12. When $k_1k_2 < 0$, $\Delta_L(t)$ has no real roots. In this case, since $\deg \Delta_L = 2$, an obstruction by Clay, Desmarais, and Naylor [7, Theorem 3.3] implies that $\pi(L)$ is not bi-orderable.

Proposition 1.17 *Suppose L is the two-bridge knot $L(p/q)$ with $p/(p-q) = [2k_1, 2k_2]$. The knot group $\pi(L)$ is bi-orderable if and only if $k_1k_2 > 0$.*

When L is a genus-one two-component link, $p/(p-q) = [2k_1, 2k_2, 2k_3]$ for some nonzero integers k_1, k_2, and k_3. The Alexander polynomial of $L(p/q)$ is

$$\begin{aligned}\Delta_L(t) &= k_1k_2k_3t^3 - (3k_1k_2k_3 + k_1 + k_3)t^2 + (3k_1k_2k_3 + k_1 + k_3)t - k_1k_2k_3\\ &= (t-1)(k_1k_2k_3t^2 - (2k_1k_2k_3 + k_1 + k_3)t + k_1k_2k_3).\end{aligned}$$

The discriminant, D, of the second factor is

$$D = 4k_1k_2k_3(k_1 + k_3) + (k_1 + k_3)^2,$$

so $D \geq 0$ if $k_1k_2k_3(k_1 + k_3) \geq 0$. It follows that $\Delta_L(t)$ has three real positive roots when $k_1k_2k_3(k_1 + k_3) \geq 0$.

Let $A = k_1k_2k_3$ and $B = 3k_1k_2k_3 + k_1 + k_3$. The coefficients of Δ_L are relatively prime precisely when $\gcd(A, B) = 1$, and $\gcd(A, B) = 1$ if and only if $\gcd(k_1, k_3) = 1$ and $\gcd(k_2, k_1 + k_3) = 1$.

Therefore, Theorem 1.12 implies the following result.

Proposition 1.18 *Suppose L is the two-component two-bridge link $L(p/q)$ with $p/(p-q) = [2k_1, 2k_2, 2k_3]$. If $\gcd(k_1, k_3) = 1$, $\gcd(k_2, k_1 + k_3) = 1$, and $k_1k_2k_3(k_1 + k_3) \geq 0$, then $\pi(L)$ is bi-orderable.*

1.5 Application to ribbon concordance

The residual torsion-free nilpotence of the commutator subgroup of a knot group has an application to ribbon concordance as well. Given two knots K_0 and K_1 in S^3, A *ribbon concordance from K_1 to K_0* is a smoothly embedded annulus C in $[0,1] \times S^3$ such that C has boundary $-(\{0\} \times K_0) \cup \{1\} \times K_1$ and C has only index 0 and 1 critical points. K_1 is said to be *ribbon concordant* to K_0, denoted $K_1 \geq K_0$, if there is a ribbon concordance from K_1 to K_0. The relation $\geq$ is clearly reflexive and transitive. Gordon [10] conjectures that $\geq$ is a partial order on knots in S^3.

Gordon gives conditions under which $\geq$ behaves antisymmetrically.

Theorem 1.19 (Gordon [10]) *If $K_0 \geq K_1$ and $K_1 \geq K_0$ and the commutator subgroup of $\pi(K_0)$ is transfinitely nilpotent, then K_0 and K_1 are ambient isotopic.*

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Remark 1.20 Transfinite nilpotence follows from residual torsion-free nilpotence (see [10] for a definition of transfinitely nilpotent).

Here, we state the following corollary.

Corollary 1.21 *If $K_1 \geq K_0$ and $K_0 \geq K_1$ and K_0 is a two-bridge knot, then K_0 and K_1 are ambient isotopic.*

Remark 1.22 Since this article's initial posting, Agol showed that Gordon's conjecture is true [1] subsuming Corollary 1.21.

1.6 Outline

The rest of this paper is devoted to the proof of Lemma 1.6. In Section 2, we illustrate the proof of Lemma 1.6 by verifying the lemma for the two-bridge knot $L(17/13)$. Section 3 investigates the properties of a presentation for the Alexander subgroup Y obtained by the Reidemeister–Schreier rewriting procedure. The proof of Lemma 1.6 is completed in Section 3.4. In Section 4, we define the cycle graph of a two-bridge link. Cycle graphs are used to prove a key lemma in Section 5.

The Appendix covers some background material on presentation matrices of modules over a principal ideal domain (PID).

2 An example

In this section, we use the two-bridge knot $K := L(17/13)$ to provide an example of the proof of Lemma 1.6. Using the Schubert normal form [29], we obtain a presentation of $\pi(K)$:

$$\pi(K) = \langle a, b \mid avb^{-1}v^{-1} \rangle,$$

where

$$v = ba^{-1}ba^{-1}b^{-1}ab^{-1}aba^{-1}ba^{-1}b^{-1}ab^{-1}a.$$

Denote the Alexander subgroup of $\pi(K)$ by Y. Using the Reidemeister–Schreier rewriting process, we obtain the following presentation of Y (see Section 3 for details):

$$Y \cong \langle \{S_k\}_{k\in\mathbb{Z}} \mid \{R_k\}_{k\in\mathbb{Z}} \rangle.$$

Here, $S_k := a^k ba^{-k-1}$ and the relators R_k are defined as follows:

$$\vdots$$

$$\begin{aligned}
R_{-1} &:= S_0S_0S_{-1}^{-1}S_{-1}^{-1}S_0S_0S_{-1}^{-1}S_{-1}^{-1}S_{-1}^{-1}S_{-2}S_{-2}S_{-1}^{-1}S_{-1}^{-1}S_{-2}S_{-2}S_{-1}^{-1}S_{-1}^{-1},\\
R_0 &:= S_1S_1S_0^{-1}S_0^{-1}S_1S_1S_0^{-1}S_0^{-1}S_0^{-1}S_{-1}S_{-1}S_0^{-1}S_0^{-1}S_{-1}S_{-1}S_0^{-1}S_0^{-1},\\
R_1 &:= S_2S_2S_1^{-1}S_1^{-1}S_2S_2S_1^{-1}S_1^{-1}S_1^{-1}S_0S_0S_1^{-1}S_1^{-1}S_0S_0S_1^{-1}S_1^{-1},
\end{aligned}$$

$$\vdots$$

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Define a sequence of groups $\{Y_n\}_{n=0}^{\infty}$ as follows:

$$\begin{aligned} Y_0 &:= \langle S_{-1}, S_0 \mid \varnothing \rangle, \\ Y_1 &:= \langle S_{-2}, S_{-1}, S_0, S_1 \mid R_{-1}, R_0 \rangle, \\ Y_2 &:= \langle S_{-3}, S_{-2}, S_{-1}, S_0, S_1, S_2 \mid R_{-2}, R_{-1}, R_0, R_1 \rangle, \\ &\vdots \end{aligned}$$

Define $\widehat{A}_1$, $\widehat{A}_2$, $\widehat{V}_1$, and $\widehat{V}_2$ as follows:

$$\begin{aligned} \widehat{A}_1 &= S_1^2 S_0^{-2}, \\ \widehat{A}_2 &= S_1, \\ \widehat{V}_1 &= S_0^{-1} S_{-1}^2 S_0^{-2} S_{-1}^2 S_0^{-2}, \\ \widehat{V}_2 &= S_0^{-2}. \end{aligned} \tag{2.1}$$

Let H_1 be the group obtained by adjoining a square root of $\widehat{V}_1^{-1}$ to Y_0 as follows:

$$H_1 := Y_0 \underset{\widehat{V}_1^{-1} = t_1^2}{*} \langle t_1 \rangle.$$

Similarly, let H_2 be the group obtained by adjoining a square root of $t_1 \widehat{V}_2^{-1}$ to H_1 as follows:

$$H_2 := H_1 \underset{t_1 \widehat{V}_2^{-1} = S_1^2}{*} \langle S_1 \rangle.$$

Thus, H_2 has the following group presentation:

$$\begin{aligned} H_2 &\cong \langle S_{-1}, S_0, S_1, t_1 \mid t_1^2 \widehat{V}_1 = 1, t_1 = S_1^2 \widehat{V}_2 \rangle \\ &\cong \langle S_{-1}, S_0, S_1 \mid (S_1^2 \widehat{V}_2)^2 \widehat{V}_1 = 1, \rangle \\ &\cong \langle S_{-1}, S_0, S_1 \mid R_0 \rangle. \end{aligned}$$

Define $\check{A}_1$, $\check{A}_2$, $\check{V}_1$, and $\check{V}_2$ as follows:

$$\begin{aligned} \check{A}_1 &= S_{-2}^2 S_{-1}^{-2}, \\ \check{A}_2 &= S_{-2}, \\ \check{V}_1 &= S_0^2 S_{-1}^{-2} S_0^2 S_{-1}^{-3}, \\ \check{V}_2 &= S_{-1}^{-2}. \end{aligned} \tag{2.2}$$

Let H_3 be the group obtained by adjoining a square root of $\check{V}_1^{-1}$ to H_2:

$$H_3 := H_2 \underset{\check{V}_1^{-1} = t_2^2}{*} \langle t_2 \rangle.$$

Let H_4 be the group obtained by adjoining a square root of $t_2 \check{V}_2^{-1}$ to H_3:

$$H_4 := H_3 \underset{t_2 \check{V}_2^{-1} = S_{-2}^2}{*} \langle S_{-2} \rangle.$$

Therefore, H_4 is isomorphic to Y_1:

$$\begin{aligned} H_4 &\cong \langle S_{-2}, S_{-1}, S_0, S_1, t_2 \mid R_0, \check{V}_1 t_2^2 = 1, t_2 = S_{-2}^2 \check{V}_2 \rangle \\ &\cong \langle S_{-2}, S_{-1}, S_0, S_1 \mid R_{-1}, R_0 \rangle \\ &\cong Y_1. \end{aligned}$$

In conclusion, Y_1 is Y_0 after adjoining roots four times, and since $R_{n\pm1}$ is R_n with all the subscripts changed by ±1, Y_{n+1} is Y_n after adjoining roots four times. Thus, for each n, Y_n embeds into Y_{n+1}. Therefore, Y is the union of an ascending chain of subgroups as follows:

$$Y_0 < Y_1 < \cdots < Y = \bigcup_{n=0}^{\infty} Y_n.$$

By Proposition 1.5, if each Y_n is para-free of the same rank, then Y is residually torsion-free nilpotent. Y_0 is clearly para-free of rank 2 since it is a rank-2 free group. We need to verify that each time we adjoin a root of an element, that element is homologically primitive. Then, by Proposition 1.8, we can conclude that each Y_n is also para-free of rank 2.

Claim For each $n \geq 0$, if Y_n is para-free of rank 2, then so is Y_{n+1}.

Proof Let n be a nonnegative integer, and suppose Y_n is para-free of rank 2. In an abuse of notation, let $\widehat{A}_1, \widehat{A}_2, \widehat{V}_1$, and $\widehat{V}_2$ be as defined in (2.1) except with the subscripts of each S_i increased by n. Similarly, let $\check{A}_1, \check{A}_2, \check{V}_1$, and $\check{V}_2$ be as defined in (2.2) except with the subscripts of each S_i decreased by n. Also, let H_1, H_2, H_3, and H_4 be the groups obtained by adjoining square roots of $\widehat{V}_1^{-1}$, $t_1\widehat{V}_2^{-1}$, $\check{V}_1^{-1}$, and $t_2\check{V}_2^{-1}$ to Y_n as before.

Let Y_n^{ab} denote the abelianization of Y_n, and let B_1 be the quotient of Y_n^{ab} obtained by killing the class of $\widehat{V}_1^{-1}$ in Y_n^{ab}. Since Y_n is para-free of rank 2, $Y_n^{\mathrm{ab}} \cong \mathbb{Z} \oplus \mathbb{Z}$. Thus,

$$B_1 \cong \mathbb{Z} \oplus \frac{\mathbb{Z}}{C\mathbb{Z}}$$

for some integer C.

Now, we view Y_n^{ab} as a $\mathbb{Z}$-module and use addition as the group operation. Y_n^{ab} is generated by $S'_{-n-1}, S'_{-n}, \ldots, S'_n$, where S'_i denotes the class of S_i in Y_n^{ab}. Using this generating set, Y_n^{ab} has a $(2n) \times (2n+2)$ presentation matrix:

$$\begin{pmatrix} 4 & -9 & 4 & & & \\ & 4 & -9 & 4 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 4 & -9 & 4 \end{pmatrix}.$$

Throughout this paper, missing entries in matrices are zeros. See the Appendix for definition and background on presentation matrices. The class of $\widehat{V}_1^{-1}$ in Y_n^{ab} is

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$-4S'_{n-1}+5S'_n$. Thus, B_1 has the following $(2n+1)\times(2n+2)$ presentation matrix, which we will also call B_1:

$$B_1=\begin{pmatrix} 4 & -9 & 4 & & & \\ & 4 & -9 & 4 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 4 & -9 & 4 \\ & & & & -4 & 5 \end{pmatrix}.$$

By Lemma A.1, the integer C is the greatest common divisor of the determinants of every $(2n+1)\times(2n+1)$ minor of B_1. By deleting the last column, we get a square minor of B_1 with determinant -4^{2n+1}. However, by deleting the first column, we see B_1 has a minor with odd determinant. (Modulo 2, the matrix obtained from B_1 by deleting the first column is the identity matrix.) Thus, $C=1$.

Therefore, B_1 is a rank-1 free abelian group. It follows that $\widehat{V}_1^{-1}$ is homologically primitive in Y_n, and H_1 is para-free of rank 2 by Proposition 1.8.

Let B_2 be the quotient of H_1^{ab} obtained by killing the class of $t_1\widehat{V}_2^{-1}$ in H_1^{ab}, the abelianization of H_1. H_1^{ab} is generated by $S'_{-n-1}, S'_{-n}, \ldots, S'_n, t'_1$, where t'_1 is the class of t_1 in H_1^{ab}. H_1^{ab} has a $(2n+1)\times(2n+3)$ presentation matrix:

$$\begin{pmatrix} 4 & -9 & 4 & & & & \\ & 4 & -9 & 4 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 4 & -9 & 4 & \\ & & & & -4 & 5 & 2 \end{pmatrix}.$$

The class of $t_1\widehat{V}_2^{-1}$ in H_1^{ab} is $2S'_n+t'_1$. Thus, B_2 has the following $(2n+2)\times(2n+3)$ presentation matrix:

$$B_2=\begin{pmatrix} 4 & -9 & 4 & & & & \\ & 4 & -9 & 4 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 4 & -9 & 4 & \\ & & & & 4 & -5 & 2 \\ & & & & & 2 & 1 \end{pmatrix}.$$

Using the 1 in the bottom-right corner, we apply a row and a column operation. Then, we kill the last row and column to get the following presentation matrix:

$$B_2\cong\begin{pmatrix} 4 & -9 & 4 & & & \\ & 4 & -9 & 4 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 4 & -9 & 4 \\ & & & & 4 & -9 \end{pmatrix}.$$

Thus, B_2 is a rank-1 free abelian group, by an argument similar to the one used for B_1. It follows that $t_1\widehat{V}_2^{-1}$ is homologically primitive in H_1, and H_2 is para-free of rank 2 by Proposition 1.8.

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Similarly, $\check{V}_1^{-1}$ and $t_2\check{V}_2^{-1}$ are homologically primitive in H_2 and H_3, respectively. Therefore, $H_4 \cong Y_{n+1}$ is para-free of rank 2. ∎

For any group G, if H is G with an nth root adjoined, then

$$H/G \cong \mathbb{Z}/n\mathbb{Z},$$

so $|H : G[H,H]| = |H : G| = n$. Thus, since for each n, Y_{n+1} is Y_n with square roots adjoined four times, $|Y_{n+1} : Y_n[Y_{n+1}, Y_{n+1}]| = 16$.

Since Y_0 is para-free of rank 2, each Y_n is para-free of rank 2 by induction. Therefore, Y is residually torsion-free nilpotent by Proposition 1.5.

3 A group presentation of the Alexander subgroup

In this section, we give a group presentation of the Alexander subgroup of an arbitrary two-bridge link group using the Reidemeister–Schreier rewriting process. From this presentation of the Alexander subgroup, we can describe the subgroup as the union of an ascending chain of subgroups which satisfy conditions (a) and (b) of Lemma 1.6 when the Alexander polynomial of the link has relatively prime coefficients.

3.1 A presentation from Reidemeister–Schreier

Consider the two-bridge link $L := L(p/q)$ where $1 \le |q| < p$ with q odd. For each integer i, define

$$\varepsilon_i := (-1)^{\lfloor \frac{iq}{p} \rfloor}. \tag{3.1}$$

Proposition 3.1 (Schubert [29]) *Given the two-bridge link $L(p/q)$,*

$$\pi(L(p/q)) \cong \langle a, b | w \rangle,$$

where a and b are classes of meridians of $L(p/q)$ and $w = a^{\varepsilon_0} b^{\varepsilon_1} \dots a^{\varepsilon_{2p-2}} b^{\varepsilon_{2p-1}}$.

Let Y be the Alexander subgroup of L. A group presentation for Y can be obtained using the Reidemeister–Schreier rewriting procedure, developed by Reidemeister [26] and Schreier [28]. The Reidemeister–Schreier rewriting procedure is described in detail in Section 2.3 of the text by Karrass, Magnus, and Solitar [12]. The application of this procedure to the situation at hand is discussed below.

Under the map $\varphi \circ h : \pi(L) \to \pi(L)/Y \cong \mathbb{Z}$ from (1.1), a and b are both sent to 1 or both sent to -1. Consider $\mathcal{A} := \{a^k\}_{k \in \mathbb{Z}}$ as a set of coset representatives for $\pi(L)/Y$. Given an element x in $\pi(L)$, let $\overline{x}$ be the coset representative of x in $\mathcal{A}$. For each $x \in \{a, b\}$ and $k \in \mathbb{Z}$, define

$$\gamma(a^k, x) := a^k x (\overline{a^k x})^{-1}.$$

Note that $\gamma(a^k, a) = 1$ and $\gamma(a^k, b) = a^k b a^{-k-1}$. Given a word $u = x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}$ with $x_i \in \{a, b\}$ and $s_i \in \{1, -1\}$ for all i, define

$$\tau(u) := \gamma(\overline{t_1}, x_1)^{s_1} \gamma(\overline{t_2}, x_2)^{s_2} \cdots \gamma(\overline{t_n}, x_n)^{s_n},$$

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where

$$t_i := \begin{cases} x_1^{s_1}\cdots x_{i-1}^{s_{i-1}} \text{ (possibly trivial)}, & s_i = 1, \\ x_1^{s_1}\cdots x_i^{s_i}, & s_i = -1. \end{cases}$$

For each integer k, define

$$S_k := \gamma(a^k, b)$$

and define

$$\mathcal{S} := \{S_k\}_{k\in\mathbb{Z}}.$$

Since, for all k, $\gamma(a^k, a) = 1$, for each word u, $\tau(u)$ is a product $S_{k_1}S_{k_2}\cdots S_{k_l}$. For each integer k, define

$$R_k := \tau(a^k w a^{-k}).$$

Define

$$\sigma_i := \begin{cases} \sum_{j=0}^{i-1}\varepsilon_j, & \text{when } i > 0, \\ \sum_{j=i}^{-1}\varepsilon_j, & \text{when } i < 0, \\ 0, & \text{when } i = 0, \end{cases} \tag{3.2}$$

for each integer i.

Proposition 3.2 *Suppose $R_0 = \tau(w) = S_{i_1}^{\eta_1}S_{i_2}^{\eta_2}\dots S_{i_n}^{\eta_n}$, where each i_j is an integer and each η_j is ± 1. Then:*

(a) *$n = p$.*
(b) *$\eta_j = \varepsilon_{2j-1}$, for each $j = 1,\dots,p$.*
(c) *$i_j = \sigma_{2j}$ if $\eta_j = 1$ and $i_j = \sigma_{2j+1}$ if $\eta_j = -1$ for each $j = 1,\dots,p$.*
(d) *For every integer k, $R_k = S_{i_1+k}^{\eta_1}S_{i_2+k}^{\eta_2}\dots S_{i_p+k}^{\eta_p}$.*

Proof Since $\gamma(a^k, a)$ is trivial, the S_i-generators in R_0 come from the b-generators in w. For (a), notice that the length of the word R_0 is the number of times b and b^{-1} appear in w, which is equal to p. By definition, η_j is equal to the exponent of the corresponding b or b^{-1} in w, which is ε_{2j-1} showing (b). Since $a = b$ modulo Y, then for any word u in a and b, $\overline{u} = a^s$ where s is the sum of the exponents of the a's and b's in u. Thus, both (c) and (d) follow by a straightforward computation. ■

Proposition 3.3 (Karrass–Magnus–Solitar [12, Theorem 2.9])

$$Y \cong \langle \{S_k\}_{k\in\mathbb{Z}} \mid \{R_k\}_{k\in\mathbb{Z}}\rangle.$$

3.2 Group presentation properties

This group presentation of Y has a few notable properties which will be of use.

Given a word W in $\mathcal{S}$, let $[W]$ denote the class of W in the free abelian group generated by $\mathcal{S}$. For each integer k, define $S'_k := [S_k]$. Denote the maximal and minimal subscripts of S appearing in the word R_0 by M and m, respectively, so that

$$[R_0] = a_M S'_M + a_{M-1}S'_{M-1} + \cdots + a_{m+1}S'_{m+1} + a_m S'_m$$

for some integers $a_m,\dots,a_M$.

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Proposition 3.4 *Suppose L is a two-bridge link, and suppose Y is the Alexander subgroup of L with presentation as defined in Section 3.1.*

(a) *For each integer n,*

$$[R_n] = a_M S'_{M+n} + a_{M-1} S'_{M-1+n} + \cdots + a_{m+1} S'_{m+1+n} + a_n S'_{m+n}.$$

(b) *Let g be the genus of L. When L is a knot, $M - m = 2g$, and when L is a link, $M - m = 2g + 1$.*

(c) *For all $j = m, \ldots, M$,*

$$a_j = \begin{cases} \underline{a}_{g+m-j}, & \text{if } m \le j \le m+g, \\ \underline{a}_{g+j-M}, & \text{if } M-g \le j \le M, \end{cases}$$

where

$$\Delta_L(t) = \underline{a}_g t^{2g} + \cdots + \underline{a}_0 t^g + \cdots + \underline{a}_g$$

when L is a knot, and

$$\Delta_L(t) = \underline{a}_g t^{2g+1} + \cdots + \underline{a}_0 t^{g+1} + \underline{a}_0 t^g + \cdots + \underline{a}_g$$

when L is a link. In particular, for all $j = 0, \ldots, M - m$,

$$a_{M-j} = a_{m+j}.$$

Proof Part (a) follows from Proposition 3.2(d).

