BULL. AUSTRAL. MATH. SOC. 55N22
VOL. 48 (1993) [337-345]

ON SPANNING SURFACES OF LINKS

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In his paper on knot cobordism groups in codimension 2, Levine develops conditions for a knotted S^n in S^{n+2} to bound a disc in B^{n+3}. In this paper some of his methods are extended to introduce a necessary condition for a classical link in S^3 to bound a surface of specified genus in B^4. In particular, this answers a question of Zeemann's about some links related to the 'Mazur link'.

The Theorem 1 is a version of a theorem of Tristram ([2]) that is itself a generalisation of the theorem of Murasugi [3]. Tristram's proof is quite different, replacing the spanning surface by a series of spherical modifications and using the concept of ribbon equivalence of links. Theorem 2 is an application of the theorem to the repeated Mazur link shown in Figure 1.

THEOREM 1. (Tristram). *Let L be an oriented link in S^3, V a connected Seifert surface for L and M the Seifert linking matrix of V. Let α be any complex number and let $H = \alpha M + \overline{\alpha} M^T$. If L bounds a connected oriented compact surface U in B^4 then*

$$\beta_1(U) \geqslant |\sigma| - \nu$$

where

$$\begin{aligned} \beta_1 &= \text{ the 1-dimensional Betti number,} \\ \sigma &= \text{ the signature of } H, \\ \nu &= \text{ the nullity of } H. \end{aligned}$$

THEOREM 2. *Let L be formed by linking S^1 in a Mazur link round a stack of k circles (Figure 1). Then L does not bound a surface of genus 0 (a puctured disc) in B^4. [It does bound a surface of genus one in B^4.]*

Received 9 November 1992

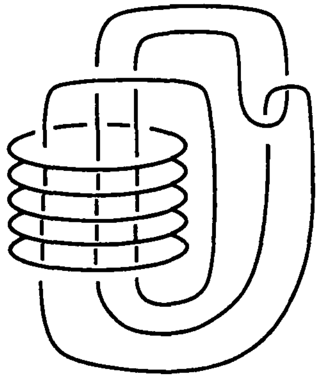

Figure 1

All homology and cohomology groups are to be taken with rational coefficients. β_i will denote the i-dimensional Betti number (that is, the dimension of the i-dimensional homology group). All manifolds will be smooth, possibly with boundary and possibly with corners.

A new proof of Theorem 1 is given, and Theorem 2 will be derived. For the proof of Theorem 1 we need some lemmas. The proofs of the lemmas closely follow Levine's proof [**1**, Lemma 2], but require a more careful look at the obstructions since the relevant cohomology groups need not vanish.

Lemma 1. *Let L be an oriented link in S^3 bounding a compact connected oriented surface U in B^4, and let V be a compact oriented surface spanning L in S^3 (a Seifert surface of L). Then there is a compact oriented 3-manifold W in B^4 with $\partial W = U \cup V$. (Note: W has a corner along L).*

Proof of Lemma 1: Let N be a tubular neighbourhood of U in B^4, chosen such that $N \cap S^3$ and $N \cap V$ are tubular neighbourhoods of L in S^3 and V respectively. Let $M = \operatorname{closure}(B^4 - N)$ and $\dot{N} = N \cap M$ in ∂M. Now $\dot{N}$ is an S^1 bundle over U and $