For each $i = 1, \ldots, 2p$, denote by w_i the word obtained from the first i generators of the relation w. Also, define

$$\theta(s) := \begin{cases} 1, & \text{if } s = 1, \\ 0, & \text{if } s = -1. \end{cases}$$

We compute the Alexander polynomial by performing Fox calculus on w with respect to b (see [9, Section 3]):

$$\begin{aligned} \frac{\partial w}{\partial b} &= a^{\varepsilon_0}\left(\frac{\partial}{\partial b}(b^{\varepsilon_1}) + b^{\varepsilon_1} a^{\varepsilon_2}\left(\frac{\partial}{\partial b}(b^{\varepsilon_3})\right) + \cdots + b^{\varepsilon_{2p-3}} a^{\varepsilon_{2p-2}}\left(\frac{\partial}{\partial b}(b^{\varepsilon_{2p-1}})\right)\cdots\right) \\ &= \sum_{i=1}^{p} w_{2i-1} \frac{\partial}{\partial b}(b^{\varepsilon_{2i-1}}) \\ &= \sum_{i=1}^{p} \varepsilon_{2i-1} w_{f(i)}, \end{aligned}$$

where

$$f(i) = 2i - \theta(\varepsilon_{2i-1}).$$

For each $i = 1, \ldots, 2p$, $\overline{w_i} = a^{\sigma_i}$. Let $t = \overline{a} = \overline{b}$. Up to multiplication by powers of t,

$$\Delta_L(t) = \varphi'\left(\frac{\partial w}{\partial b}\right) = \sum_{i=1}^{p} \varepsilon_{2i-1} t^{\sigma_{f(i)}}, \tag{3.3}$$

where $\varphi' : \mathbb{Z}[\pi(L)] \to \mathbb{Z}[t]$ is the map induced by $\varphi \circ h$.

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By Proposition 3.2,

$$R_k = S^{\varepsilon_1}_{\sigma_{f(1)}+k} S^{\varepsilon_3}_{\sigma_{f(2)}+k} \cdots S^{\varepsilon_{2p-1}}_{\sigma_{f(p)}+k},$$

so

$$\begin{aligned}[R_k] &= \varepsilon_1 S'_{\sigma_{f(1)}+k} + \varepsilon_3 S'_{\sigma_{f(2)}+k} + \cdots + \varepsilon_{2p-1} S'_{\sigma_{f(p)}+k} \\ &= \sum_{i=1}^{p} \varepsilon_{2i-1} S'_{\sigma_{f(i)}+k}.\end{aligned} \tag{3.4}$$

The degree of Δ_L is $2g$ when L is a knot and $2g+1$ when L is a link [8, 21, 22]. Thus, parts (b) and (c) follow from (3.3) and (3.4). ■

3.3 An ascending chain of subgroups

With the group presentation from Proposition 3.3, we can describe Y as an ascending chain of subgroups.

Define Y_0 to be the free group

$$Y_0 := \langle S_m, S_{m+1}, \dots, S_{M-1} \,|\, \varnothing \rangle, \tag{3.5}$$

and define Y_n to be the group with presentation

$$Y_n := \langle S_{m-n}, S_{m-n+1}, \dots, S_{M+n-1} \mid R_{-n}, \dots, R_{n-1} \rangle \tag{3.6}$$

for each positive integer n.

Y_{n+1} is Y_n with two extra generators, S_{m-n-1} and S_{M+n}, and two extra relators, R_{-n-1} and R_n. It turns out that all of the appearances of S_{M+n} in R_n are contained in nested repeating patterns of words. Similarly, all of the appearances of S_{m-n-1} in R_{-n-1} are contained in nested repeating patterns of words. Given an explicit two-bridge link, one can find these patterns easily, as we did in Section 2 for $L(17/13)$, yet showing that these patterns exist for an arbitrary two-bridge link is much more complicated.

Once it is established that these patterns exist, however, it follows that for each nonnegative integer n, Y_{n+1} is Y_n after adjoining roots a finite number of times. This implies that each Y_n embeds into Y_{n+1}. Since Y is the direct limit of the sequence of Y_n's, Y is the union of the ascending chain of Y_n's. When the coefficients of Δ_L are relatively prime, the elements whose roots are adjoining are homologically primitive.

The following lemma explicitly describes the relator R_n as nested patterns of repeating words. For simplicity of notation, let $\delta = \pm 1$.

Lemma 3.5 *For each integer n, there exist a positive integer N, sequences of words in* $\mathcal{S}$,

$$\widehat{A}_0, \widehat{A}_1, \dots, \widehat{A}_N,$$

$$\widehat{V}_1, \dots, \widehat{V}_N,$$

and

$$\widehat{W}_1, \dots, \widehat{W}_N,$$

and a sequence of positive integers $n_1, \dots, n_N$ *such that all of the following hold:*

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(M1) $\widehat{A}_0$ *is a cyclic permutation of* R_n.

(M2) $\widehat{A}_N = S^{\delta}_{M+n}$.

(M3) *For each* $i = 1, \dots, N$,

$$\widehat{W}_i^{-1}\widehat{A}_{i-1}\widehat{W}_i = \widehat{A}_i^{n_i}\widehat{V}_i.$$

(M4) *For each* $i = 1, \dots, N$, $\widehat{V}_i$ *and* $\widehat{W}_i$ *are contained in the subgroup generated by the set*

$$\{S_{m+n}, S_{m+n+1}, \dots, S_{M+n-1}\}.$$

(M5) *For each* $i = 1, \dots, N$, *there is some* l *with* $m < l \le M$ *and integers* $b_l, \dots, b_M$ *(which depend on i) such that*

$$[\widehat{A}_i] = \sum_{j=l}^{M} b_j S'_{j+n} = b_l S'_{l+n} + b_{l+1}S'_{l+n+1} + \cdots + b_M S'_{M+n}$$

with $|b_{l+j}| = |b_{M-j}|$.

Also, there are sequences

$$\check{A}_0, \check{A}_1, \dots, \check{A}_N,$$
$$\check{V}_1, \dots, \check{V}_N,$$

and

$$\widecheck{W}_1, \dots, \widecheck{W}_N,$$

such that:

(m1) $\check{A}_0$ *is a cyclic permutation of* R_n.

(m2) $\check{A}_N = S^{\delta}_{m+n}$.

(m3) *For each* $i = 1, \dots, N$,

$$\widecheck{W}_i^{-1}\check{A}_{i-1}\widecheck{W}_i = \check{A}_i^{n_i}\check{V}_i.$$

(m4) *For each* $i = 1, \dots, N$, $\check{V}_i$ *and* $\widecheck{W}_i$ *are contained in the subgroup generated by the set*

$$\{S_{m+n+1}, \dots, S_{M+n-1}, S_{M+n}\}.$$

(m5) *For each* $i = 1, \dots, N$, *there is some* l' *with* $m \le l' < M$, *and integers* $b_m, \dots, b_{l'}$ *(which depend on i) such that*

$$[\check{A}_i] = \sum_{j=m}^{l'} b_j S'_{j+n} = b_m S'_{m+n} + \cdots + b_{l'} S'_{l'+n}$$

with $|b_{m+j}| = |b_{l'-j}|$.

Remark 3.6 Y_1 is obtained from Y_0 by adding $2N$ roots. In order of increasing index, each $\widehat{A}_i$ is added as the n_ith root of some element, then each $\check{A}_i$ is added as an n_ith root. The conditions (M5) and (m5) are used to show that the elements whose roots are added are homologically primitive.

Lemma 3.5 is proved in Section 5.7.

Proposition 3.7 *The Alexander subgroup Y of any oriented two-bridge link is a union of an ascending chain of subgroups*

$$Y_0 < Y_1 < Y_2 < \cdots < Y_i < \cdots < \bigcup_{n=1}^{\infty} Y_n \cong Y,$$

where Y_{n+1} is obtained from Y_n by adjoining a finite number of roots.

Proof Define the sequence $Y_0, Y_1, Y_2, \ldots$ as in (3.5) and (3.6). Consider Y_n for some nonnegative integer n:

$$Y_n = \langle S_{m-n}, \ldots, S_{M+n-1} \mid R_{-n}, \ldots, R_{n-1} \rangle$$

and

$$Y_{n+1} = \langle S_{m-n-1}, \ldots, S_{M+n} \mid R_{-n-1}, \ldots, R_n \rangle.$$

By Lemma 3.5, there are an integer N, sequences of words

$$\widehat{A}_0, \ldots, \widehat{A}_N,$$

$$\widehat{V}_1, \ldots, \widehat{V}_N,$$

and

$$\widehat{W}_1, \ldots, \widehat{W}_N,$$

and a sequence of integers

$$n_1, \ldots, n_N$$

satisfying (M1)–(M4).

Let $\langle t_i \rangle$ be an infinite cyclic group generated by t_i for each $i = 1, \ldots, N$. Also, let t_0 be the identity element of Y_n.

Define

$$H_0 = Y_n, \tag{3.7}$$

and for each $i = 1, \ldots, N$, recursively define

$$H_i := H_{i-1} \underset{\widehat{h}_i = t_i^{n_i}}{*} \langle t_i \rangle, \tag{3.8}$$

where

$$\widehat{h}_i = \widehat{W}_i^{-1} t_{i-1} \widehat{W}_i \widehat{V}_i^{-1}. \tag{3.9}$$

We know that $\widehat{h}_i$ is an element of H_{i-1} since $\widehat{V}_i$ and $\widehat{W}_i$ only use generators in $\{S_{m+n}, \ldots, S_{M+n-1}\}$ by Lemma 3.5(M4).

We can write the following presentation for H_N:

$$H_N \cong \langle S_{m-n}, \ldots, S_{M+n-1}, t_1, \ldots, t_N \mid R_{-n}, \ldots, R_{n-1}, \{\widehat{h}_i^{-1} t_i^{n_i}\}_{i=1}^{N} \rangle. \tag{3.10}$$

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For each $i = 1, \dots, N$, $\widehat{h}_i^{-1} t_i^{n_i} = 1$, so by (3.9),

$$1 = t_{i-1}^{-1} \widehat{W}_i t_i^{n_i} \widehat{V}_i \widehat{W}_i^{-1}. \tag{3.11}$$

Now, we find a new presentation of H_N by altering the one in (3.10). Since by Lemma 3.5(M2), $\widehat{A}_N$ is S_{M+n}^{δ}, we can add the generator S_{M+n} and identify it with t_N^{δ} by adding the relation $t_N^{-1}\widehat{A}_N$. By backward substitution using Lemma 3.5(M3) and (3.11),

$$t_{i-1} = \widehat{W}_i \widehat{A}_i^{n_i} \widehat{V}_i \widehat{W}_i^{-1} = \widehat{A}_{i-1}$$

for each $i = N, \dots, 1$. Thus, each of the relations $\widehat{h}_i^{-1} t_i^{n_i}$ in (3.10) is equivalent to $t_i^{-1} A_i$ for $i = 0, \dots, N-1$. In particular, since t_0 is trivial, $A_0 = 1$. After these alterations, H_N has the following presentation:

$$H_N \cong \langle S_{m-n}, \dots, S_{M+n}, t_1, \dots, t_N \mid R_{-n}, \dots, R_{n-1}, \widehat{A}_0, t_1^{-1}\widehat{A}_1, \dots, t_N^{-1}\widehat{A}_N \rangle.$$

We can now use the relations $t_1^{-1}\widehat{A}_1, \dots, t_N^{-1}\widehat{A}_N$ to eliminate the generators $t_1, \dots, t_N$. Since A_0 is a cyclic permutation of R_n by Lemma 3.5(M1), A_0 can be replaced by R_n producing the following presentation:

$$H_N \cong \langle S_{m-n}, \dots, S_{M+n} \mid R_{-n}, \dots, R_n \rangle.$$

Likewise, by Lemma 3.5, there are sequences of words

$$\check{A}_0, \dots, \check{A}_N,$$
$$\check{V}_1, \dots, \check{V}_N,$$

and

$$\widecheck{W}_1, \dots, \widecheck{W}_N,$$

satisfying (m1)–(m4).

For each $i = 1, \dots, N$, define

$$H_{i+N} := H_{i+N-1} \underset{\check{h}_i = t_i^{n_i}}{*} \langle t_i \rangle, \tag{3.12}$$

where

$$\check{h}_i = \widecheck{W}_i^{-1} t_{i-1} \widecheck{W}_i \check{V}_i^{-1},$$

$$H_{2N} \cong \langle S_{m-n}, \dots, S_{M+n}, t_1, \dots, t_N \mid R_{-n}, \dots, R_n, \{\check{h}_i^{-1} t_i^{n_i}\}_{i=1}^N \rangle.$$

We can identify t_N with $\check{A}_N$ which is S_{m-n-1}^{δ} by Lemma 3.5(m2). By backward substitution using (m1)–(m3) of Lemma 3.5,

$$\begin{aligned} H_{2N} &\cong \langle S_{m-n-1}, \dots, S_{M+n}, t_1, \dots, t_N \mid R_{-n}, \dots, R_{n-1}, \check{A}_0, t_1^{-1}\check{A}_1, \dots, t_N^{-1}\check{A}_N \rangle \\ &\cong \langle S_{m-n-1}, \dots, S_{M+n} \mid R_{-(n+1)}, \dots, R_n \rangle \\ &\cong Y_{n+1}. \end{aligned} \tag{3.13}$$

Consider Y_n and Y_{n+1} for a nonnegative integer n. For each $i = 0, \dots, 2N-1$, H_i embeds into H_{i+1} since H_{i+1} is a free product of H_i and $\mathbb{Z}$ amalgamated along infinite cyclic subgroups. Let $\varphi_i : H_i \to H_{i+1}$ be the embedding which maps $S_k \mapsto S_k$ and $t_k \mapsto$

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t_k for all k. The composition $f_n = \varphi_{2N-1} \circ \cdots \circ \varphi_0$ is an embedding of Y_n into Y_{n+1} which maps $S_k \mapsto S_k$ for all k.

Thus, we have the following sequence of embeddings:

$$Y_0 \xhookrightarrow{f_0} Y_1 \xhookrightarrow{f_1} Y_2 \xhookrightarrow{f_2} \cdots \xhookrightarrow{f_{n-1}} Y_n \xhookrightarrow{f_n} \cdots.$$

The Alexander subgroup Y is the direct limit of this sequence. Since each f_n is an embedding, Y is a union of an ascending chain of subgroups as desired. ■

3.4 Proof of Lemma 1.6

We now turn our attention to proving Lemma 1.6. First, we state a more precise and detailed version of Lemma 1.6.

Lemma 3.8 *Suppose that Y is the Alexander subgroup of a two-bridge link whose Alexander polynomial has relatively prime coefficients so that Y is an ascending chain of subgroups*

$$Y_0 < Y_1 < Y_2 < \cdots < Y = \bigcup_{n=1}^{\infty} Y_n$$

as defined in (3.5) and (3.6). For each n :

(a) *Y_n is para-free of the rank $M - m$ and*
(b) *$|Y_{n+1} : Y_n[Y_{n+1}, Y_{n+1}]| = \underline{a}_g^2$, where $\underline{a}_g$ is the leading coefficient of the Alexander polynomial of L.*

Proof First, we show (a). Y_0 is a para-free of rank $M - m$ since it is a rank $M - m$ free group. Suppose that for some $n \geq 0$, Y_n is para-free of rank $M - m$. By Lemma 3.5, there is are integer N, sequences of words

$$\widehat{A}_0, \ldots, \widehat{A}_N,$$
$$\widehat{V}_1, \ldots, \widehat{V}_N,$$

and

$$\widehat{W}_1, \ldots, \widehat{W}_N,$$

and a sequence of integers

$$n_1, \ldots, n_N,$$

satisfying (M1)–(M4).

Define $H_0, \ldots, H_{2N}$ as in (3.7), (3.8), and (3.12), so $H_{2N} \cong Y_{n+1}$ as in (3.13).

Suppose H_{k-1} is para-free of rank $M - m$ for some k such that $0 < k \leq N$, so $H_{k-1}^{\mathrm{ab}} \cong \mathbb{Z}^{M-m}$. Define

$$B := \frac{H_{k-1}}{\langle \widehat{h}_k \rangle [H_{k-1}, H_{k-1}]} \cong \mathbb{Z}^{M-m-1} \oplus \frac{\mathbb{Z}}{C\mathbb{Z}},$$

where

$$\widehat{h}_k = \widehat{W}_k^{-1} t_{k-1} \widehat{W}_k \widehat{V}_k^{-1}$$

and C is an integer. If $B \cong \mathbb{Z}^{M-m-1}$, then $\widehat{h}_k$ is homologically primitive in H_{k-1}, and inductively, by Proposition 1.8, each H_k is para-free of rank $M-m$.

By Proposition 3.4, $H_0^{\text{ab}} = Y_n^{\text{ab}}$ has a $2n \times (2n+M-m)$ presentation matrix:

$$\begin{pmatrix} a_m & a_{m+1} & \cdots & a_{M-1} & a_M & & & & \\ & \ddots & & \ddots & & \ddots & & & \\ & & & a_m & a_{m+1} & \cdots & a_{M-1} & a_M \end{pmatrix}.$$

H_{k-1} is H_0 with the n_j root of $\widehat{h}_j$ added for each $j = 1, \ldots, k-1$. Thus, B is H_0^{ab} after killing the classes $[\widehat{h}_j^{-1} t_j^{n_j}]$ for each $j = 1, \ldots, k-1$ and killing the class $[\widehat{h}_k^{-1}]$. B is generated by $S'_{m-n}, \ldots, S'_{M+n-1}, t'_1, \ldots, t'_{k-1}$, where t'_j is the class $[t_j]$. Using these generators, B has the following $(2n+k) \times (2n+k+M-m-1)$ presentation matrix:

$$\begin{pmatrix} a_m & a_{m+1} & \cdots & a_{M-1} & a_M & & & & & & \\ & \ddots & & \ddots & & \ddots & & & & & \\ & & a_m & a_{m+1} & \cdots & a_{M-1} & a_M & & & & \\ & & 0 & \longleftarrow & [\widehat{V}_1] & \longrightarrow & n_1 & & & & \\ & & 0 & \longleftarrow & [\widehat{V}_2] & \longrightarrow & -1 & n_2 & & & \\ & & 0 & \longleftarrow & [\widehat{V}_3] & \longrightarrow & 0 & -1 & n_3 & & \\ & & & & \vdots & & & & \ddots & \ddots & \\ & & 0 & \longleftarrow & [\widehat{V}_{k-1}] & \longrightarrow & 0 & \cdots & 0 & -1 & n_{k-1} \\ & & 0 & \longleftarrow & [\widehat{V}_k] & \longrightarrow & 0 & & \cdots & 0 & -1 \end{pmatrix}.$$

Applying the row operations $\mathbf{row}_j + n_{j+1}\mathbf{row}_{j+1} \to \mathbf{row}_j$ for each row $j = 2n+k-1, \ldots, 2n+1$ results in the matrix

$$\begin{pmatrix} a_m & a_{m+1} & \cdots & a_{M-1} & a_M & & & & & & \\ & \ddots & & \ddots & & \ddots & & & & & \\ & & a_m & a_{m+1} & \cdots & a_{M-1} & a_M & & & & \\ & & 0 & \longleftarrow & [U_1] & \longrightarrow & 0 & & & & \\ & & 0 & \longleftarrow & [U_2] & \longrightarrow & -1 & 0 & & & \\ & & 0 & \longleftarrow & [U_3] & \longrightarrow & 0 & -1 & 0 & & \\ & & & & \vdots & & & & \ddots & \ddots & \\ & & 0 & \longleftarrow & [U_{k-1}] & \longrightarrow & 0 & \cdots & 0 & -1 & 0 \\ & & 0 & \longleftarrow & [U_k] & \longrightarrow & 0 & & \cdots & 0 & -1 \end{pmatrix},$$

where

$$[U_j] = [\widehat{V}_j] + n_j([\widehat{V}_{j+1}] + n_{j+1}([\widehat{V}_{j+2}] + \cdots + n_{k-2}([\widehat{V}_{k-1}] + n_{k-1}[\widehat{V}_k])\cdots).$$

Eliminating the last $k-1$ rows and columns results in the $(2n+1) \times (2n+M-m)$ presentation matrix D:

$$D = \begin{pmatrix} a_m & a_{m+1} & \cdots & a_{M-1} & a_M & & & & \\ & a_m & a_{m+1} & \cdots & a_{M-1} & a_M & & & \\ & & \ddots & & \ddots & & \ddots & & \\ & & & a_m & a_{m+1} & \cdots & a_{M-1} & a_M \\ & & & & c_m & c_{m+1} & \cdots & c_{M-1} \end{pmatrix},$$

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where

$$[U_1] = c_m S'_{m+n} + c_{m+1} S'_{m+n+1} + \cdots + c_{M-1} S'_{M+n-1}.$$

By Lemma 3.5(M5), for some l with $m < l \leq M$, there are integers $b_l, \ldots, b_M$ such that

$$[\widehat{A}_k] = \sum_{j=l}^{M} b_j S'_{j+n} \tag{3.14}$$

and $|b_{l+j}| = |b_{M-j}|$.

Claim 1 For each $j = m, \ldots, M-1$,

$$c_j = \begin{cases} a_j, & \text{when } m \leq j < l, \\ a_j - (\prod_{s=1}^{k} n_s) b_j, & \text{when } l \leq j < M-1. \end{cases}$$

■

From the row operations,

$$\begin{aligned}
[U_1] &= [\widehat{V}_1] + n_1([\widehat{V}_2] + n_2([\widehat{V}_3] + \cdots + n_{k-2}([\widehat{V}_{k-1}] + n_{k-1}[\widehat{V}_k])\cdots)) \\
&= [\widehat{V}_1] + n_1[\widehat{V}_2] + n_1 n_2 [\widehat{V}_3] + \cdots + \left(\prod_{s=1}^{k-2} n_s\right)[\widehat{V}_{k-1}] + \left(\prod_{s=1}^{k-1} n_s\right)[\widehat{V}_k] \\
&= \sum_{j=1}^{k}\left(\prod_{s=1}^{j-1} n_s\right)[\widehat{V}_j].
\end{aligned}$$

We use the convention that any empty product $\prod_{j=1}^{0}(x_j)$ is 1. By Lemma 3.5(M3), $\widehat{V}_j = \widehat{A}_j^{-n_j}\widehat{W}_j^{-1}\widehat{A}_{j-1}\widehat{W}_j$, so $[\widehat{V}_j] = [\widehat{A}_{j-1}] - n_j[\widehat{A}_j]$. Thus,

$$\begin{aligned}
\sum_{j=1}^{k}\left(\prod_{s=1}^{j-1} n_s\right)[\widehat{V}_j] &= \sum_{j=1}^{k}\left(\prod_{s=1}^{j-1} n_s\right)([\widehat{A}_{j-1}] - n_j[\widehat{A}_j]) \\
&= \sum_{j=1}^{k}\left(\prod_{s=1}^{j-1} n_s\right)[\widehat{A}_{j-1}] - \sum_{j=1}^{k}\left(\prod_{s=1}^{j} n_s\right)[\widehat{A}_j] \\
&= [\widehat{A}_0] - \left(\prod_{s=1}^{k} n_s\right)[\widehat{A}_k].
\end{aligned}$$

Therefore, since $\widehat{A}_0$ is a cyclic permutation of R_n by Lemma 3.5(M1),

$$[U_1] = [R_n] - \left(\prod_{s=1}^{k} n_s\right)[\widehat{A}_k]. \tag{3.15}$$

The statement of the claim follows from Proposition 3.4(a), (3.14), and (3.15).

By Lemma A.1, C is the gcd of all the $(2n+1) \times (2n+1)$ minors of D. Suppose a prime d divides C, so d divides the determinant of every $(2n+1) \times (2n+1)$ minor of D. The determinant of the minor of D given by the first $2n+1$ columns is a_m^{2n+1}, so d divides a_m.

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Claim 2 There is some $(2n+1) \times (2n+1)$ minor of D whose determinant is not divisible by d.

By Proposition 3.4(c), the integers $a_m, \ldots, a_M$ are the coefficients of the Alexander polynomial. Since the coefficients of $\Delta_L(t)$ are relatively prime, there is some coefficient that d does not divide. Let $m+i$ be the minimal index such that d does not divide a_{m+i}. We prove this claim in two cases.

Case 1. Suppose at least one of the following holds:

- $m+i < l$,
- d divides some n_s with $s \leq k$, or
- d divides b_j for all $j = l, \ldots, i$.

Then, either $m+i < l$ or d must divide $(\prod_{s=1}^{k} n_s) b_j$ for all $j = l, \ldots, m+i$. By Claim 1, d divides c_j when $j < m+i$ and d does not divide c_{m+i}.

Let E be the $(2n+1) \times (2n+1)$ minor of D consisting of the $n+1$ consecutive columns starting with the first column which with a_{m+i} (or c_{m+i} if $n=0$) at the top. Thus, working modulo d, we have the following minor:

$$E = \begin{pmatrix} a_{m+i} & * & * & \cdots & * & * \\ 0 & a_{m+i} & * & \cdots & * & * \\ 0 & 0 & a_{m+i} & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{m+i} & * \\ 0 & 0 & 0 & \cdots & 0 & c_{m+i} \end{pmatrix}.$$

Since d does not divide a_{m+i} or c_{m+i}, d cannot divide $\det(E)$.

Case 2. Suppose all of the following hold:

- $l \leq m+i$,
- d does not divide any n_s with $s \leq k$, and
- there is some $j \leq m+i$ such that d does not divide b_j.

Let F_1 be the $(2n+1) \times 2n$ minor given by the $2n$ consecutive columns with the coefficient a_{M-i}. By Proposition 3.4(c), $a_{m+j} = a_{M-j}$ for all $j = 0, \ldots, M-m$, so $M-i$ is the maximal index such that d divides a_{M-i}. Thus, modulo d, F_1 has the following form:

$$F_1 = \begin{pmatrix} a_{M-i} & 0 & 0 & \cdots & 0 \\ * & a_{M-i} & 0 & \cdots & 0 \\ * & * & a_{M-i} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & a_{M-i} \\ * & * & * & \cdots & * \end{pmatrix}.$$

We need to find a column in D with the first $2n$ entries divisible by d and the last entry not divisible by d.

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Let $l + i'$ be the minimal index such that d does not divide $b_{l+i'}$, so $l + i' \leq m + i$.

Since d does not divide $b_{l+i'}$ and $b_{l+i'} = b_{M-i'}$, d does not divide $b_{M-i'}$. By Lemma 3.5(M4), for all j, the coefficient of S'_{M+n} in $[\widehat{V}_j]$ is zero, so by (3.15),

$$a_M = b_M \prod_{s=1}^{k} n_s.$$

Since $a_m = a_M$ and d divides a_m, d must also divide b_M. Therefore, d divides b_l, so $i' > 0$ and $M - i' \leq M - 1$.

Since $M - i' \leq M - 1$, there is some column F_2 which ends with $c_{M-i'}$. Every other entry in F_2 is 0 or a_j for some $j > M - i'$. Since $l + i' \leq m + i$ and $m < l$,

$$0 < l - m \leq i - i',$$

so $M - i < M - i'$. Thus, by Claim 1, d does not divide $c_{M-i'}$, and for all $j > M - i'$, d divides a_j.

Combine F_1 and F_2 to get a $(2n+1) \times (2n+1)$ minor F of D. Working modulo d, we have the minor:

$$F = \begin{pmatrix} a_{M-i} & 0 & 0 & \cdots & 0 & 0 \\ * & a_{M-i} & 0 & \cdots & 0 & 0 \\ * & * & a_{M-i} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & a_{M-i} & 0 \\ * & * & * & \cdots & * & c_{M-i'} \end{pmatrix}.$$

Since d does not divide a_{M-i} or $c_{M-i'}$, d cannot divide $\det(F)$.

In conclusion, there are no primes which divide every determinant of $(2n+1) \times (2n+1)$ submatrices of D, so $C = 1$. Thus, $B \cong \mathbb{Z}^{M-m-1}$, and H_k is para-free of rank $M - m$. By induction, H_N is para-free of rank $M - m$.

By a similar induction argument, $H_N, \ldots, H_{2N}$ are also para-free of rank $M - m$. Therefore, $Y_{n+1} \cong H_{2N}$ is para-free of rank $M - m$, so by induction Y_n is para-free of rank $M - m$ for each nonnegative integer n.

For (b), consider the group $Y_{n+1}/Y_n[Y_{n+1}, Y_{n+1}]$, which is an abelian group with the following presentation:

$$\frac{Y_{n+1}}{Y_n[Y_{n+1}, Y_{n+1}]} \cong \langle S'_{m-n-1}, \ldots, S'_{M+n} \mid [R_{-n-1}], \ldots, [R_n], S'_{m-n}, \ldots, S'_{M+n-1} \rangle.$$

By Proposition 3.4,

$$[R_j] = \underline{a}_g S'_{M+j} + \underline{a}_{g-1} S'_{M-1+j} + \cdots + \underline{a}_{g-1} S'_{m+1+j} + \underline{a}_g S'_{m+j}.$$

After eliminating the generators $S'_{m-n}, \ldots, S'_{M+n-1}$, we have that

$$\frac{Y_{n+1}}{Y_n[Y_{n+1}, Y_{n+1}]} \cong \langle S'_{m-n-1}, S'_{M+n} \mid \underline{a}_g S'_{M-n-1}, \underline{a}_g S'_{m+n} \rangle,$$

so

$$\left| Y_{n+1}/Y_n[Y_{n+1}, Y_{n+1}] \right| = \left| \frac{\mathbb{Z}}{\underline{a}_g \mathbb{Z}} \oplus \frac{\mathbb{Z}}{\underline{a}_g \mathbb{Z}} \right| = \underline{a}_g^2.$$

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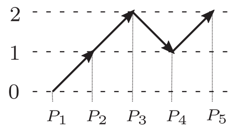

Figure 4: The incremental path Γ.

4 Cycle graphs

Explicitly, Lemma 3.5 is about nested patterns of repeating words in the relator R_0. However, this pattern is inherited from patterns in the sequences of ε_i's and σ_i's defined in (3.1) and (3.2). In the spirit of Hirasawa and Murasugi [11], graphs are used in order to gain intuition about how the sequences of ε_i's and σ_i's behave; however, the construction here slightly differs from the one Hirasawa and Murasugi used.

4.1 Incremental paths and cycles

A *graded directed graph* is a connected directed graph Γ with map $\mathrm{gr} : V(\Gamma) \to \mathbb{Z}$ called the *grading*. Here, $V(\Gamma)$ denotes the set of vertices of Γ. Two graded directed graphs Γ and Γ' are *isomorphic* if there is a directed graph isomorphism $f : \Gamma \to \Gamma'$ such that for every vertex P in Γ, $\mathrm{gr}(f(P)) = \mathrm{gr}(P)$. Γ and Γ' are called *relatively isomorphic* if there is a directed graph isomorphism $f : \Gamma \to \Gamma'$ and an integer k such that for every vertex P in Γ, $\mathrm{gr}(f(P)) = \mathrm{gr}(P) + k$.

An *incremental path* is a graded directed path graph Γ where the gradings of adjacent vertices differ by ± 1. Similarly, an *incremental cycle* is a graded directed cycle graph Γ where the gradings of adjacent vertices differ by ± 1. An edge (P, P') in an incremental path or cycle is *positive* if $\mathrm{gr}(P') - \mathrm{gr}(P) = +1$ and *negative* if $\mathrm{gr}(P') - \mathrm{gr}(P) = -1$.

Example 4.1 Let Γ be a directed graph with five vertices $P_1, \ldots, P_5$, and edges $(P_1, P_2), \ldots, (P_4, P_5)$. Define a grading on the vertices as follows:

$$\mathrm{gr}(P_1) = 0, \mathrm{gr}(P_2) = 1, \mathrm{gr}(P_3) = 2, \mathrm{gr}(P_4) = 1, \mathrm{gr}(P_5) = 2.$$

Γ is an incremental path (see Figure 4).

Let Γ and Γ' be two incremental paths in which the grading of the last vertex in Γ is equal to the grading of the first vertex in Γ'. Define the *concatenation* of Γ and Γ', denoted $\Gamma * \Gamma'$, to be the graded directed graph obtained by identifying the last vertex in Γ with the first vertex in Γ' (see Figure 5).

If the gradings of the first and last vertices in Γ are the same, Γ is called *closable* and the *closure of* Γ, $\mathrm{cl}(\Gamma)$, is defined to be the incremental cycle obtained by identifying the first and last vertices in Γ.

4.2 Cycle graphs of co-prime pairs

Ultimately, Lemma 3.5 is a statement about the sequences of ε_i's and σ_i's for co-prime pairs of integers. As computed in Proposition 3.2, the ith S-generator in R_0

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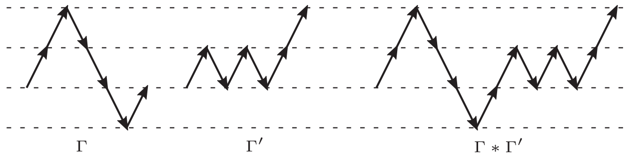

Figure 5: The concatenation of Γ and Γ′.

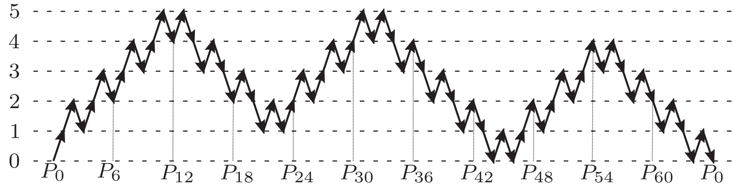

Figure 6: $\overline{\Gamma}(33,23)$.

is determined by the values of σ_{2i-1} and σ_{2i}. Here, we construct a graph to analyze the sequences of ε_i's and σ_i's.

We call a pair of integers (p,q) a *relevant co-prime pair* if p and q are co-prime, q is odd, and $p > |q| > 0$. Define the sequences ε_i and σ_i as in (3.1) and (3.2) for each integer i. Define the incremental path $\Gamma(p,q)$ as follows: The vertex set of $\Gamma(p,q)$ is $\{P_0,\dots,P_{2p}\}$, and the edge set of $\Gamma(p,q)$ is

$$E(\Gamma(p,q)) = \{(P_0,P_1),(P_1,P_2),\dots,(P_{2p-1},P_{2p})\}.$$

The grading of each vertex is defined by $\mathrm{gr}(P_i) = \sigma_i$. $\Gamma(p,q)$ is always closable, and the *cycle graph of p and q*, $\overline{\Gamma}(p,q)$, is defined to be $\mathrm{cl}(\Gamma(p,q))$. When studying $\overline{\Gamma}(p,q)$, it is convenient to think of its vertices $\{P_0,\dots,P_{2p-1}\}$ being indexed by elements of $\mathbb{Z}/(2p\mathbb{Z})$. See Figure 6 for example.

Proposition 4.2 *Let (p,q) be a relevant co-prime pair. The cycle graphs $\overline{\Gamma}(p,q)$ and $\overline{\Gamma}(p,-q)$ are relatively isomorphic.*

Proof Let $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ be the sequence of signs of (p,q) defined in (3.1). For each integer i, define

$$\varepsilon_i := (-1)^{\lfloor \frac{-iq}{p} \rfloor},$$

which is the sequence of signs of $(p,-q)$. Let q' be the unique integer such that $0 < q' < 2p$ and $q'q \equiv p-1$ modulo $2p$, so $q'q = p-1+2pk$ for some integer k.

We claim that the following equivalence holds:

$$\varepsilon_i = \varepsilon_{i+q'}. \tag{4.1}$$

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Consider the following computation:

$$\left\lfloor \frac{-iq}{p} \right\rfloor = \begin{cases} -\lfloor \frac{iq}{p} \rfloor, & iq \bmod p = 0, \\ -\lfloor \frac{iq}{p} \rfloor - 1, & iq \bmod p \neq 0, \end{cases}$$

$$\left\lfloor \frac{(i+q')q}{p} \right\rfloor = \left\lfloor \frac{iq+q'q}{p} \right\rfloor = \left\lfloor \frac{iq+p-1+2pk}{p} \right\rfloor = \left\lfloor \frac{iq-1}{p} \right\rfloor + 2k + 1$$
$$= \begin{cases} \lfloor \frac{iq}{p} \rfloor + 2k, & iq \bmod p = 0, \\ \lfloor \frac{iq}{p} \rfloor + 2k + 1, & iq \bmod p \neq 0. \end{cases}$$

We get the following equivalences modulo 2:

$$-\left\lfloor \frac{iq}{p} \right\rfloor \equiv \left\lfloor \frac{iq}{p} \right\rfloor + 2k, \qquad \pmod 2$$
$$-\left\lfloor \frac{iq}{p} \right\rfloor - 1 \equiv \left\lfloor \frac{iq}{p} \right\rfloor + 1 + 2k. \qquad \pmod 2$$

Thus,

$$(-1)^{\lfloor \frac{-iq}{p} \rfloor} = (-1)^{\lfloor \frac{(i+q')q}{p} \rfloor}.$$

For each integer $i = 0, \ldots, 2p$, define

$$\varsigma_i := \sum_{j=0}^{i-1} \varepsilon_i,$$

which are the gradings of the vertices of $\overline{\Gamma}(p,-q)$. By (4.1),

$$\varsigma_i = \sigma_{i+q'} - \sigma_{q'}$$

for every positive integer i. Since the σ_i's are the gradings of the vertices of $\overline{\Gamma}(p,q)$, it follows that $\overline{\Gamma}(p,q)$ and $\overline{\Gamma}(p,-q)$ are relatively isomorphic. ■

4.3 Structure of cycle graphs

Given an incremental cycle Γ, a *positive(negative) k-segment* is a set of k consecutive positive(negative) edges in Γ which are followed and preceded by negative(positive) edges (see Figure 7a). For each relevant co-prime integer pair (p,q), $\overline{\Gamma}(p,q)$ is the closure of the concatenation of segments of alternating sign as follows:

$$\overline{\Gamma}(p,q) = \mathrm{cl}(\Lambda_0 * \Lambda_1 * \cdots * \Lambda_{n-1}).$$

As a convention, let Λ_0 denote the segment in $\overline{\Gamma}(p,q)$ containing the edge (P_0, P_1).

Propositions 4.3 and 4.4 are analogs of the properties proved in Section 6 of Hirasawa and Murasugi's paper [11].

Proposition 4.3 *Let (p,q) be a relevant co-prime pair with $q > 0$. Denote the vertices of $\overline{\Gamma}(p,q)$ by $P_0, \ldots, P_{2p-1}$ as defined in Section 4.2, and let*

$$\overline{\Gamma}(p,q) = cl(\Lambda_0 * \Lambda_1 * \cdots * \Lambda_{n-1}),$$

(a) A negative 3-segment (b) A 2-block of length 4

Figure 7: Examples of a segment and a block.

where $\Lambda_0, \ldots, \Lambda_{n-1}$ are segments. Also, let κ and ξ be integers such that $p = \kappa q + \xi$ and $0 \le \xi < q$:

(a) *The number of segments n in $\overline{\Gamma}(p,q)$ is equal to $2q$.*
(b) *P_i is at the beginning of a segment precisely when $iq \bmod p < q$.*
(c) *P_i is at the beginning of a κ-segment precisely when $\xi \le iq \bmod p < q$, and P_i is at the beginning of a $(\kappa+1)$-segment precisely when $iq \bmod p < \xi$.*
(d) *Λ_0 is a positive $(\kappa+1)$-segment.*
(e) *The number of $(\kappa+1)$-segments in $\overline{\Gamma}(p,q)$ is 2ξ .*

Proof For (a), the number of segments in $\overline{\Gamma}(p,q)$ corresponds to the number of distinct floored quotients $\lfloor \frac{iq}{p} \rfloor$ there are when $i = 0, \ldots, 2p-1$. Since $p > q$, these quotients range from 0 to $2q-1$ without skipping, so there are exactly $2q$ segments.

A segment begins precisely when

$$\lfloor \frac{(i-1)q}{p} \rfloor \ne \lfloor \frac{iq}{p} \rfloor,$$

which happens when $(iq \bmod p) < q$, proving (b).

For (c), suppose P_i is the beginning of a k-segment. k is the smallest positive integer such that

$$\lfloor \frac{iq}{p} \rfloor \ne \lfloor \frac{(i+k)q}{p} \rfloor,$$

so

$$(iq \bmod p) + (k-1)q < p$$

and

$$(iq \bmod p) + kq \ge p.$$

When $\xi \le (iq \bmod p) < q$, $k = \kappa$. Likewise, when $(iq \bmod p) < \xi$, $k = \kappa + 1$.

Since P_i is the beginning of a segment, $iq \bmod p < q$, so exactly one of either $\xi \le (iq \bmod p) < q$ or $(iq \bmod p) < \xi$ is true. This determines precisely when κ- and $(\kappa+1)$-segments occur.

For part (d), it follows from (c) that Λ_0 is a $(\kappa+1)$-segment. Since ε_0 is positive, Λ_0 is a positive segment.

Part (e) immediately follows from (c). ■

When $q = 1$, $\varepsilon_i = 1$ for $0 \le i < p$ and $\varepsilon_i = -1$ for $p \le i < 2p$. Thus, $\overline{\Gamma}(p,q)$ is the concatenation of two κ-segments. When $q > 1$, $\overline{\Gamma}(p,q)$ has more interesting structure.

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A *k-block of length l* in $\overline{\Gamma}(p,q)$ is a sequence of l consecutive k-segments that is not preceded or followed by a k-segment (see Figure 7b). A k-block of length 1 is called an *isolated block.*

Proposition 4.4 *Let (p,q) be a relevant co-prime pair with $q > 1$, and let $P_0, \dots, P_{2p-1}$ be the vertices of $\overline{\Gamma}(p,q)$ as defined in Section 4.2. Let κ, ξ, κ', and ξ' be integers such that*

$$p = \kappa q + \xi \text{ with } 0 < \xi < q \tag{4.2}$$

and

$$q = \kappa' \xi + \xi' \text{ with } 0 \le \xi' < \xi. \tag{4.3}$$

(a) *All of the κ-blocks in $\overline{\Gamma}(p,q)$ have length κ' or $\kappa' - 1$.*
(b) *If P_j is the start of a κ-block, then when*

$$q - \xi' \le jq \bmod p < q,$$

the κ-blocks have length κ' and when

$$q - \xi \le jq \bmod p < q - \xi',$$

the κ-blocks have length $\kappa' - 1$.
(c) *If $\kappa' \ge 2$, then all the $(\kappa+1)$-blocks in $\overline{\Gamma}(p,q)$ are isolated.*
(d) *If $\kappa' = 1$, then all the κ-blocks in $\overline{\Gamma}(p,q)$ are isolated.*

Proof Similar to the proof of Proposition 4.3, this proposition is just a matter of determining when κ-blocks and $(\kappa+1)$-blocks appear in $\overline{\Gamma}(p,q)$.

Suppose P_i is the beginning of a $(\kappa+1)$-segment. The next segment begins at P_j where $j = i + \kappa + 1$, and by (4.2),

$$\begin{aligned} jq \bmod p &= ((i+\kappa+1)q) \bmod p \\ &= (iq + \kappa q + q) \bmod p \\ &= (iq + p - \xi + q) \bmod p \\ &= ((iq \bmod p) + q - \xi) \bmod p. \end{aligned}$$

Since P_i is the beginning of a $(\kappa+1)$-segment, $(iq \bmod p) < \xi$ by Proposition 4.3(c), so

$$q - \xi \le (iq \bmod p) + q - \xi < q < p. \tag{4.4}$$

Thus,

$$jq \bmod p = (iq \bmod p) + q - \xi. \tag{4.5}$$

For (a) and (b), suppose a κ-block starts at vertex P_j. The length of the κ-block starting at P_j is the smallest positive integer n, such that $P_{s(n)}$ is the start of a $(\kappa+1)$-block where $s(k) = j + k\kappa$, so n is the smallest positive integer such that

$$0 \le s(n)q \bmod p\xi < \xi.$$

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By (4.2),

$$\begin{aligned} s(k)q \bmod p &= (j + k\kappa)q \bmod p \\ &= (jq + k\kappa q) \bmod p \\ &= (jq + kp - k\xi) \bmod p \\ &= ((jq \bmod p) - k\xi) \bmod p. \end{aligned}$$

By (4.4) and (4.5), since P_j is the beginning of a κ-segment,

$$q - \xi \le jq \bmod p < q.$$

We compute the length n for each of the two cases $q - \xi \le (jq \bmod p) < q - \xi'$ and $q - \xi' \le (jq \bmod p) < q$.

Suppose that

$$q - \xi' \le jq \bmod p < q. \tag{4.6}$$

By (4.3),

$$((jq \bmod p) - \kappa'\xi = ((jq \bmod p) - q + \xi'$$

and

$$0 \le ((jq \bmod p) - q + \xi' < \xi',$$

so

$$0 \le s(\kappa')q \bmod p < \xi' < \xi.$$

Thus, $n \le \kappa'$.

Suppose $k \le \kappa' - 1$. By (4.3) and (4.6),

$$\begin{aligned} \xi &\le ((jq \bmod p) - q + \xi' + \xi \\ &= ((jq \bmod p) - \kappa'\xi + \xi \\ &= ((jq \bmod p) - (\kappa' - 1)\xi, \end{aligned}$$

so

$$\xi \le ((jq \bmod p) - k\xi < q.$$

Thus,

$$\xi \le s(k)q \bmod p < q,$$

so $n \ge \kappa'$. Therefore, $n = \kappa'$.

Suppose

$$q - \xi \le (jq \bmod p) < q - \xi'.$$

By (4.3),

$$((jq \bmod p) - (\kappa' - 1)\xi = ((jq \bmod p) - q + \xi' + \xi$$

and

$$0 \le \xi' \le ((jq \bmod p) - q + \xi' + \xi < \xi,$$

so

$$0 \le s(\kappa' - 1)q \bmod p < \xi.$$

Thus, $n \le \kappa' - 1$.

Suppose $k \le \kappa' - 2$. By (4.3) and (4.6),

$$\begin{aligned} \xi &\le ((jq \bmod p) - q + \xi' + 2\xi \\ &= ((jq \bmod p) - (\kappa' - 2)\xi, \end{aligned}$$

so

$$\xi \le ((jq \bmod p) - k\xi < q.$$

Thus,

$$\xi \le s(k)q \bmod p < q,$$

so $n \ge \kappa' - 1$. Therefore, $n = \kappa' - 1$. Thus, all of the κ-blocks have length κ' or $\kappa' - 1$.

For (c), suppose that $\kappa' \ge 2$. By (4.3),

$$q - \xi = (\kappa' - 1)\xi + \xi',$$

and since $\kappa' \ge 2$,

$$\xi \le \xi + \xi' \le q - \xi,$$

so by (4.4),

$$\xi \le (iq \bmod p) + q - \xi < q.$$

Thus, by (4.5),

$$\xi \le jq \bmod p < q.$$

By Proposition 4.3(c), P_j must be the beginning of a κ-segment, so $(\kappa + 1)$-segments cannot occur consecutively. Therefore, $(\kappa + 1)$-blocks are isolated.

Statement (d) follows immediately from (a). ∎

4.4 Reducing cycle graphs

Let (p, q) be a relevant co-prime pair with $q > 1$. Let κ, ξ, κ', and ξ' be defined as in Proposition 4.4, and decomposition $\overline{\Gamma}(p, q)$ into segments $\Lambda_0, \dots, \Lambda_{2q-1}$ as follows:

$$\overline{\Gamma}(p, q) = \mathrm{cl}(\Lambda_0 * \cdots * \Lambda_{2q-1}). \tag{4.7}$$

Again, Λ_0 is the segment containing the edge (P_0, P_1). By Proposition 4.3(e), 2ξ of the segments in (4.7) are $(\kappa + 1)$-segments. Let $j_0, \dots, j_{2\xi-1}$ be the indices in ascending order of the $(\kappa + 1)$-segments in (4.7). Define a reduction of $\overline{\Gamma}(p, q)$, denoted $R(\overline{\Gamma})(p, q)$, to be the following graded directed cycle graph with 2ξ vertices $Q_0, \dots, Q_{2\xi-1}$ with edge set

$$\{(Q_0, Q_1), (Q_1, Q_2), \dots, (Q_{2\xi-2}, Q_{2\xi-1}), (Q_{2\xi-1}, Q_0)\}.$$

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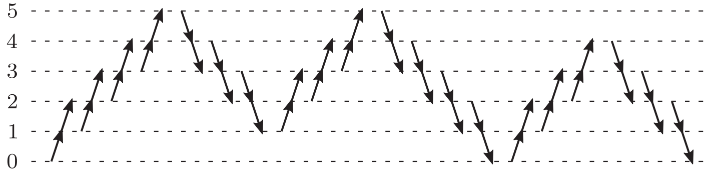
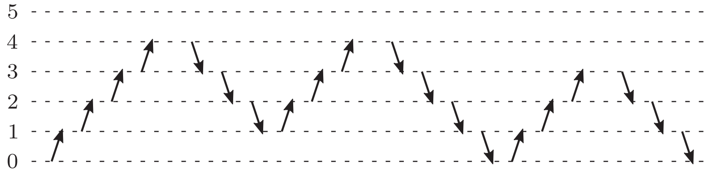
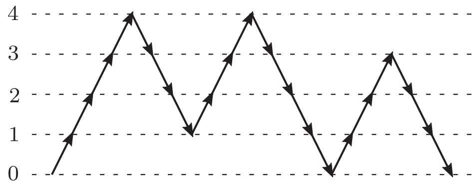


(a) All the 1-segments have been removed from $\overline{\Gamma}(33, 23)$; see Figure 6.



(b) The 2-segments have been replaced by edges.



(c) The resulting graph $R(\overline{\Gamma})(33, 23)$ is isomorphic to $\overline{\Gamma}(10, 3)$.

Figure 8: Reducing $\overline{\Gamma}(33, 23)$.

Define $\mathrm{gr}(Q_0) = 0$. For each $i = 1, \dots, 2\xi - 1$, define $\mathrm{gr}(Q_i) := \mathrm{gr}(Q_{i-1}) + 1$ when Λ_{j_i} is a positive segment, and $\mathrm{gr}(Q_i) := \mathrm{gr}(Q_{i-1}) - 1$ when Λ_{j_i} is a negative segment. Essentially, $R(\overline{\Gamma})(p, q)$ is $\overline{\Gamma}(p, q)$ with the κ-segments removed and the $(\kappa + 1)$-segments replaced with edges according to the sign of the segment. For example, see Figure 8.

Lemma 4.5 *Let (p, q) be a relevant co-prime pair with $q > 1$ and $\xi > 1$. Define p^* to be ξ, and define q^* as follows:*

$$q^* := \begin{cases} \xi', & \text{when } \kappa' \text{ is even,} \\ \xi' - \xi, & \text{when } \kappa' \text{ is odd,} \end{cases}$$

(a) *p^* is always positive and q^* is always odd.*
(b) *$R(\overline{\Gamma})(p, q)$ is isomorphic to $\overline{\Gamma}(p^*, q^*)$.*

Proof For (a), clearly, $p^* = \xi$ is positive. Also, notice that q is odd and

$$\xi' = q - \kappa' \xi.$$

If κ' is even, then $q^* = \xi'$ is odd. If κ' is odd, then ξ' and ξ must have opposite parities, so $q^* = \xi' - \xi$ is odd.

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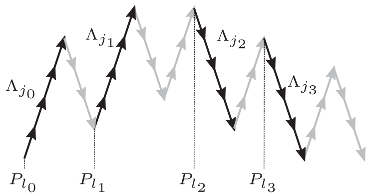

Figure 9: The $(\kappa+1)$-segments of $\overline{\Gamma}(17,5)$. The indices of the segments are $j_0 = 0$, $j_1 = 2$, $j_2 = 5$, and $j_3 = 7$. The indices of the vertices at the beginning of each $(\kappa+1)$-segment are $l_0 = 0$, $l_1 = 7$, $l_2 = 17$, and $l_3 = 24$.

For (b), consider $\overline{\Gamma}(p,q)$. By definition, $R(\overline{\Gamma})(p,q)$ has 2ξ edges and 2ξ vertices. Let $\{Q_0,\dots,Q_{2\xi-1}\}$ be the vertex set of $R(\overline{\Gamma})(p,q)$, and let $\{P_0^*,\dots,P_{2\xi-1}^*\}$ be the vertex set of $\overline{\Gamma}(p^*,q^*)$. Since $R(\overline{\Gamma})(p,q)$ and $\overline{\Gamma}(p^*,q^*)$ are cycle graphs with the same number of vertices, there is an ungraded directed graph isomorphism between them mapping $Q_i \mapsto P_i^*$. Since $\mathrm{gr}(Q_0)$ and $\mathrm{gr}(P_0^*)$ are both 0 by definition, it only remains to show that

$$\mathrm{gr}(Q_{i+1}) - \mathrm{gr}(Q_i) = \mathrm{gr}(P_{i+1}^*) - \mathrm{gr}(P_i^*)$$

for each $i = 0,\dots,2\xi-1$.

For $i = 0,\dots,2\xi-1$, define

$$\varepsilon_i := \mathrm{gr}(Q_{i+1}) - \mathrm{gr}(Q_i)$$

and

$$\eta_i := (-1)^{\lfloor \frac{i\xi'}{\xi} \rfloor}.$$

If $q^* = \xi'$, then

$$\mathrm{gr}(P_{i+1}^*) - \mathrm{gr}(P_i^*) = \eta_i,$$

and if $q^* = \xi' - \xi$, then

$$\mathrm{gr}(P_{i+1}^*) - \mathrm{gr}(P_i^*) = (-1)^{\lfloor \frac{i(\xi'-\xi)}{\xi} \rfloor} = (-1)^i \eta_i.$$

Lt l_i be the index of the vertex in $\overline{\Gamma}(p,q)$ at the beginning of Λ_{j_i} (see Figure 9). By definition of $R(\overline{\Gamma})(p,q)$, ε_i is positive precisely when Λ_{j_i} is a positive segment. Thus, $\varepsilon_{i+1} = \varepsilon_i$ when Λ_{j_i} and $\Lambda_{j_{i+1}}$ are separated by an even number of κ-segments, and $\varepsilon_{i+1} = -\varepsilon_i$ when Λ_{j_i} and $\Lambda_{j_{i+1}}$ are separated by an odd number of κ-segments. The desired result will follow from three claims.

Claim 1 Whenever $0 \le (i\xi' \bmod \xi) < \xi - \xi'$,

$$\eta_{i+1} = \eta_i,$$

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and whenever $(i\xi' \bmod \xi) \geq \xi - \xi'$,

$$\eta_{i+1} = -\eta_i.$$

■

When $0 \leq (i\xi' \bmod \xi) < \xi - \xi'$, there are integers s and t with

$$i\xi' = s\xi + t \text{ and } 0 \leq t < \xi - \xi',$$

so

$$s\xi \leq (i+1)\xi' = s\xi + t + \xi' < (s+1)\xi.$$

Thus,

$$\eta_{i+1} = (-1)^s = \eta_i.$$

When $(i\xi' \bmod \xi) \geq \xi - \xi'$, there are integers s and t with

$$i\xi' = s\xi + t \text{ and } \xi - \xi' \leq t < \xi,$$

so

$$(s+1)\xi \leq (i+1)\xi' = s\xi + t + \xi' < (s+1)\xi + \xi' < (s+2)\xi.$$

Thus,

$$\eta_{i+1} = (-1)^{s+1} = -\eta_i.$$

Claim 2 The segments Λ_{j_i} and $\Lambda_{j_{i+1}}$ are separated by a κ-block of length κ' when

$$\xi - \xi' \leq (l_i q \bmod p) < \xi$$

and a κ-block of length $\kappa' - 1$ (possibly zero) when

$$0 \leq (l_i q \bmod p) < \xi - \xi'.$$

By Proposition 4.4(b), every κ-block begins at a vertex P_l where

$$q - \xi \leq (lq \bmod p) < q.$$

The length of the block is κ' when

$$q - \xi' \leq (lq \bmod p) < q, \tag{4.8}$$

and the length is $\kappa' - 1$ when

$$q - \xi \leq (lq \bmod p) < q - \xi'. \tag{4.9}$$

The vertex at the end of the segment Λ_{j_i} is the same as the vertex at the beginning the segment Λ_{j_i+1}, so Λ_{j_i+1} begins at the vertex with index $l' := l_i + \kappa + 1$. By Proposition 4.3(b),

$$0 \leq l_i q \bmod p + q - \xi < q < p,$$

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so

$$\begin{aligned} l'q \bmod p &= (l_i + \kappa + 1)q \bmod p \\ &= (l_i q \bmod p + q - \xi) \bmod p \\ &= l_i q \bmod p + q - \xi. \end{aligned}$$

By (4.8), Λ_{j_i} and $\Lambda_{j_{i+1}}$ are separated by a κ-block of length κ' when

$$q - \xi' \le (l'q \bmod p) < q,$$

so

$$\xi - \xi' \le (l_i q \bmod p) < \xi.$$

By (4.9), Λ_{j_i} and $\Lambda_{j_{i+1}}$ are separated by a κ-block of length $\kappa' - 1$ when

$$q - \xi \le (l'q \bmod p) < q - \xi',$$

so

$$0 \le (l_i q \bmod p) < \xi - \xi'.$$

Claim 3 For each $i = 0, \dots, 2\xi - 1$,

$$l_i q \bmod p = i\xi' \bmod \xi.$$

P_{l_i} and $P_{l_{i+1}}$ are separated by a $(\kappa+1)$-segment and a κ-block. Therefore, when the length of the κ-block is κ',

$$l_{i+1} = l_i + (\kappa + 1) + \kappa'\kappa,$$

so

$$\begin{aligned} l_{i+1} q \bmod p &= (l_i q + \kappa q + q + \kappa'\kappa q) \bmod p \\ &= (l_i q \bmod p + \xi' - \xi) \bmod p. \end{aligned}$$

The last equality follows from (4.2) and (4.3). By Claim 2,

$$0 \le l_i q \bmod p + \xi' - \xi < \xi' < p.$$

Therefore,

$$l_{i+1} q \bmod p = l_i q \bmod p + \xi' - \xi. \tag{4.10}$$

When the length of the κ-block is $\kappa' - 1$,

$$l_{i+1} = l_i + (\kappa + 1) + (\kappa' - 1)\kappa = l_i + 1 + \kappa'\kappa,$$

so

$$\begin{aligned} l_{i+1} q \bmod p &= (l_i q + q + \kappa'\kappa q) \bmod p \\ &= (l_i q \bmod p + \xi') \bmod p. \end{aligned}$$

By Claim 2,

$$0 < \xi' \le l_i q \bmod p + \xi' < \xi < p.$$

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Therefore,

$$l_{i+1}q \bmod p = l_i q \bmod p + \xi'. \tag{4.11}$$

If either (4.10) or (4.11) holds,

$$l_{i+1}q \bmod p = (l_i q \bmod p + \xi') \bmod \xi,$$

so since $l_0 = 0$,

$$l_i q \bmod p = i\xi' \bmod \xi$$

for each $i = 0, \ldots, 2\xi - 1$ by induction. This completes the proof of the claim.

Suppose κ' is even. When Λ_{i+1} and Λ_i are separated by a κ-block of length $\kappa' - 1$, Λ_{i+1} and Λ_i have the same sign, so

$$\varepsilon_{i+1} = \varepsilon_i.$$

By the three claims,

$$0 \le (i\xi' \bmod \xi) < \xi - \xi',$$

so

$$\eta_{i+1} = \eta_i.$$

When Λ_{i+1} and Λ_i are separated by a κ-block of length κ', Λ_{i+1} and Λ_i have opposite signs, so

$$\varepsilon_{i+1} = -\varepsilon_i.$$

By the three claims,

$$(i\xi' \bmod \xi) \ge \xi - \xi',$$

so

$$\eta_{i+1} = -\eta_i.$$

Since $\varepsilon_0 = \eta_0 = 1$, for every $i = 0, \ldots, 2\xi - 1$,

$$\varepsilon_i = \eta_i,$$

so when $q^* = \xi'$,

$$\mathrm{gr}(P^*_{i+1}) - \mathrm{gr}(P^*_i) = \eta_i = \varepsilon_i = \mathrm{gr}(Q_{i+1}) - \mathrm{gr}(Q_i).$$

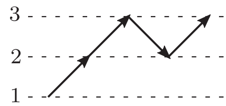
Suppose κ' is odd. When Λ_{i+1} and Λ_i are separated by a κ-block of length κ', then $\varepsilon_{i+1} = \varepsilon_i$. When Λ_{i+1} and Λ_i are separated by a κ-block of length $\kappa' - 1$, then $\varepsilon_{i+1} = -\varepsilon_i$.

Thus, by the claims, $\varepsilon_{i+1} = \varepsilon_i$ when $\eta_{i+1} = -\eta_i$, and $\varepsilon_{i+1} = -\varepsilon_i$ when $\eta_{i+1} = \eta_i$. Again, $\varepsilon_0 = \eta_0 = 1$. Therefore, for every $i = 0, \ldots, 2\xi - 1$,

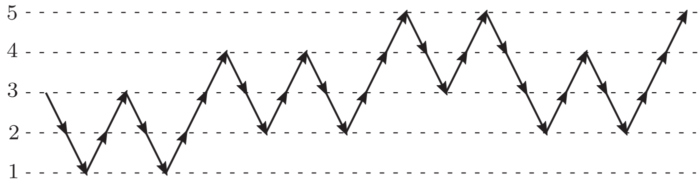
$$\varepsilon_i = (-1)^i \eta_i,$$

so when $q^* = \xi' - \xi$, then

$$\mathrm{gr}(P^*_{i+1}) - \mathrm{gr}(P^*_i) = (-1)^i \eta_i = \varepsilon_i = \mathrm{gr}(Q_{i+1}) - \mathrm{gr}(Q_i).$$



(a) An incremental path Γ



(b) $\tilde{E}(\Gamma, 2, 3, +1)$

Figure 10: Expanding an incremental path.

Example 4.6 Consider the co-prime pair $(33, 23)$. $R(\overline{\Gamma})(33, 23)$ is isomorphic to $\Gamma(10, 3)$ (see Figure 8).

4.5 Expanding cycle graphs

We can also reverse the reduction process. Let Γ be an incremental path with vertices $P_0, \ldots, P_n$ indexed such that (P_i, P_{i+1}) is an edge in Γ for each $i = 0, \ldots, n-1$. Let s and b be positive integers, and let $e = \pm 1$. Define $\tilde{E}(\Gamma, s, b, e)$ to be the incremental path graph constructed as follows:

(1) Create an $(s+1)$-segment, Λ_i, for each edge (P_i, P_{i+1}) in Γ. Choose Λ_i to be positive or negative according to the sign of the edge (P_i, P_{i+1}).
(2) Between each pair Λ_i and Λ_{i+1}, for $i = 0, \ldots, n-2$, add an s-block of length b or $b-1$. The length of the s-block is odd if the edges Λ_i and Λ_{i+1} have the same sign, and the length is even if Λ_i and Λ_{i+1} have opposite signs. Also, the first s-segment in the block has sign opposite of the sign of Λ_i.
(3) Add another s-block to the beginning of Λ_0 of length b or $b-1$ depending on the signs of Λ_0 and e following the same convention as the previous step. Also, the first s-segment in the block has sign opposite of e.
(4) Finally, set the grading of the first vertex Q_0 as follows:

$$\text{gr}(Q_0) = \begin{cases} \text{gr}(P_0) + s, & \text{when } e \text{ and } (P_0, P_1) \text{ are both positive,} \\ \text{gr}(P_0) - s, & \text{when } e \text{ and } (P_0, P_1) \text{ are both negative,} \\ \text{gr}(P_0), & \text{when } e \text{ and } (P_0, P_1) \text{ have opposite sign.} \end{cases} \tag{4.12}$$

For example, see Figure 10.

We begin by investigating the gradings of the vertices in $\tilde{E}(\Gamma, s, b, e)$.

Lemma 4.7 *Let Q_0 be the vertex at the beginning of $\tilde{E}(\Gamma, s, b, e)$. For $i = 1, \ldots, n$, let Q_i be the vertex at the end of $(s+1)$-segment Λ_{i-1} as in the definition of $\tilde{E}$.*

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For each $i = 1, \ldots, n$*:*

(a) *If the signs of* Λ_{i-1} *and e are the same, then*

$$gr(Q_i) - gr(Q_0) = gr(P_i) - gr(P_0).$$

(b) *If* Λ_{i-1} *is positive and e is negative, then*

$$gr(Q_i) - gr(Q_0) = gr(P_i) - gr(P_0) + s.$$

(c) *If* Λ_{i-1} *is negative and e is positive, then*

$$gr(Q_i) - gr(Q_0) = gr(P_i) - gr(P_0) - s.$$

Proof Let Γ' be the subgraph of $\tilde{E}(\Gamma, s, b, e)$ starting at Q_0 and ending at Q_i. Γ' is the concatenation of sum number of $(s+1)$- and s-segments. Let D^+ and D^- be the number of positive or, respectively, negative $(s+1)$-segments in Γ'. Likewise, let d^+ and d^- be the number of positive or, respectively, negative s-segments in Γ'. Note that D^+ and D^- are also the number of positive and negative edges separating P_0 and P_i in Γ, so

$$D^+ - D^- = \mathrm{gr}(P_i) - \mathrm{gr}(P_0).$$

Suppose Λ_{i-1} and e have the same sign, then the number of positive segments in Γ' is equal to the number of negative segments, so

$$D^+ + d^+ = D^- + d^-.$$

Thus,

$$\begin{aligned}\mathrm{gr}(Q_i) - \mathrm{gr}(Q_0) &= D^+(s+1) - D^-(s+1) + d^+s - d^-s \\ &= (D^+ + d^+)s - (D^- + d^-)s + D^+ - D^- \\ &= D^+ - D^- \\ &= \mathrm{gr}(P_i) - \mathrm{gr}(P_0).\end{aligned}$$

Suppose Λ_{i-1} is positive and e is negative, then the total number of positive segments in Γ' is one more than the total number of negative segments, so

$$\begin{aligned}\mathrm{gr}(Q_i) - \mathrm{gr}(Q_0) &= D^+(s+1) - D^-(s+1) + d^+s - d^-s \\ &= (D^+ + d^+)s - (D^- + d^-)s + D^+ - D^- \\ &= s + D^+ - D^- \\ &= \mathrm{gr}(P_i) - \mathrm{gr}(P_0) + s.\end{aligned}$$

Suppose Λ_{i-1} is negative and e is positive, then the total number of positive segments in Γ' is one less than the total number of negative segments, so

$$\begin{aligned}\mathrm{gr}(Q_i) - \mathrm{gr}(Q_0) &= D^+(s+1) - D^-(s+1) + d^+s - d^-s \\ &= (D^+ + d^+)s - (D^- + d^-)s + D^+ - D^- \\ &= -s + D^+ - D^- \\ &= \mathrm{gr}(P_i) - \mathrm{gr}(P_0) - s.\end{aligned}$$

■

From this, we can show that concatenation behaves well under expansion.

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Lemma 4.8 *Suppose* Γ *and* Γ' *are incremental paths where the last vertex in* Γ *has the same grading as the first vertex in* Γ'. *Let* e' *be the sign of the last edge in* Γ. *For any positive integers* s *and* b *and any sign* $e = \pm 1$,

$$\tilde{E}(\Gamma * \Gamma', s, b, e) \cong \tilde{E}(\Gamma, s, b, e) * \tilde{E}(\Gamma', s, b, e').$$

Proof The conclusion will be true by definition of the expansion procedure as long as $\tilde{E}(\Gamma, s, b, e)$ and $\tilde{E}(\Gamma', s, b, e')$ can be concatenated. Thus, our goal is to show that the last vertex in $\tilde{E}(\Gamma, s, b, e)$ has the same grading as the first vertex in $\tilde{E}(\Gamma', s, b, e')$. This can be done by computing the gradings of $\tilde{E}(\Gamma * \Gamma', s, b, e)$ for many cases depending on the signs of e, the last edge in Γ, and the first edge in Γ'.

For example, suppose e, the last edge in Γ, and the first edge in Γ' are all positive. Let P_0 and P_n be the first and last vertices of Γ. Let P'_0 be the first vertex in Γ' so $\mathrm{gr}(P_n) = \mathrm{gr}(P'_0)$. Let Q_0 and Q_n be the first and last vertices of $\tilde{E}(\Gamma, s, b, e)$. Finally, let Q'_0 be the first vertex in $\tilde{E}(\Gamma', s, b, e')$.

By (4.12),

$$\mathrm{gr}(Q'_0) = \mathrm{gr}(P'_0) + s = \mathrm{gr}(P_n) + s.$$

By Lemma 4.7,

$$\begin{aligned}\mathrm{gr}(Q_n) &= \mathrm{gr}(P_n) - \mathrm{gr}(P_0) + \mathrm{gr}(Q_0)\\ &= \mathrm{gr}(Q'_0) - s - \mathrm{gr}(P_0) + \mathrm{gr}(P_0) + s\\ &= \mathrm{gr}(Q'_0).\end{aligned}$$

The proofs of all the other cases are similar. ∎

Let Γ be a closable incremental path, and let e be the sign of the last edge in Γ. For any two positive integers s and b, define

$$E(\Gamma, s, b) := \tilde{E}(\Gamma, s, b, e).$$

When Γ is closable, $E(\Gamma, s, b)$ is also closable.

Suppose Γ' is a closable incremental path such that $\mathrm{cl}(\Gamma) \cong \mathrm{cl}(\Gamma')$. By construction,

$$\mathrm{cl}(E(\Gamma, s, b)) \cong \mathrm{cl}(E(\Gamma', s, b)) \tag{4.13}$$

for all positive integers s and b.

For an incremental cycle $\overline{\Gamma}$, define

$$E(\overline{\Gamma}, s, b) := \mathrm{cl}(E(\Gamma, s, b)),$$

where Γ is any incremental path such that $\mathrm{cl}(\Gamma) \cong \overline{\Gamma}$. By (4.13), $E(\overline{\Gamma}, s, b)$ is well defined.

Reduction and expansion are naturally opposite operations.

Proposition 4.9 *Suppose* (p, q) *is a relevant co-prime pair with* $q > 1$. *Define* κ *and* κ' *as in* (4.2) *and* (4.3):

$$E(R(\overline{\Gamma})(p, q), \kappa, \kappa') \cong \overline{\Gamma}(p, q).$$

Proof By Proposition 4.3, $\overline{\Gamma}(p,q)$ is the concatenation of κ-segments and $(\kappa+1)$-segments. The reduction R replaces $(\kappa+1)$-segments with single edges of the same sign. The expansion E transforms all the edges back into $(\kappa+1)$-segments.

By Proposition 4.4(a), the $(\kappa+1)$-segments of $\overline{\Gamma}(p,q)$ are separated by κ-blocks of length κ' or $\kappa'-1$ (possibly zero). The blocks in $\overline{\Gamma}(p,q)$ have even length precisely when the preceding and following $(\kappa+1)$-segments have opposite sign.

R removes these κ-blocks, and E restores them. The signs of consecutive edges in $R(\overline{\Gamma}(p,q))$ correspond to the signs of the preceding and following $(\kappa+1)$-segments in $\overline{\Gamma}(p,q)$, so the length of each κ-block after the expansion will be the same as it was before the reduction.

It remains to check that gradings are preserved. Consider the edge in $R(\overline{\Gamma}(p,q))$ corresponding to Λ_0 in $\overline{\Gamma}(p,q)$ as labeled in (4.7). Label the vertices at the beginning and end of this edge P_0 and P_1, respectively.

By the definition of R, the grading of P_0 is equal to the grading of the vertex at the beginning of Λ_0.

Consider Λ_0', the $(\kappa+1)$-segment resulting from expansion of the edge after P_0. Let Q_0 be the grading at the end of the Λ_0' as in Lemma 4.7, and let Q_1' be the vertex at the beginning of Λ_0'.

Now, we show that $\operatorname{gr}(P_0)=\operatorname{gr}(Q_1')$. The edge after P_0 is always positive since it corresponds to Λ_0. Thus,

$$\operatorname{gr}(P_1)=\operatorname{gr}(P_0)-1 \tag{4.14}$$

and

$$\operatorname{gr}(Q_1)-\operatorname{gr}(Q_1')=\kappa+1. \tag{4.15}$$

When the edge before P_0 is also positive,

$$\begin{aligned}\operatorname{gr}(Q_1')-\operatorname{gr}(P_0)&=\operatorname{gr}(Q_1')-\operatorname{gr}(Q_1)+\operatorname{gr}(Q_1)-\operatorname{gr}(Q_0)+\operatorname{gr}(Q_0)-\operatorname{gr}(P_0)\\&=(-\kappa-1)+(\operatorname{gr}(P_1)-\operatorname{gr}(P_0))+(\operatorname{gr}(P_0)+\kappa)-\operatorname{gr}(P_0)\\&=\operatorname{gr}(P_1)-\operatorname{gr}(P_0)-1\\&=0.\end{aligned}$$

The second equality follows from (4.15), Lemma 4.7, and (4.12). The last equality follows from (4.14). Similarly, when the edge before P_0 is negative,

$$\begin{aligned}\operatorname{gr}(Q_1')-\operatorname{gr}(P_0)&=\operatorname{gr}(Q_1')-\operatorname{gr}(Q_1)+\operatorname{gr}(Q_1)-\operatorname{gr}(Q_0)+\operatorname{gr}(Q_0)-\operatorname{gr}(P_0)\\&=(-\kappa-1)+(\operatorname{gr}(P_1)-\operatorname{gr}(P_0)+\kappa)+\operatorname{gr}(P_0)-\operatorname{gr}(P_0)\\&=0.\end{aligned}$$ ∎

Given an arbitrary relevant co-prime pair (p^*,q^*) and integers s and b, the expansion $E(\overline{\Gamma}(p^*,q^*),s,b)$ may not be $\overline{\Gamma}(p,q)$ for any co-prime (p,q) with q odd. Consider the pair $(5,3)$. Suppose $E(\overline{\Gamma}(5,3),2,3)\cong\overline{\Gamma}(p,q)$ for some pair (p,q). Define κ, κ', ξ, and ξ' for (p,q) as in (4.2) and (4.3). Since the sizes of the segments of $E(\overline{\Gamma}(5,3),2,3)$ are either 2 or 3 and the blocks have length 3 or 2, κ must be 2, and κ' must be 3. By Proposition 4.9, $\overline{\Gamma}(5,3)\cong R(\overline{\Gamma})(p,q)$. By Lemma 4.5, $q^*=3$ is

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equal to ξ or $\xi' - \xi$. Since $\xi' - \xi$ cannot be positive, $\xi = 3$. Also, by Proposition 4.5, $p \bmod q = \xi = 5$. Thus, $q = 3(5) + 3 = 18$, which is not odd.

5 Proof of Lemma 3.5

In this section, we reinterpret Lemma 3.5 as a set of properties of the cycle graph $\overline{\Gamma}(p,q)$. These properties will hold for simple relevant co-prime pairs (p,q) with $q = 1$ or $(p \bmod q) = 1$. Then, it is shown that these conditions hold for any relevant co-prime pair of integers p and q with p positive and q odd by a strong induction argument using the relative isomorphism between $\overline{\Gamma}(p,q)$ and $\overline{\Gamma}(p,-q)$ and the reduction from $\overline{\Gamma}(p,q)$ to $R(\overline{\Gamma})(p,q)$.

5.1 Making words from graphs

Given an incremental path Γ, a word $\rho(\Gamma)$ in $\mathcal{S}$ can be defined as follows: Let $\{P_1,\ldots,P_n\}$ be the vertices of Γ indexed so that the edge (P_i,P_{i+1}) is in Γ. For $i = 2,\ldots,n$, let $s_i = \mathrm{gr}(P_i) - \mathrm{gr}(P_{i-1})$ and let $N_i = \mathrm{gr}(Q_i) + \theta(s_i)$ where $\theta(1) = 1$ and $\theta(-1) = 0$. Define

$$\rho(\Gamma) := \begin{cases} S_{N_3}^{s_3} S_{N_5}^{s_5} \cdots S_{N_k}^{s_k}, & \text{if } n > 2 \text{ and } \mathrm{gr}(P_1) \text{ is even,} \\ S_{N_2}^{s_2} S_{N_4}^{s_4} \cdots S_{N_k}^{s_k}, & \text{if } n > 1 \text{ and } \mathrm{gr}(P_1) \text{ is odd,} \\ 1, & \text{otherwise,} \end{cases} \tag{5.1}$$

where $k = n-1$, if $n \equiv \mathrm{gr}(P_1)$ modulo 2, and $k = n$, if $n \not\equiv \mathrm{gr}(P_1)$ modulo 2. Given a two-bridge link $L(p/q)$, by Proposition 3.2, $\rho(\Gamma(p,q))$ is the word R_0.

Lemma 5.1 *Given incremental paths Γ and Γ' such that the last vertex of Γ has the same grading as the first vertex of Γ',*

$$\rho(\Gamma * \Gamma') = \rho(\Gamma)\rho(\Gamma').$$

Proof Let $\{P_1,\ldots,P_n\}$ and $\{P'_1,\ldots,P'_{n'}\}$ be the vertex sets for incremental paths Γ and Γ', respectively. Also, define $N_2,\ldots,N_n$ and $s_2,\ldots,s_n$ for Γ as in the definition of ρ. Similarly, define $N'_2,\ldots,N'_{n'}$ and $s'_2,\ldots,s'_{n'}$ for Γ'. Let $\Gamma'' = \Gamma * \Gamma'$, which has length $n + n' - 1$, and define $N''_2,\ldots,N''_{n+n'-1}$ and $s''_2,\ldots,s''_{n+n'-1}$ for Γ'' as the analogous integers are defined for Γ and Γ'.

This result is just a matter of computing $\rho(\Gamma * \Gamma')$ for each case of (5.1) for Γ and Γ'. For example, suppose $\mathrm{gr}(P_1)$ and n are even, $n > 2$, and $n' > 1$. Then, since n is even,

$$\mathrm{gr}(P'_1) = \mathrm{gr}(P_n) \equiv (\mathrm{gr}(P_1) + n - 1) \equiv \mathrm{gr}(P_1) + 1, \qquad (\mathrm{mod}\ 2)$$

so since $\mathrm{gr}(P_1)$ is even, $\mathrm{gr}(P'_1)$ is odd. Thus,

$$\rho(\Gamma) = S_{N_3}^{s_3} S_{N_5}^{s_5} \cdots S_{N_{n-1}}^{s_{n-1}}$$

and

$$\rho(\Gamma') = S_{N_2}^{s_2} S_{N_4}^{s_4} \cdots S_{N_k}^{s_k},$$

where $k = n'$ when n' is even and $k = n' - 1$ when n' is odd.

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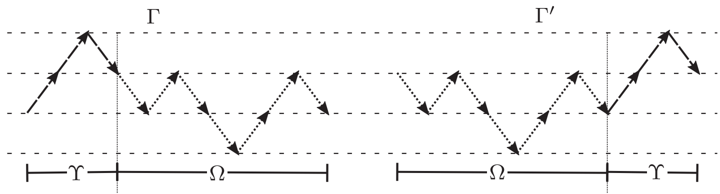

Figure 11: Closable graphs Γ and Γ' with isomorphic closures with the subgraphs Υ (dashed) and Ω (dotted) shown.

For each $i = 1, \dots, n + n' - 1$,

$$\mathrm{gr}(P_i'') = \begin{cases} \mathrm{gr}(P_i), & \text{when } 1 \le i \le n, \\ \mathrm{gr}(P'_{i-n+1}), & \text{when } n \le i \le n + n' - 1. \end{cases}$$

Thus, when $2 \le i \le n$, $s_i'' = s_i$, and $N_i'' = N_i$, and when $n + 1 \le i \le n + n' - 1$, $s_i'' = s_{i-n+1}$, and $N_i'' = N_{i-n+1}$. Therefore,

$$\rho(\Gamma * \Gamma') = S_{N_3}^{s_3} S_{N_5}^{s_5} \cdots S_{N_{n-1}}^{s_{n-1}} S_{N_2'}^{s_2'} S_{N_4'}^{s_4'} \cdots S_{N_k'}^{s_k'} = \rho(\Gamma)\rho(\Gamma').$$

The proofs of all the other cases are similar. ■

Lemma 5.2 *Given two closable incremental paths Γ and Γ' such that $cl(\Gamma)$ is isomorphic to $cl(\Gamma')$, then $\rho(\Gamma)$ and $\rho(\Gamma')$ are cyclic permutations of each other.*

Proof Since Γ and Γ' have isomorphic closures, they must have the same number of vertices. Let $\{P_0, \dots, P_n\}$ and $\{P_0', \dots, P_n'\}$ be the vertices in order of Γ and Γ', respectively. Let $\{Q_0, \dots, Q_{n-1}\}$ be the vertex set of $\mathrm{cl}(\Gamma)$ chosen such that $\mathrm{gr}(Q_i) = \mathrm{gr}(P_i)$ for $i = 0, \dots, n - 1$. Likewise, let $\{Q_0', \dots, Q_{n-1}'\}$ be the vertex set of $\mathrm{cl}(\Gamma')$ chosen such that $\mathrm{gr}(Q_i') = \mathrm{gr}(P_i')$ for $i = 0, \dots, n - 1$.

Since $\mathrm{cl}(\Gamma) \cong \mathrm{cl}(\Gamma')$, there is a directed graph isomorphism from $f : \mathrm{cl}(\Gamma) \cong \mathrm{cl}(\Gamma')$, which preserves gradings. Let k be the index of the vertex in $\mathrm{cl}(\Gamma)$ such that $f(Q_k) = Q_0'$. If $k = 0$, then f maps Q_i to Q_i' for each $i = 0, \dots, n - 1$. It follows that $\mathrm{gr}(P_i) = \mathrm{gr}(P_i')$ for each $i = 0, \dots, n$ so $\rho(\Gamma) = \rho(\Gamma')$.

Suppose $k \ne 0$. Let Υ be the subgraph of Γ induced by $P_0, \dots, P_k$, and let Ω be the subgraph of Γ induced by $P_k, \dots, P_n$. Since $f(Q_0) = Q'_{n-k}$ and $f(Q_k) = Q_0'$, the subgraph of Γ' induced by $P'_{n-k}, \dots, P_n'$ must be isomorphic to Υ, and the subgraph of Γ' induced by $P_0', \dots, P'_{n-k}$ must be isomorphic to Ω. Thus, $\Gamma \cong \Upsilon * \Omega$ and $\Gamma' \cong \Omega * \Upsilon$ (see Figure 11 for example). Therefore, $\rho(\Gamma) = \rho(\Upsilon)\rho(\Omega)$ and $\rho(\Gamma') = \rho(\Omega)\rho(\Upsilon)$. ■

5.2 Summits and bottoms in cycle graphs

Let (p, q) be a relevant co-prime pair, and define M and m for $L(p/q)$ as in Section 3. Our goal is to prove Lemma 3.5. By Proposition 3.2(d), it is sufficient to show Lemma 3.5 for the relator R_0. Thus, we are interested in the appearances of S_M^δ and S_m^δ in the word R_0. When M is odd, the ith S-generator of R_0 is S_M^δ precisely when $\sigma_{2i} = M + 1$, and when M is even, the ith S-generator of R_0 is S_M^δ when $\sigma_{2i-1} = M + 1$. Thus,

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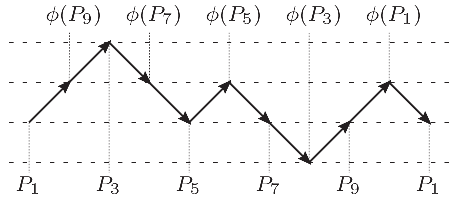

Figure 12: A symmetric incremental cycle. The first and last vertices are identified. ϕ is the unique order-reversing bijection defined by $\phi(P_1) = P_{10}$.

appearances of S_M^δ in R_0 correspond to the indices when σ_i is maximal. Similarly, the ith S-generator of R_0 is S_m^δ precisely when $\sigma_{2i-1} = m$ when m is odd or $\sigma_{2i} = m$ when m is even. Thus, appearances S_m^δ in R_0 correspond to the indices when σ_i is minimal.

A vertex, P, in a graded graph Γ is called a *summit* if $\mathrm{gr}(P) \geq \mathrm{gr}(Q)$ for any vertex Q in Γ. Similarly, P is called a *bottom* if $\mathrm{gr}(P) \leq \mathrm{gr}(Q)$ for any vertex Q in Γ. For each relevant co-prime pair (p, q), the grading of a summit of $\Gamma(p, q)$ is always $M + 1$ and the grading of a bottom of $\Gamma(p, q)$ is always m. Furthermore, the appearances of S_M in R_0 correspond precisely to the summits in $\Gamma(p, q)$, and the appearances of S_m correspond to bottoms.

5.3 Symmetric incremental paths and cycles

It is useful to know when an incremental cycle is relatively isomorphic to itself after rotating 180° and reversing its edges. More precisely, we call an incremental cycle Γ *symmetric* if there is a bijection $\phi : V(\Gamma) \to V(\Gamma)$ such that:

(1) (P, Q) is an edge of Γ if and only if $(\phi(Q), \phi(P))$ is an edge of Γ for any two vertices P and Q in Γ and

(2) for some integer k, $\mathrm{gr}(P) + \mathrm{gr}(\phi(P)) = k$ for every vertex P in Γ.

An incremental path Γ is called *symmetric* if $\mathrm{cl}(\Gamma)$ is symmetric (see Figure 12). The symmetry of incremental paths and cycles plays an important role in investigating properties (M5) and (m5) of Lemma 3.5.

5.4 Reinterpretation of Lemma 3.5

Here, we reinterpret Lemma 3.5 in terms of incremental paths and cycles. Given a closable incremental path Γ and a positive integer n, define Γ^n to be the concatenation of n copies of Γ. We call a relevant co-prime pair $(\underline{p}, q)$ an *pre-RTFN pair* if for some incremental path Γ whose closure is isomorphic to $\overline{\Gamma}(p, q)$, there are a positive integer N, sequences of subgraphs of Γ,

$$\Gamma_0, \ldots, \Gamma_N,$$

$$\Upsilon_1, \ldots, \Upsilon_N,$$

and

$$\Omega_1, \ldots, \Omega_N,$$

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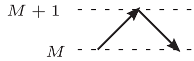
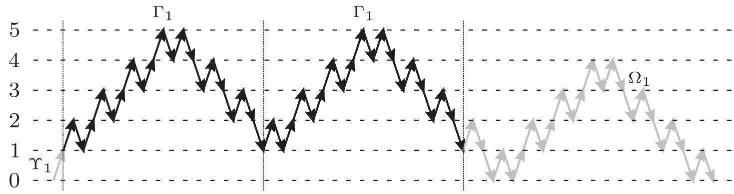
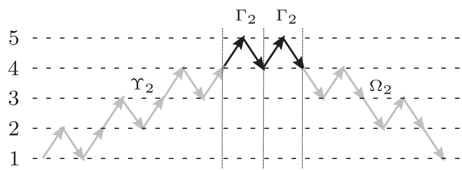


Figure 13: The graph Γ_{top}.



(a) $\Gamma_0 = \Gamma(33, 23)$ with Υ_1 and Ω_1 in gray



(b) Γ_1 with Υ_2 and Ω_2 in gray

Figure 14: $(33, 23)$ is a pre-RTFN pair.

and a sequence of positive integers

$$n_1, \ldots, n_N$$

such that the following conditions are satisfied:

(R1) $\Gamma_0 = \Gamma$.
(R2) Γ_N is isomorphic to the graph Γ_{top} defined in Figure 13.
(R3) For each $i = 1, \ldots, N$,

$$\Gamma_{i-1} \cong \Upsilon_i * \Gamma_i^{n_i} * \Omega_i.$$

(R4) For each $i = 1, \ldots, N$, no summits of $\mathrm{cl}(\Gamma)$ appear in Υ_i or Ω_i.
(R5) For each $i = 0, \ldots, N$, Γ_i is symmetric, and when $i \geq 1$, Γ_i contains no bottoms of $\mathrm{cl}(\Gamma)$.

For example, Figure 14 demonstrates that $(33, 23)$ is a pre-RTFN pair.

Lemma 5.3 *(p, q) is a pre-RTFN pair if and only if $(p, -q)$ is a pre-RTFN pair.*

Proof By Proposition 4.2, $\Gamma(p, q)$ and $\Gamma(p, -q)$ have relatively isomorphic closures, so the conclusion of the lemma follows immediately. ■

Lemma 5.4 *Suppose (p, q) is a relevant co-prime pair. If (p, q) is a pre-RTFN pair, then $L(p/q)$ satisfies Lemma 3.5.*

Proof By Proposition 3.2(d), it is sufficient to show that Lemma 3.5 holds for R_0. Let (p, q) be a pre-RTFN pair. Then, we have a graph Γ whose closure is isomorphic to $\overline{\Gamma}(p, q)$ satisfying (R1)–(R5).

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For each $i = 0, \dots, N$, define

$$\widehat{A}_i := \rho(\Gamma_i),$$

and when $i > 0$, define

$$\widehat{V}_i := \rho(\Omega_i)\rho(\Upsilon_i) \text{ and } \widehat{W}_i := \rho(\Upsilon_i).$$

Proof of (M1) and (M2) By (R1), $\mathrm{cl}(\Gamma_0)$ is isomorphic to $\overline{\Gamma}(p,q)$. Therefore, by Lemma 5.2, $\widehat{A}_0 = \rho(\Gamma_0)$ is a cyclic permutation of $\rho(\Gamma(p,q))$ which is R_0. ∎

By (R2),

$$A_N = \rho(\Gamma_N) = S_M^{\delta}.$$

Proof of (M3) Suppose i is an integer with $1 \le i \le N$. By (R3),

$$\Gamma_{i-1} \cong \Upsilon_i * \Gamma_i^{n_i} * \Omega_i.$$

Therefore,

$$\begin{aligned}
\widehat{A}_{i-1} &= \rho(\Gamma_{i-1}) \\
&= \rho(\Upsilon_i * \Gamma_i^{n_i} * \Omega_i) \\
&= \rho(\Upsilon_i)\rho(\Gamma_i)^{n_i}\rho(\Omega_i) \\
&= \rho(\Upsilon_i)\rho(\Gamma_i)^{n_i}\rho(\Omega_i)\rho(\Upsilon_i)\rho(\Upsilon_i)^{-1} \\
&= \widehat{W}_i \widehat{A}_i^{n_i} \widehat{V}_i \widehat{W}_i^{-1},
\end{aligned}$$

so

$$\widehat{W}_i^{-1}\widehat{A}_{i-1}\widehat{W}_i = \widehat{A}_i^{n_i}\widehat{V}_i.$$ ∎

Proof of (M4) For each vertex P in $\Gamma(p,q)$, $m \le P \le M+1$. Thus, since for each $i = 1, \dots, N$, $\widehat{W}_i$ and $\widehat{V}_i$ are subgraphs of $\Gamma(p,q)$, $\rho(\widehat{W}_i)$ and $\rho(\widehat{V}_i)$ are contained in the subgroup generated by $\{S_m, \dots, S_M\}$. Since no summits of Γ appear in Υ_i or Ω_i, S_M^{δ} cannot appear in $\widehat{V}_i$ or $\widehat{W}_i$. ∎

Proof of (M5) Suppose i is an integer with $0 \le i \le N$. The maximum grading of a vertex in Γ_i is $M+1$. Let l be the minimum grading of a vertex in Γ_i. For some integer coefficients $b_l, b_{l+1} \dots, b_M$,

$$[\rho(\Gamma_i)] = b_l S_l' + b_{l+1}S_{l+1}' + \cdots + b_M S_M'.$$

Our goal is to show that for each $j = 0, \dots, M-l$, $|b_{l+j}| = |b_{M-j}|$. ∎

The vertices of $\mathrm{cl}(\Gamma_i)$ can be classified into four types according to Figure 15. Define $\nu_{(**)}(n)$ to be the number vertices in $\mathrm{cl}(\Gamma_i)$ of type $(**)$ with grading n.

Suppose $n = l, \dots, M$. When n is even, S_n always has exponent -1 in $\rho(\Gamma_i)$, and S_n^{-1} appears precisely when there is negative edge followed by a vertex in $\mathrm{cl}(\Gamma_i)$ with grading n, so

$$|b_n| = \nu_{(--)}(n) + \nu_{(-+)}(n). \tag{5.2}$$

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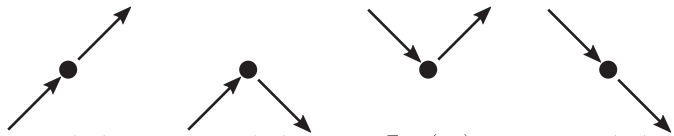

Figure 15: The four vertex types.

Similarly, when n is odd, S_n always has exponent 1 in $\rho(\Gamma_i)$, and S_n appears precisely when there is a vertex in $\mathrm{cl}(\Gamma_i)$ with grading n followed by a positive edge, so

$$|b_n| = v_{(++)}(n+1) + v_{(+-)}(n+1). \tag{5.3}$$

Since Γ_i is symmetric by (R5), there is an order-reversing bijection ϕ of the vertex set of $\mathrm{cl}(\Gamma_i)$ such that $\mathrm{gr}(P) + \mathrm{gr}(\phi(P)) = l + M + 1$ for each vertex P in $\mathrm{cl}(\Gamma_i)$. Furthermore, P and $\phi(P)$ have types rotated 180° with arrows reversed (see Figure 16). As a consequence,

$$\begin{aligned} v_{(--)}(n) &= v_{(--)}(l+M+1-n),\\ v_{(-+)}(n) &= v_{(+-)}(l+M+1-n),\\ v_{(++)}(n) &= v_{(++)}(l+M+1-n),\\ v_{(+-)}(n) &= v_{(-+)}(l+M+1-n). \end{aligned} \tag{5.4}$$

Each positive edge connects a vertex of type $(*+)$ to a vertex of type $(+*)$. Likewise, each negative edge connects a vertex of type $(*-)$ to a vertex of type $(-*)$ (see Figure 17). Thus,

$$\begin{aligned} v_{(++)}(n) + v_{(-+)}(n) &= v_{(++)}(n+1) + v_{(+-)}(n+1),\\ v_{(--)}(n) + v_{(+-)}(n) &= v_{(--)}(n-1) + v_{(-+)}(n-1). \end{aligned} \tag{5.5}$$

The incremental path Γ_i is closable, and the gradings of adjacent vertices in Γ_i differ by ± 1. It follows that every time Γ_i passes from below to above some grading level at a vertex, Γ_i must pass from above to below the same grading level at some other vertex. Thus, in each grading n,

$$v_{(++)}(n) = v_{(--)}(n). \tag{5.6}$$

Now, we show that $|b_{l+j}| = |b_{M-j}|$. Let j be an integer such that $0 \le j \le M - l$. When $l + j$ and $M - j$ are both even,

$$\begin{aligned} |b_{l+j}| &= v_{(--)}(l+j) + v_{(-+)}(l+j)\\ &= v_{(--)}(M-j+1) + v_{(+-)}(M-j+1)\\ &= v_{(--)}(M-j) + v_{(-+)}(M-j)\\ &= |b_{M-j}| \end{aligned}$$

by (5.2), (5.4), and (5.5).

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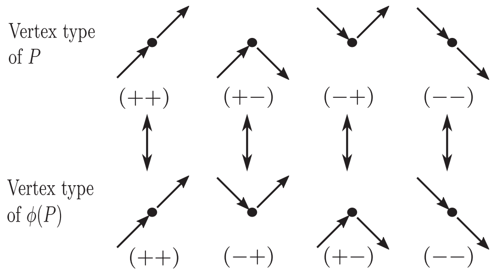

Figure 16: The effect of ϕ on vertex type.

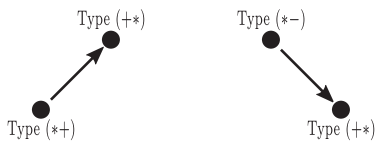

Figure 17: Vertex types of adjacent vertices.

When $l+j$ and $M-j$ are odd,

$$\begin{aligned}|b_{l+j}| &= v_{(++)}(l+j+1) + v_{(+-)}(l+j+1)\\ &= v_{(++)}(M-j) + v_{(-+)}(M-j)\\ &= v_{(++)}(M-j+1) + v_{(+-)}(M-j+1)\\ &= |b_{M-j}|\end{aligned}$$

by (5.3)–(5.5).

When $l+j$ is even and $M-j$ is odd,

$$\begin{aligned}|b_{l+j}| &= v_{(--)}(l+j) + v_{(-+)}(l+j)\\ &= v_{(--)}(M-j+1) + v_{(+-)}(M-j+1)\\ &= v_{(++)}(M-j+1) + v_{(+-)}(M-j+1)\\ &= |b_{M-j}|\end{aligned}$$

by (5.2), (5.4), (5.6), and (5.3).

When $l+j$ is odd and $M-j$ is even,

$$\begin{aligned}|b_{l+j}| &= v_{(++)}(l+j+1) + v_{(+-)}(l+j+1)\\ &= v_{(++)}(M-j) + v_{(-+)}(M-j)\\ &= v_{(--)}(M-j) + v_{(-+)}(M-j)\\ &= |b_{M-j}|\end{aligned}$$

by (5.3), (5.4), (5.6), and (5.2).

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When $i \geq 1$, no bottoms appear in Γ_i, so $l > m$.

Proof of (m1)–(m5) Since $\Gamma_0 = \Gamma$ is symmetric, there is an order-reversing bijection $\overline{\phi}$ on the vertices of $\mathrm{cl}(\Gamma)$ such that

$$\mathrm{gr}(P) + \mathrm{gr}(\overline{\phi}(P)) = m + M + 1$$

for each vertex P in $\mathrm{cl}(\Gamma)$. Thus, $\overline{\phi}$ induces a map on the subgraphs of $\mathrm{cl}(\Gamma)$.

For each $i = 0, \ldots, N$, define

$$\breve{A}_i := \rho(\overline{\phi}(\Gamma_{N-i})),$$

and when $i > 0$, define

$$\breve{V}_i := \rho(\overline{\phi}(\Upsilon_{N-i}))\rho(\overline{\phi}(\Omega_{N-i})) \text{ and } \breve{W}_i := \rho(\overline{\phi}(\Omega_{N-i})).$$

(m1)–(m5) follow from proofs similar to the those used for (M1)–(M5). ∎

5.5 Using reductions for induction

Suppose (p, q) is a relevant co-prime pair with $q > 1$ and with $(p \bmod q) \neq 1$. By Lemma 4.5, $R(\overline{\Gamma})(p, q)$ is isomorphic to $\overline{\Gamma}(p^*, q^*)$ for some relevant co-prime pair (p^*, q^*) defined as in Lemma 4.5. Along with Lemma 5.3, $\overline{\Gamma}(p, q)$ can be simplified through a sequence of reductions and relative isomorphisms to $\overline{\Gamma}(p_0, q_0)$ such that $q_0 = 1$ or $(p \bmod q) = 1$.

Example 5.5

$$\overline{\Gamma}(119, 43) \overset{R}{\to} \overline{\Gamma}(33, -23) \overset{rel.}{\cong} \overline{\Gamma}(33, 23) \overset{R}{\to} \overline{\Gamma}(10, 3).$$

The goal now is to show that when (p^*, q^*) is a pre-RTFN pair, (p, q) is also a pre-RTFN pair.

5.6 Leading and trailing vertices

Call a vertex in $\overline{\Gamma}(p, q)$ at the end of a $(\kappa + 1)$-segment a *leading vertex*, and a vertex at the beginning of a $(\kappa + 1)$-segment a *trailing vertex* (see Figure 18). Let P be a leading vertex in $\overline{\Gamma}(p, q)$, and let Λ_L be the $(\kappa + 1)$-segment of $\overline{\Gamma}(p, q)$ immediately preceding P. Define $f_L(P)$ to be the vertex at the end of the edge in $R(\overline{\Gamma})(p, q)$ corresponding to Λ_L. Let P be a trailing vertex in $\overline{\Gamma}(p, q)$, and let Λ_T be the $(\kappa + 1)$-segment of $\overline{\Gamma}(p, q)$ immediately following P. Define $f_T(P)$ to be the vertex at the beginning of the edge in $R(\overline{\Gamma})(p, q)$ corresponding to Λ_T. When the path from a leading vertex P_A to a trailing vertex P_B is a κ-block, $f_L(P_A) = f_T(P_B)$.

f_L is a bijection from the leading vertices of $\Gamma(p, q)$ to the vertex set of $R(\overline{\Gamma})(p, q)$, and f_T is a bijection from the trailing vertices of $\Gamma(p, q)$ to the vertex set of $R(\overline{\Gamma})(p, q)$. Let P^* be a vertex in $R(\overline{\Gamma})(p, q)$. Since $f_L^{-1}(P^*)$ and $f_T^{-1}(P^*)$ are separated by a κ-block of length κ' or $\kappa' - 1$, the gradings of $f_L^{-1}(P^*)$ and $f_T^{-1}(P^*)$ are either the same of differ by $\pm\kappa$.

Any vertex in $\overline{\Gamma}(p, q)$ at the end of a positive (or negative) segment is called a *peak* (resp. *valley*). There is a relationship between the gradings of the vertices in $\overline{\Gamma}(p, q)$ and $R(\overline{\Gamma})(p, q)$.

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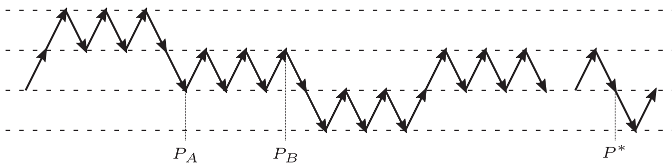

Figure 18: P_A is a leading vertex of $\overline{\Gamma}(13,11)$, and P_B is a trailing vertex of $\overline{\Gamma}(13,11)$ (left). $f_L(P_A) = f_T(P_B) = P^*$ in $R(\overline{\Gamma})(13,11)$ (right).

Proposition 5.6 *Let P and Q be leading vertices of* $\overline{\Gamma}(p,q)$.

(1) *If P and Q are both peaks or both valleys, then*

$$gr(f_L(P)) - gr(f_L(Q)) = gr(P) - gr(Q).$$

(2) *If P is a valley and Q is a peak, then*

$$gr(f_L(P)) - gr(f_L(Q)) = gr(P) - gr(Q) - \kappa.$$

(3) *If P is a peak and Q is a valley, then*

$$gr(f_L(P)) - gr(f_L(Q)) = gr(P) - gr(Q) + \kappa.$$

Proof This follows from Lemma 4.7. Let Γ^* be the unique path subgraph of $R(\overline{\Gamma})(p,q)$ beginning with $f_L(P)$ and ending $f_L(Q)$. Let e be the sign of the edge in $R(\overline{\Gamma})(p,q)$ preceding $f_L(P)$. Let Γ be $\tilde{E}(\Gamma^*,\kappa,\kappa',e)$. Finally, let Q_0 be the first vertex in Γ.

The vertices P and Q are at the end of $(\kappa+1)$-segments, which we will denote as Λ_P and Λ_Q, respectively. The first vertex in Γ^* is $f_L(P)$. Also, e and Λ_P will always have the same sign. Therefore,

$$\mathrm{gr}(P) - \mathrm{gr}(Q_0) = 0.$$

Thus,

$$\begin{aligned} \mathrm{gr}(P) - \mathrm{gr}(Q) &= (\mathrm{gr}(P) - \mathrm{gr}(Q_0)) - (\mathrm{gr}(Q) - \mathrm{gr}(Q_0)) \\ &= \mathrm{gr}(Q_0) - \mathrm{gr}(Q). \end{aligned} \tag{5.7}$$

Also, the first vertex P_0 in Γ^* is $f_L(P)$, so

$$\mathrm{gr}(P_0) - \mathrm{gr}(f_L(Q)) = \mathrm{gr}(f_L(P)) - \mathrm{gr}(f_L(Q)). \tag{5.8}$$

The signs of e are determined by whether P is a peak or a valley:

$$e = \begin{cases} 1, & P \text{ is a peak,} \\ -1, & P \text{ is a valley.} \end{cases}$$

Also, Λ_Q is positive when Q is a peak and negative when Q is a valley. The desired result follows from Lemma 4.7, (5.7), and (5.8). ∎

Corollary 5.7 *Given a leading vertex P of* $\overline{\Gamma}(p,q)$, *P is a summit of* $\overline{\Gamma}(p,q)$ *if and only if* $f_L(P)$ *is a summit of* $R(\overline{\Gamma})(p,q)$.

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Proof Suppose a leading vertex P of $\overline{\Gamma}(p,q)$ is a summit. Consider a vertex in $R(\overline{\Gamma})(p,q)$ which is $f_L(Q)$ for some leading vertex Q of $\overline{\Gamma}(p,q)$. Since P is a summit, $\mathrm{gr}(P)-\mathrm{gr}(Q)\geq 0$. Since P is a peak, by Proposition 5.6, either

$$\mathrm{gr}(f_L(P))-\mathrm{gr}(f_L(Q))=\mathrm{gr}(P)-\mathrm{gr}(Q)\geq 0$$

or

$$\mathrm{gr}(f_L(P))-\mathrm{gr}(f_L(Q))=\mathrm{gr}(P)-\mathrm{gr}(Q)+\kappa\geq 0.$$

Thus, $f_L(P)$ is a summit of $R(\overline{\Gamma})(p,q)$.

Conversely, suppose for some leading vertex P of $\overline{\Gamma}(p,q)$ that $f_L(P)$ is a summit of $R(\overline{\Gamma})(p,q)$. Let Q be a summit of $\overline{\Gamma}(p,q)$, so $\mathrm{gr}(P)-\mathrm{gr}(Q)\leq 0$. Since Q is a peak, by Proposition 5.6,

$$\mathrm{gr}(P)-\mathrm{gr}(Q)\geq \mathrm{gr}(f_L(P))-\mathrm{gr}(f_L(Q))\geq 0.$$

The last inequality is true since $f_L(P)$ is a summit. Thus, $\mathrm{gr}(P)=\mathrm{gr}(Q)$, so since Q is a summit of $\overline{\Gamma}(p,q)$, P is also. ∎

5.7 Proof of Lemma 3.5

We now have everything we need to show that every relevant co-prime pair (p,q) with p positive and q odd is a pre-RTFN pair. For each relevant co-prime pair, we need to find a positive integer N, sequences of subgraphs of $\Gamma(p,q)$

$$\Gamma_0,\ldots,\Gamma_N,$$

$$\Upsilon_1,\ldots,\Upsilon_N,$$

and

$$\Omega_1,\ldots,\Omega_N,$$

and integers

$$n_1,\ldots,n_N,$$

satisfying (R1)–(R5). We prove this using a strong induction starting with the base cases below.

Let Γ be an incremental path, and let P,P' be vertices in Γ. Define $\omega(\Gamma,P',P)$, the unique path in $\mathrm{cl}(\Gamma)$ from P' to P.

Lemma 5.8 *Let (p,q) be a relevant co-prime pair with p and q positive and q odd. If $q=1$ or $(p \bmod q)=1$, then (p,q) is a pre-RTFN pair.*

Proof $\Gamma(p,q)$ has $2p+1$ vertices $P_0\ldots,P_{2p}$.

When $q=1$, the grading are

$$\mathrm{gr}(P_i)=\begin{cases} i, & 0\leq i\leq p,\\ 2p-i, & p\leq i\leq 2p.\end{cases}$$

See Figure 19a.

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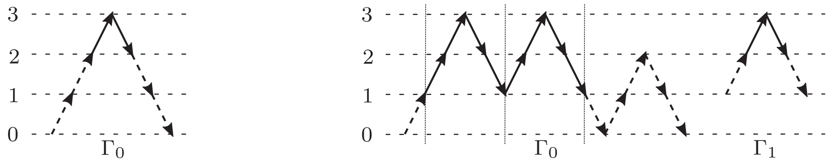

(a) $\Gamma(3,1)$ (left) only has one summit. The solid arrows indicate Γ_1.

(b) $\Gamma(7,3)$ (right) has two summits both in one 2-block of length 2. The solid arrows indicate Γ_1 and Γ_2 (in Γ_1).

Figure 19: Cycle graphs with $q = 1$ and $(p \bmod q) = 1$.

Make the following choices of subgraphs and integers:

- Let $N = 1$.
- Let $\Gamma_0 = \Gamma(p,q)$.
- Let $\Gamma_1 = \Gamma_{\text{top}}$.
- Let $n_1 = 1$.
- Let $\Upsilon_1 = \omega(\Gamma(p,q), P_0, P_{p-1})$.
- Let $\Omega_1 = \omega(\Gamma(p,q), P_{p+1}, P_{2p})$.

It is clear that (R1) and (R2) are satisfied.

$$\begin{aligned}\Gamma_0 = \Gamma(p,q) &= \omega(\Gamma(p,q), P_0, P_\kappa) * \omega(\Gamma(p,q), P_\kappa, P_{\kappa+2}) * \omega(\Gamma(p,q), P_{\kappa+2}, P_{2p})\\ &= \Upsilon_1 * \omega(\Gamma(p,q), P_\kappa, P_{\kappa+2}) * \Omega_1\\ &= \Upsilon_1 * \Gamma_1 * \Omega_1.\end{aligned}$$

Thus, (R3) is satisfied.

The grading of a summit of $\Gamma(p,1)$ is p. Since the maximum grading of a vertices in Υ_1 or Ω_1 is $p-1$, Υ_1 or Ω_1 contain no summits of $\Gamma(p,1)$, so (R4) is satisfied.

Consider the map $\phi : \overline{\Gamma}(p,1) \to \overline{\Gamma}(p,1)$ defined by

$$\phi(P_i) := \begin{cases} P_{p-i}, & 0 \le i \le p, \\ P_{3p-i}, & p < i < 2p, \end{cases}$$

When $0 \le i \le p$,

$$\text{gr}(P_i) + \text{gr}(\phi(P_i)) = i + p - i = p.$$

When $p < i < 2p$,

$$\text{gr}(P_i) + \text{gr}(\phi(P_i)) = 2p - i + 2p - (3p - i) = p,$$

so Γ_0 is symmetric.

Since $\Gamma_1 \cong \Gamma_{\text{top}}$, its closure has two vertices: one graded $p-1$ and one graded p. Consider the map $\phi : \text{cl}(\Gamma_1) \to \text{cl}(\Gamma_1)$, which exchanges the two vertices. For each vertex P in $\text{cl}(\Gamma_1)$,

$$\text{gr}(P) + \text{gr}(\phi(P)) = p + p - 1 = 2p - 1,$$

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so Γ_1 is symmetric. Also, the minimum grading of a vertex in $\Gamma(p,1)$ is 0, and the minimum grading of a vertex in Γ_1 is p. Therefore, no bottoms of $\Gamma(p,1)$ are in Γ_1. Thus, (R5) is satisfied.

When $p \bmod q = 1$, define κ and κ' as in (4.2) and (4.3). Since $p \bmod q = 1$, $\kappa' = q$, which is odd. By Proposition 4.3(a) and (e), $\overline{\Gamma}(p,q)$ has two $(\kappa+1)$-segments, and has $2q-2$ κ-segments. By Proposition 4.4(a), the κ-segments must be contained in two κ-blocks of length $q-1$. It follows that $\Gamma(p,q)$ is the concatenation of a positive $(\kappa+1)$-segment, a κ-block of length $q-1$, a negative $(\kappa+1)$-segment, followed by another κ-block of length $q-1$ (see Figure 19b).

Explicitly, the gradings are

$$\mathrm{gr}(P_i) = \begin{cases} 0, & i = 0 \leq i \leq \kappa+1, \\ j\kappa+2-i, & (j-1)\kappa+1 \leq i \leq j\kappa+1 \text{ and } \\ & j = 2,4,\dots,q-1, \\ i-j\kappa, & j\kappa+1 \leq i \leq (j+1)\kappa+1 \text{ and } \\ & j = 2,\dots,q-1, \\ p+\kappa+1-i, & p \leq i \leq p+\kappa+1, \\ i-p-(j-1)\kappa-1, & p+(j-1)\kappa+1 \leq i \leq p+j\kappa+1 \text{ and } \\ & j = 2,\dots,q-1, \\ p+(j+1)\kappa+1-i, & p+j\kappa+1 \leq i \leq p+(j+1)\kappa+1 \text{ and } \\ & j = 2,\dots,q-1. \end{cases}$$

When $\kappa = 1$, make the following choices:

- Let $N = 1$.
- Let $\Gamma_0 = \Gamma(p,q)$.
- Let $\Gamma_1 = \Gamma_{\mathrm{top}}$.
- Let $n_1 = (q+1)/2$.
- Let $\Upsilon_1 = \omega(\Gamma(p,q), P_0, P_1)$.
- Let $\Omega_1 = \omega(\Gamma(p,q), P_{1+2n_1}, P_{2p})$.

When $\kappa > 1$, make the following choices:

- Let $N = 2$.
- Let $\Gamma_0 = \Gamma(p,q)$.
- Let $\Gamma_1 = \omega(\Gamma(p,q), P_1, P_{1+2\kappa})$.
- Let $\Gamma_2 = \Gamma_{\mathrm{top}}$.
- Let $n_1 = (q+1)/2$.
- Let $n_2 = 1$.
- Let $\Upsilon_1 = \omega(\Gamma(p,q), P_0, P_1)$.
- Let $\Omega_1 = \omega(\Gamma(p,q), P_{1+2\kappa n_1}, P_{2p})$.
- Let $\Upsilon_2 = \omega(\Gamma(p,q), P_1, P_\kappa)$.
- Let $\Omega_2 = \omega(\Gamma(p,q), P_{\kappa+2}, P_{2\kappa+1})$.

Again, it is clear that (R1) and (R2) are satisfied.

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When $\kappa = 1$,

$$\begin{aligned}\Gamma_0 = \Gamma(p,q) &= \omega(\Gamma(p,q),P_0,P_1) * \omega(\Gamma(p,q),P_1,P_{1+2n_1}) * \omega(\Gamma(p,q),P_{1+2n_1},P_{2p})\\ &= \Upsilon_1 * \Gamma_1^{n_1} * \Omega_1.\end{aligned}$$

When $\kappa > 1$,

$$\begin{aligned}\Gamma_0 = \Gamma(p,q) &= \omega(\Gamma(p,q),P_0,P_1) * \omega(\Gamma(p,q),P_1,P_{1+2\kappa n_1}) * \omega(\Gamma(p,q),P_{1+2\kappa n_1},P_{2p})\\ &= \Upsilon_1 * \Gamma_1^{n_1} * \Omega_1\end{aligned}$$

and

$$\begin{aligned}\Gamma_1 &= \omega(\Gamma(p,q),P_1,P_{1+2\kappa})\\ &= \omega(\Gamma(p,q),P_1,P_\kappa) * \omega(\Gamma(p,q),P_\kappa,P_{\kappa+2}) * \omega(\Gamma(p,q),P_{\kappa+2},P_{2\kappa+1})\\ &= \Upsilon_2 * \Gamma_2 * \Omega_2.\end{aligned}$$

Thus, (R3) is satisfied.

The grading of a summit of $\Gamma(p,1)$ is $\kappa+1$. The maximum grading of a vertex in Υ_1 is 1, and the maximum grading of a vertex in Υ_1, Ω_1, or Ω_2 is κ. Thus, Υ_1, Υ_2, Ω_1, or Ω_2 contains no summits of $\Gamma(p,q)$, so (R4) is satisfied.

Consider the map $\phi : \overline{\Gamma}(p,q) \to \overline{\Gamma}(p,q)$ defined by

$$\phi(P_i) := \begin{cases} P_{\kappa+1-i}, & 0 \le i \le \kappa+1,\\ P_{2p+\kappa+1-i}, & \kappa+1 < i < 2p.\end{cases}$$

When $0 \le i \le \kappa+1$,

$$\mathrm{gr}(P_i) + \mathrm{gr}(\phi(P_i)) = i + \kappa + 1 - i = \kappa + 1.$$

When $p \le i \le p+\kappa+1$,

$$p \le 2p + \kappa + 1 - i \le p + \kappa + 1.$$

Thus,

$$\mathrm{gr}(P_i) + \mathrm{gr}(\phi(P_i)) = p + \kappa + 1 - i + p + \kappa + 1 - (2p + \kappa + 1 - i) = \kappa + 1.$$

Let $j \in \{2,4,\dots,q-1\}$.

When $(j-1)\kappa + 1 \le i \le j\kappa + 1$,

$$2p + \kappa + 1 - (j\kappa + 1) \le 2p + \kappa + 1 - i \le 2p + \kappa + 1 - ((j-1)\kappa + 1).$$

Since $p = q\kappa + 1$, we can substitute $2p = p + q\kappa + 1$ on each side to obtain

$$p + (q+1-j)\kappa + 1 \le 2p + \kappa + 1 - i \le p + (q+2-j)\kappa + 1.$$

Let $l = q + 1 - j$. Thus,

$$p + l\kappa + 1 \le 2p + \kappa + 1 - i \le p + (l+1)\kappa + 1.$$

Therefore,

$$\begin{aligned}\mathrm{gr}(P_i) + \mathrm{gr}(\phi(P_i)) &= j\kappa + 2 - i + p + (l+1)\kappa + 1 - (2p + \kappa + 1 - i)\\ &= (j+l)\kappa + 2 - p\\ &= q\kappa + 1 - p + \kappa + 1\\ &= \kappa + 1.\end{aligned}$$

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When $j\kappa + 1 \le i \le (j+1)\kappa + 1$,

$$2p + \kappa + 1 - ((j+1)\kappa + 1) \le 2p + \kappa + 1 - i \le 2p + \kappa + 1 - (j\kappa + 1).$$

We substitute $2p = p + q\kappa + 1$ on each side to obtain

$$p + (q-j)\kappa + 1 \le 2p + \kappa + 1 - i \le p + (q-j+1)\kappa + 1.$$

Thus,

$$p + (l-1)\kappa + 1 \le 2p + \kappa + 1 - i \le p + l\kappa + 1.$$

Therefore,

$$\begin{aligned} \mathrm{gr}(P_i) + \mathrm{gr}(\phi(P_i)) &= i - j\kappa + 2p + \kappa + 1 - i - p - (l-1)\kappa - 1 \\ &= -(j+l)\kappa + p + \kappa + \kappa \\ &= -(q+1)\kappa + p + \kappa + \kappa \\ &= p - q\kappa + \kappa \\ &= \kappa + 1. \end{aligned}$$

When $p + j\kappa + 1 \le i \le p + (j+1)\kappa + 1$, there is some integer c such that

$$(j-1)\kappa + 1 \le c \le j\kappa + 1$$

and $P_i = \phi(P_c)$. Since ϕ^2 is the identity map,

$$\mathrm{gr}(P_i) + \mathrm{gr}(\phi(P_i)) = \mathrm{gr}(\phi(P_c)) + \mathrm{gr}(P_c) = \kappa + 1.$$

When $p + (j-1)\kappa + 1 \le i \le p + j\kappa + 1$, there is some integer c such that

$$j\kappa + 1 \le c \le (j+1)\kappa + 1$$

and $P_i = \phi(P_c)$. Similar to the previous case,

$$\mathrm{gr}(P_i) + \mathrm{gr}(\phi(P_i)) = \mathrm{gr}(\phi(P_c)) + \mathrm{gr}(P_c) = \kappa + 1.$$

Therefore, Γ_0 is symmetric.

The choices of Γ_1 and Γ_2 are relatively isomorphic to either $\Gamma(\kappa, 1)$ or Γ_{top} which are symmetric. Therefore, Γ_1 and Γ_2 are symmetric.

When $(p \bmod q) = 1$, the minimum grading of a vertex in $\Gamma(p,q)$ is 0. The minimum grading for a vertex in Γ_1 is 1. The minimum grading for a vertex in Γ_2 is κ. Therefore, no bottoms of $\Gamma(p,q)$ are contained in Γ_1 or Γ_2. Thus, (R5) is satisfied.

In conclusion, $\Gamma(p,q)$ is a pre-RTFN pair when $q = 1$ or $(p \bmod q) = 1$. ■

Let (p,q) be a relevant co-prime pair with $q > 1$ and $(p \bmod q) > 1$, and let (p^*, q^*) be the co-prime pair defined by Lemma 4.5. Suppose (p^*, q^*) is a pre-RTFN pair, so there are a positive integer N^* and subgraphs

$$\Gamma_0^*, \ldots, \Gamma_{N^*}^*,$$

$$\Upsilon_1^*, \ldots, \Upsilon_{N^*}^*,$$

and

$$\Omega_1^*, \ldots, \Omega_{N^*}^*,$$

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and integers

$$n_1^*, \ldots, n_{N^*}^*,$$

satisfying (R1)–(R5).

To show that (p,q) is a pre-RTFN pair, we need to define N, the subgraphs $\{\Gamma_i\}_{i=0}^N$, $\{\Upsilon_i\}_{i=1}^N$, and $\{\Omega_i\}_{i=1}^N$, and the integers $\{n_i\}_1^N$ for (p,q). This choice depends on how expansion affects the nested repeating pattern of summits in $\overline{\Gamma}(p^*,q^*)$.

Define κ and κ' as in (4.2) and (4.3), so $\overline{\Gamma}(p,q) \cong E(\overline{\Gamma}(p^*,q^*),\kappa,\kappa')$ by Proposition 4.9. Suppose Γ^* is a proper subgraph of $\Gamma(p^*,q^*)$, so Γ^* naturally embeds into $\overline{\Gamma}(p^*,q^*)$. Let e^* be the sign of the edge immediately preceding Γ^* in $\overline{\Gamma}(p^*,q^*)$. For simplicity of notation, we define

$$\tilde{E}(\Gamma^*) := \tilde{E}(\Gamma^*,\kappa,\kappa',e^*),$$

and when Γ^* is closable, define

$$E(\Gamma^*) := E(\Gamma^*,\kappa,\kappa').$$

Notice that $E(\Gamma^*)$ and $\tilde{E}(\Gamma^*)$ are not always the same.

Ideally, when $i = 0,\ldots,N^*$, we want to define Γ_i to be $E(\Gamma_i^*)$ and set n_i equal to n_i^*. Then, we examine the structure of $E(\Gamma_{N^*}^*)$. The hope is that since the expansion operation is compatible with concatenation (see Lemma 4.8), we can leverage the pre-RTFN pair properties of the Γ_i^*, Υ_i^*, and Ω_i^* sequences to prove that Γ_i, Υ_i, and Ω_i also satisfy the pre-RTFN properties. This turns out to be more subtle than one might first expect.

For all $i = 1,\ldots,N^*$,

$$\Gamma_{i-1}^* \cong \Upsilon_i^* * (\Gamma_i^*)^{n_i^*} * \Omega_i^*$$

by (R3) for (p^*,q^*). By Lemma 4.8,

$$\tilde{E}(\Gamma_{i-1}^*) \cong \tilde{E}(\Upsilon_i^*) * \tilde{E}((\Gamma_i^*)^{n_i^*}) * \tilde{E}(\Omega_i^*). \tag{5.9}$$

However, if Γ_i is $E(\Gamma_i^*)$, then $\Gamma_i^{n_i}$ is $E(\Gamma_i^*)^{n_i^*}$, and $E(\Gamma_i^*)^{n_i^*}$ may not be equal to $\tilde{E}((\Gamma_i^*)^{n_i^*})$. We show that they can be made equal by adding or removing κ edges. See Figure 20.

Consider $i \in \{1,\ldots,N^*\}$. Define $\tilde{\Gamma}_i := \tilde{E}((\Gamma_i^*)^{n_i^*})$. Let $\tilde{\Gamma}_i^+$ be $\tilde{\Gamma}_i$ with the κ edges in $\Gamma(p,q)$ preceding the first vertex of $\tilde{E}((\Gamma_i^*)^{n_i^*})$ added. Let $\tilde{\Gamma}_i^-$ be $\tilde{\Gamma}_i$ with the first κ vertices with their incident edges removed.

Lemma 5.9 *One of $\tilde{\Gamma}_i$, $\tilde{\Gamma}_i^+$, or $\tilde{\Gamma}_i^-$ is isomorphic to $E(\Gamma_i^*)^{n_i^*}$.*

Proof Consider $(\Gamma_i^*)^{n_i}$ as a subgraph of $\overline{\Gamma}(p^*,q^*)$. For all but the first Γ_i^* in $(\Gamma_i^*)^{n_i}$, $\tilde{E}(\Gamma_i^*) \cong E(\Gamma_i^*)$, so

$$\tilde{\Gamma}_i \cong \tilde{E}(\Gamma_i^*) * (E(\Gamma_i^*))^{n_i^*-1}.$$

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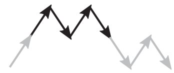
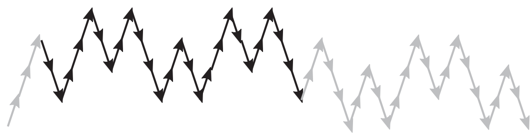
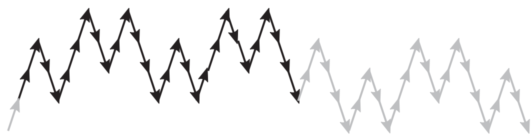


(a) The graph $R(\overline{\Gamma})(26, 11) = \overline{\Gamma}(4, 3)$ with $(\Gamma_1^*)^2$ in black and Υ_1^* and Ω_1^* in gray.



(b) The graph $\Gamma(26, 11) = E(\Gamma(4, 3))$ with $\tilde{E}((\Gamma_1^*)^2)$ in black and $E(\Upsilon_1^*)$ and $E(\Omega_1^*)$ in gray.



(c) The graph $\Gamma_0 = \Gamma(26, 11)$ with $\Gamma_1^2 = E(\Gamma_1^*)^2$ in black and Υ_1 and Ω_1 in gray.

Figure 20: Expanding $\Gamma(4, 3)$ to $\Gamma(26, 11)$.

Let e_0 be the sign of the first edge in Γ_i^*, and define

$$\Gamma_{\text{short}} := \begin{cases} \tilde{E}(\Gamma_i^*, \kappa, \kappa', +1), & (e_0 \text{ is 1 and } \kappa' \text{ is even) or } (e_0 \text{ is -1 and } \kappa' \text{ is odd)} \\ \tilde{E}(\Gamma_i^*, \kappa, \kappa', -1), & (e_0 \text{ is 1 and } \kappa' \text{ is odd) or } (e_0 \text{ is -1 and } \kappa' \text{ is even)} \end{cases}$$

and

$$\Gamma_{\text{long}} := \begin{cases} \tilde{E}(\Gamma_i^*, \kappa, \kappa', -1), & (e_0 \text{ is 1 and } \kappa' \text{ is even) or } (e_0 \text{ is -1 and } \kappa' \text{ is odd)}, \\ \tilde{E}(\Gamma_i^*, \kappa, \kappa', +1), & (e_0 \text{ is 1 and } \kappa' \text{ is odd) or } (e_0 \text{ is -1 and } \kappa' \text{ is even)}. \end{cases}$$

Notice that Γ_{long} is a κ-segment concatenated with Γ_{short}. Each of $\tilde{E}(\Gamma_i^*)$ and $E(\Gamma_i^*)$ is isomorphic to Γ_{long} or Γ_{short}.

When $\tilde{E}(\Gamma_i^*)$ is isomorphic to $E(\Gamma_i^*)$, $\tilde{\Gamma}_i$ is isomorphic to $E(\Gamma_i^*)^{n_i^*}$.

When $\tilde{E}(\Gamma_i^*)$ is isomorphic to Γ_{short} and $E(\Gamma_i^*)$ is isomorphic to Γ_{long}, $\tilde{\Gamma}_i^-$ is isomorphic to $E(\Gamma_i^*)^{n_i^*}$.

When $\tilde{E}(\Gamma_i^*)$ is isomorphic to Γ_{long} and $E(\Gamma_i^*)$ is isomorphic to Γ_{short}, $\tilde{\Gamma}_i^+$ is isomorphic to $E(\Gamma_i^*)^{n_i^*}$. ∎

Now, we analyze the structure of $E(\Gamma_{N^*}^*)$. In particular, we want to know where the summits $\overline{\Gamma}(p, q)$ are located. By Corollary 5.7, the leading summits of $\overline{\Gamma}(p, q)$ correspond to the summits of $\overline{\Gamma}(p^*, q^*)$. Now, we consider the nonleading summits in $\overline{\Gamma}(p, q)$. Let d be κ' or $\kappa' - 1$ whichever is even.

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Let Γ_{top} and Γ^*_{top} be defined for (p,q) and (p^*,q^*), respectively, as shown in Figure 13. By (R2), $\Gamma^*_{N^*}$ is isomorphic to Γ^*_{top}. By definition, $E(\Gamma^*_{\mathrm{top}})$ is the concatenation of a κ-block of even length, a positive $(\kappa+1)$-segment, another κ-block of even length, and a negative $(\kappa+1)$-segment. It follows that every summit in $\overline{\Gamma}(p^*,q^*)$ corresponds to $d/2+1$ summits in $\Gamma(p,q)$.

We are ready to define N, $\{\Gamma_i\}_0^N$, $\{\Upsilon_i\}_0^N$, $\{\Omega_i\}_0^N$ and $\{n_i\}_1^N$. Let Γ be an incremental path with vertex set $\{V_0,\dots,V_n\}$, and let Υ be a connected subgraph of Γ with vertices $V_i,\dots,V_{i+k}$. Define $\mathrm{Left}(\Gamma,\Upsilon)$ to be $\omega(\Gamma,V_0,V_i)$, and define $\mathrm{Right}(\Gamma,\Upsilon)$ to be $\omega(\Gamma,V_{i+k},V_n)$.

For each $i=1,\dots,N^*$, exactly one of the subgraphs $\tilde{\Gamma}_i$, $\tilde{\Gamma}_i^+$, or $\tilde{\Gamma}_i^-$ is isomorphic to $E(\Gamma_i^*)^{n_i^*}$ by Lemma 5.9. Call this subgraph Γ_i'. For each $i=0,\dots,N^*$, we make the following choices.

- Let $\Gamma_i = E(\Gamma_i^*)$.
- Let $n_i = n_i^*$.
- Let $\Upsilon_i = \mathrm{Left}(\Gamma_{i-1},\Gamma_i')$.
- Let $\Omega_i = \mathrm{Right}(\Gamma_{i-1},\Gamma_i')$.

Note that since Γ_{i-1} is $E(\Gamma^*_{i-1})$, Γ_i' is a subgraph of Γ_{i-1} by (5.9).

Let $\{Q_0,\dots,Q_n\}$ be the vertex set of Γ_{N^*}. Since $\Gamma_{N^*}=E(\Gamma^*_{N^*})$,

$$n = 2(\kappa + d\kappa + 1).$$

Suppose $\kappa'=1$, so $n=2(\kappa+1)$.

- Let $N=N^*+1$.
- Let $\Gamma_N=\Gamma_{\mathrm{top}}$.
- Let $n_N = d/2+1$.
- Let $\Upsilon_N=\omega(\Gamma_{N^*-1},Q_0,Q_\kappa)$.
- Let $\Omega_N=\omega(\Gamma_{N^*-1},Q_{\kappa+2},Q_n)$.

Suppose $\kappa=1$.

- Let $N=N^*+1$.
- Let $\Gamma_N=\Gamma_{\mathrm{top}}$.
- Let $n_N = d/2+1$.
- Let $\Upsilon_N=\omega(\Gamma_{N^*-1},Q_d,Q_{d+1})$.
- Let $\Omega_N=\omega(\Gamma_{N^*-1},Q_{n-1},Q_n)$.

Suppose $\kappa'>1$ and $\kappa>1$.

- Let $N=N^*+2$.
- Let Γ_{N-1} be a positive κ-segment followed by a negative κ-segment.
- Let $\Gamma_N=\Gamma_{\mathrm{top}}$.
- Let $n_{N-1}=d/2+1$.
- Let $n_N=1$.
- Let $\Upsilon_{N-1}=\omega(\Gamma_{N^*-1},Q_{d\kappa},Q_{d\kappa+1})$.
- Let $\Omega_{N-1}=\omega(\Gamma_{N^*-1},Q_{2d\kappa+3},Q_n)$.
- Let $\Upsilon_N=\omega(\Gamma_{N^*-1},Q_{d\kappa+1},Q_{d\kappa+\kappa})$.
- Let $\Omega_N=\omega(\Gamma_{N^*-1},Q_{d\kappa+\kappa+2},Q_{d\kappa+2\kappa+1})$.

Lemma 5.10 *The integers $\{n_i\}_{i=1}^N$ and the subgraphs $\{\Gamma_i\}_{i=0}^N$, $\{\Upsilon_i\}_{i=1}^N$, and $\{\Omega_i\}_{i=1}^N$ satisfy* (R1)–(R4).

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Proof Since $\Gamma_0^* \cong \Gamma(p^*, q^*)$,

$$\Gamma_0 \cong E(\Gamma(p^*, q^*)) \cong \Gamma(p, q),$$

so (R1) is satisfied.

By definition, $\Gamma_N \cong \Gamma_{\text{top}}$, so (R2) is satisfied.

When $N^* < i \le N$, (R3) and (R4) are satisfied by proofs similar to those in Lemma 5.8.

Suppose $i \in \{1, \dots, N^*\}$:

$$\begin{aligned}\Gamma_{i-1} &= \text{Left}(\Gamma_{i-1}, \Gamma_i') * \Gamma_i' * \text{Right}(\Gamma_{i-1}, \Gamma_i') \\ &\cong \Upsilon_i * E(\Gamma_i^*)^{n_i^*} * \Omega_i \\ &= \Upsilon_i * \Gamma_i^{n_i} * \Omega_i.\end{aligned}$$

Therefore, (R3) is satisfied.

$\tilde{E}((\Gamma_i^*)^{n_i^*})$ is Γ_i' possibly with κ edges added to or removed from the beginning. Also,

$$\Upsilon_i * \Gamma_i' * \Omega_i \cong \Gamma_{i-1} \cong \tilde{E}(\Gamma_{i-1}^*) \cong \tilde{E}(\Upsilon_i^*) * \tilde{E}((\Gamma_i^*)^{n_i^*}) * \tilde{E}(\Omega_i^*).$$

It follows that Ω_i is $\tilde{E}(\Omega_i^*)$ and Υ_i is $\tilde{E}(\Upsilon_i^*)$ with possibly κ edges added to or removed from the end (see Figure 20).

Since no summits of $\Gamma(p^*, q^*)$ are in Υ_i^*, there are no summits of $\Gamma(p, q)$ in $\tilde{E}(\Upsilon_i^*)$. It follows that if Υ_i is equal to $\tilde{E}(\Upsilon_i^*)$ or is $\tilde{E}(\Upsilon_i^*)$ with edges removed, then Υ_i contains no summits of $\Gamma(p, q)$.

Consider the case when Υ_i is $\tilde{E}(\Upsilon_i^*)$ with a κ-segment added. Let P be the vertex at the end of $\tilde{E}(\Upsilon_i^*)$. If the segment added is negative, then the gradings of the vertices added to $\tilde{E}(\Upsilon_i^*)$ are less than $\text{gr}(P)$, so they cannot be summits.

If the segment added is positive, then P is at the end of either a κ-segment or a $(\kappa+1)$-segment. In either case, the maximum grading of a vertex in $\tilde{E}(\Upsilon_i^*)$ is at least $\text{gr}(P) + \kappa$. Since none of the vertices of $\tilde{E}(\Upsilon_i^*)$ are summits of $\Gamma(p, q)$, the grading of a summit of $\Gamma(p, q)$ must be bigger than $\text{gr}(P) + \kappa$. Since only κ edges are being added, the gradings of the vertices added to $\tilde{E}(\Upsilon_i^*)$ are no bigger than $\text{gr}(P) + \kappa$, so they cannot be summits of $\Gamma(p, q)$. Thus, there are no summits in Υ_i.

Since no summits are in Ω_i^*, there are no summits $\tilde{E}(\Omega_i^*) \cong \Omega_i$. Therefore, (R4) is satisfied. ∎

Lemma 5.11 *The subgraphs $\{\Gamma_i\}_0^N$ satisfy (R5).*

Proof First, we show that Γ_i has no bottoms for each $i = 1, \dots, N$. Since $N^* \ge 1$, $\Gamma_1 = E(\Gamma_1^*)$. Since Γ_1^* has no bottoms, Γ_1 does not have bottoms. When $1 \le i \le N$,

$$\Gamma_{i-1} \cong \Upsilon_i \Gamma_i^{n_i} * \Omega_i,$$

so Γ_i is a subgraph of Γ_1. Therefore, Γ_i has no bottoms.

Suppose $0 \le i \le N$. Here, we show that Γ_i is symmetric. When $i > N^*$, Γ_i is either the concatenation of a positive κ-segment and a negative κ-segment or Γ_{top}. In both cases, Γ_i can be shown to be symmetric by an argument similar to those used in the proof of Lemma 5.8.

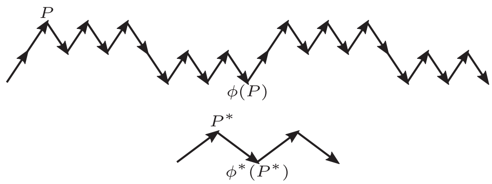

Figure 21: The incremental cycles $\mathsf{cl}(\Gamma_i)$ (top) and $\mathsf{cl}(\Gamma_i^*)$ (bottom) are shown. P is a leading vertex, and $f_L(P)$ is denoted P^*. $\phi(P)$ is a trailing vertex, and $\phi^*(P^*) = f_T(\phi(P))$.

Suppose $0 \le i \le N^*$. In this case, $\Gamma_i = E(\Gamma_i^*)$. Our goal is to show that since Γ_i^* is symmetric, Γ_i is also symmetric.

Since Γ_i^* is symmetric, there are an order-reversing bijection ϕ^* on the set of vertices of $\mathsf{cl}(\Gamma_i^*)$ and an integer k^* such that for each P^* in $\mathsf{cl}(\Gamma_i^*)$,

$$\mathrm{gr}(P^*) + \mathrm{gr}(\phi^*(P^*)) = k^*. \tag{5.10}$$

Let V_L and V_T be the sets of leading and trailing vertices of $\mathsf{cl}(\Gamma_i)$, respectively, and let V^* be the vertex set of $\mathsf{cl}(\Gamma_i^*)$. Define ϕ to be the unique order-reversing bijection on the vertices of $\mathsf{cl}(\Gamma_i)$ such that the following diagram commutes:

$$\begin{array}{ccc} V_L & \xrightarrow{\phi|_{V_L}} & V_T \\ {\scriptstyle f_L}\big\downarrow & & \big\downarrow{\scriptstyle f_T} \\ V^* & \xrightarrow{\phi^*} & V^* \end{array}$$

In particular, ϕ maps leading vertices bijectively to trailing vertices (see Figure 21). Let P_S be a leading summit of Γ_i, and let $P_S^* = f_L(P_S)$ in Γ_i^*.

Let $k = \mathrm{gr}(P_S) + \mathrm{gr}(\phi(P_S))$, and let P be an arbitrary vertex in Γ_i. The goal is to show that $\mathrm{gr}(P) + \mathrm{gr}(\phi(P)) = k$, which is done in four cases.

Case 1. Suppose P is a leading vertex and $P^* := f_L(P)$ has the same vertex type as P, either a peak (type $(-+)$) or valley (type $(+-)$). Recall from Figure 16 how the automorphism ϕ^* affects vertex type. If P^* is of type $(-+)$, then $\phi^*(P^*)$ is of type $(+-)$, and if P^* is of type $(+-)$, then $\phi^*(P^*)$ is of type $(-+)$. Therefore, either $f_L^{-1}(P^*)$ and $f_T^{-1}(P^*)$ are both peaks and $f_L^{-1}(\phi^*(P^*))$ and $f_T^{-1}(\phi^*(P^*))$ are both valleys or $f_L^{-1}(P^*)$ and $f_T^{-1}(P^*)$ are both valleys and $f_L^{-1}(\phi^*(P^*))$ and $f_T^{-1}(\phi^*(P^*))$ are both peaks. In either case,

$$\mathrm{gr}(f_L^{-1}(\phi^*(P^*))) = \mathrm{gr}(f_T^{-1}(\phi^*(P^*))). \tag{5.11}$$

Thus,

$$\begin{aligned} \mathrm{gr}(P) + \mathrm{gr}(\phi(P)) - k &= \mathrm{gr}(P) - \mathrm{gr}(P_S) + \mathrm{gr}(\phi(P)) - \mathrm{gr}(\phi(P_S)) \\ &= \mathrm{gr}(f_L^{-1}(P^*)) - \mathrm{gr}(f_L^{-1}(P_S^*)) \\ &\quad + \mathrm{gr}(\phi(f_L^{-1}(P^*))) - \mathrm{gr}(\phi(f_L^{-1}(P_S^*))) \\ &= \mathrm{gr}(f_L^{-1}(P^*)) - \mathrm{gr}(f_L^{-1}(P_S^*)) \\ &\quad + \mathrm{gr}(f_T^{-1}(\phi^*(P^*))) - \mathrm{gr}(f_T^{-1}(\phi^*(P_S^*))). \end{aligned}$$

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Summits are of type $(-+)$, so by (5.11),

$$\mathrm{gr}(f_T^{-1}(\phi^*(P^*))) - \mathrm{gr}(f_T^{-1}(\phi^*(P_S^*))) = \mathrm{gr}(f_L^{-1}(\phi^*(P^*))) - \mathrm{gr}(f_L^{-1}(\phi^*(P_S^*))).$$

By Proposition 5.6 and (5.10),

$$\begin{aligned}
\mathrm{gr}(P) + \mathrm{gr}(\phi(P)) - k &= \mathrm{gr}(f_L^{-1}(P^*)) - \mathrm{gr}(f_L^{-1}(P_S^*)) \\
&\quad + \mathrm{gr}(f_L^{-1}(\phi^*(P^*))) - \mathrm{gr}(f_L^{-1}(\phi^*(P_S^*))) \\
&= \mathrm{gr}(P^*) - \mathrm{gr}(P_S^*) + \mathrm{gr}(\phi^*(P^*)) - \mathrm{gr}(\phi^*(P_S^*)) \\
&= \mathrm{gr}(P^*) + \mathrm{gr}(\phi^*(P^*)) - (\mathrm{gr}(P_S^*) + \mathrm{gr}(\phi^*(P_S^*))) \\
&= k^* - k^* = 0.
\end{aligned}$$

Therefore,

$$\mathrm{gr}(P) + \mathrm{gr}(\phi(P)) = k.$$

Case 2. Suppose P is a leading peak and $P^* := f_L(P)$ has type $(++)$. In this case, $f_L^{-1}(P^*)$ and $f_L^{-1}(\phi^*(P^*))$ are both peaks and $f_T^{-1}(P^*)$ and $f_T^{-1}(\phi^*(P^*))$ are both valleys. Thus,

$$\mathrm{gr}(f_L^{-1}(\phi^*(P^*))) = \mathrm{gr}(f_T^{-1}(\phi^*(P^*))) + \kappa,$$

and

$$\begin{aligned}
\mathrm{gr}(P) + \mathrm{gr}(\phi(P)) - k &= \mathrm{gr}(P) - \mathrm{gr}(P_S) + \mathrm{gr}(\phi(P)) - \mathrm{gr}(\phi(P_S)) \\
&= \mathrm{gr}(f_L^{-1}(P^*)) - \mathrm{gr}(f_L^{-1}(P_S^*)) \\
&\quad + \mathrm{gr}(\phi(f_L^{-1}(P^*))) - \mathrm{gr}(\phi(f_L^{-1}(P_S^*))) \\
&= \mathrm{gr}(f_L^{-1}(P^*)) - \mathrm{gr}(f_L^{-1}(P_S^*)) \\
&\quad + \mathrm{gr}(f_T^{-1}(\phi^*(P^*))) - \mathrm{gr}(f_T^{-1}(\phi^*(P_S^*))) \\
&= \mathrm{gr}(f_L^{-1}(P^*)) - \mathrm{gr}(f_L^{-1}(P_S^*)) \\
&\quad + \mathrm{gr}(f_L^{-1}(\phi^*(P^*))) - \mathrm{gr}(f_L^{-1}(\phi^*(P_S^*))) \\
&= \mathrm{gr}(P^*) - \mathrm{gr}(P_S^*) + \mathrm{gr}(\phi^*(P^*)) - \mathrm{gr}(\phi^*(P_S^*)) \\
&= 0.
\end{aligned}$$

Case 3. Suppose P is a leading valley and $P^* := f_L(P)$ has type $(--)$. In this case, $f_T^{-1}(P^*)$ and $f_T^{-1}(\phi^*(P^*))$ are both peaks and $f_L^{-1}(P^*)$ and $f_L^{-1}(\phi^*(P^*))$ are both valleys. Thus,

$$\mathrm{gr}(f_L^{-1}(\phi^*(P^*))) = \mathrm{gr}(f_T^{-1}(\phi^*(P^*))) - \kappa,$$

and

$$\begin{aligned}
\mathrm{gr}(P) + \mathrm{gr}(\phi(P)) - k &= \mathrm{gr}(P) - \mathrm{gr}(P_S) + \mathrm{gr}(\phi(P)) - \mathrm{gr}(\phi(P_S)) \\
&= \mathrm{gr}(f_L^{-1}(P^*)) - \mathrm{gr}(f_L^{-1}(P_S^*)) \\
&\quad + \mathrm{gr}(\phi(f_L^{-1}(P^*))) - \mathrm{gr}(\phi(f_L^{-1}(P_S^*))) \\
&= \mathrm{gr}(f_L^{-1}(P^*)) - \mathrm{gr}(f_L^{-1}(P_S^*)) \\
&\quad + \mathrm{gr}(f_T^{-1}(\phi^*(P^*))) - \mathrm{gr}(f_T^{-1}(\phi^*(P_S^*)))
\end{aligned}$$

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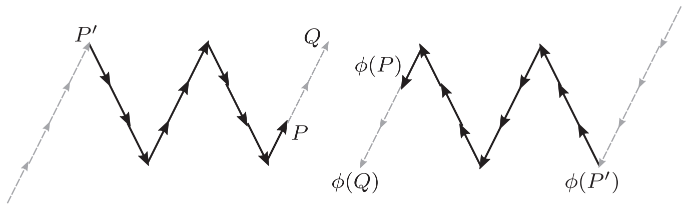

Figure 22: In solid black, the subgraphs $\omega(\mathrm{cl}(\Gamma_i), P', P)$ (left) and $\omega(\mathrm{cl}(\Gamma_i), \phi(P), \phi(P'))$ (right) are shown. The dashed gray arrows are other edges in $\mathrm{cl}(\Gamma_i)$. The case shown is when P' is a peak.

$$\begin{aligned} &= \mathrm{gr}(f_L^{-1}(P^*)) - \mathrm{gr}(f_L^{-1}(P_S^*)) \\ &\quad + \mathrm{gr}(f_L^{-1}(\phi^*(P^*))) - \mathrm{gr}(f_L^{-1}(\phi^*(P_S^*))) \\ &= \mathrm{gr}(P^*) - \mathrm{gr}(P_S^*) + \mathrm{gr}(\phi^*(P^*)) - \mathrm{gr}(\phi^*(P_S^*)) \\ &= 0. \end{aligned}$$

Case 4. Suppose P is not a leading vertex. Let P' be the leading vertex in $\mathrm{cl}(\Gamma_i)$ such that the length of the path $\omega(\mathrm{cl}(\Gamma_i), P', P)$ is minimal. It follows that $\omega(\mathrm{cl}(\Gamma_i), P', P)$ is isomorphic to a subgraph of a κ-block as in Figure 22. In particular, there are no leading vertices between P' and P in $\mathrm{cl}(\Gamma_i)$; therefore, there are no trailing vertices between $\phi(P)$ and $\phi(P')$ in $\mathrm{cl}(\Gamma_i)$, so $\omega(\mathrm{cl}(\Gamma_i), \phi(P), \phi(P'))$ is also isomorphic to a subgraph of a κ-block.

Let Q be the closest vertex (in the forward direction) to P with grading $\mathrm{gr}(Q) = \mathrm{gr}(P')$. When P' is a peak, Q is a peak. Likewise, when P' is a valley, Q is a valley. Define d be the distance (going forward) from P' to Q. Since P is in a κ-block which starts at P', Q and P lie on the same segment, so

$$\mathrm{gr}(Q) - \mathrm{gr}(P) = \begin{cases} d, & \text{when } Q \text{ is a peak,} \\ -d, & \text{when } Q \text{ is a valley;} \end{cases}$$

also, $\phi(Q)$ and $\phi(P)$ lie on the same segment, so

$$\mathrm{gr}(\phi(Q)) - \mathrm{gr}(\phi(P)) = \begin{cases} -d, & \text{when } Q \text{ is a peak,} \\ d, & \text{when } Q \text{ is a valley.} \end{cases}$$

If P' and Q are peaks, then

$$\mathrm{gr}(P) = \mathrm{gr}(Q) - d = \mathrm{gr}(P') - d$$

and

$$\mathrm{gr}(\phi(P)) = \mathrm{gr}(\phi(Q)) + d = \mathrm{gr}(\phi(P')) + d.$$

If P' and Q are valleys, then

$$\mathrm{gr}(P) = \mathrm{gr}(Q) + d = \mathrm{gr}(P') + d$$

and

$$\mathrm{gr}(\phi(P)) = \mathrm{gr}(\phi(Q)) - d = \mathrm{gr}(\phi(P')) - d.$$

In both cases,

$$\mathrm{gr}(P) + \mathrm{gr}(\phi(P)) = \mathrm{gr}(P') + \mathrm{gr}(\phi(P')) = k.$$

Therefore, for every vertex P in $\mathrm{cl}(\Gamma_i)$, $\mathrm{gr}(P) + \mathrm{gr}(\phi(P)) = k$, so Γ_i is symmetric. ∎

Proof of Lemma 3.5 By Lemma 5.4, it is sufficient to show that every relevant co-prime pair is a pre-RTFN pair.

Let (p, q) be a relevant co-prime pair. If $q = 1$ or $(p \bmod q) = 1$ with q positive, then (p, q) is a pre-RTFN pair by Lemma 5.8. If $q = -1$, then (p, q) is a pre-RTFN pair by Lemma 5.3.

Suppose $|q| \neq 1$ and $(p \bmod q) > 1$, and assume every relevant co-prime pair (p', q') with $|q'| < |q|$ is a pre-RTFN pair. When q is positive, define the relevant co-prime pair (p^*, q^*) as in Lemma 4.5. Since $|q^*| < |q|$, (p^*, q^*) is a pre-RTFN pair. By Lemmas 5.10 and 5.11, (p, q) is also pre-RTFN pair. When q is negative, the pair $(p, -q)$ is a pre-RTFN pair by the above argument. Thus, (p, q) is a pre-RTFN pair by Lemma 5.3.

By strong induction, every relevant co-prime pair (p, q) with p positive and q odd is a pre-RTFN pair. ∎

A Background on presentation matrices

Let R be a PID. Suppose X is an R-module with presentation

$$\langle x_1, \dots, x_n | s_1, \dots, s_m \rangle.$$

For each i,

$$s_i = \sum_{j=1}^{n} r_{i,j} x_j,$$

where each $r_{i,j}$ is in R. The matrix of $r_{i,j}$ coefficients

$$\begin{pmatrix} r_{1,1} & \cdots & r_{1,n} \\ \vdots & & \vdots \\ r_{m,1} & \cdots & r_{m,n} \end{pmatrix}$$

is called a *presentation matrix* of X. Suppose A is a presentation matrix of X. Performing row and column operations on A will always produce another presentation matrix of X.

Using row and column operations, any matrix over a PID can be put in the form

$$\left(\begin{array}{ccc|c} d_1 & & & \\ & \ddots & & 0 \\ & & d_k & \\ \hline & 0 & & 0 \end{array}\right),$$

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where each d_i is nonzero and d_i divides d_{i+1} for each $i = 1, \dots, k-1$. This is called the *Smith normal form* of a matrix.

When A is the presentation matrix of X and $d_1, \dots, d_k$ are the diagonal entries of the Smith normal form of A,

$$X \cong R^{n-k} \oplus \frac{R}{d_1 R} \oplus \cdots \oplus \frac{R}{d_k R}. \tag{A.1}$$

The d_i which are not units are the invariant factors of X.

The following lemma plays a key role in showing that elements in a para-free group are homologically primitive.

Lemma A.1 *Suppose X is an R-module with an $m \times n$ presentation matrix A of full rank. If the greatest common divisor of every $m \times m$ minor of A is a unit, then X is a free R-module. Otherwise, the greatest common divisor of every $m \times m$ minor of A is equal to the product of the invariant factors of X up to multiplication by a unit.*

Proof Let B be the Smith normal form of A. Since A has full row rank, B has no extra rows of zeros, so B has the following form:

$$B = \left(\begin{array}{ccc|c} d_1 & & & \\ & \ddots & & 0 \\ & & d_m & \end{array} \right).$$

For any $m \times n$ matrix with entries in R, the greatest common divisor of its $m \times m$ minors is invariant under row and column operations up to multiplication by a unit. Therefore, up to a unit, the greatest common divisor of the $m \times m$ minors of A is $\prod_{i=1}^m d_i$. When $\prod_{i=1}^m d_i$ is a unit, each d_i is a unit, so by (A.1), X is a free R-module. If $\prod_{i=1}^m d_i$ is not a unit, it is the product of the invariant factors of X up to multiplication by a unit. ■

Acknowledgment The author would like to thank Cameron Gordon for his guidance and encouragement throughout this project. The author would also like to thank Ahmad Issa for providing the example of the knot with all real positive roots. The author would also like to thank Hannah Turner for many helpful writing suggestions and support. The author would also like to thank the Canadian Journal of Mathematics referee for noticing a key oversight in the statement of Lemma 3.5 and for the many other excellent editorial suggestions made in their very thorough review. The author would also like to thank Jae Choon Cha, Charles Livingston, and Allison Moore for creating and maintaining *KnotInfo* [14] and *LinkInfo* [15], which were invaluable to this project.

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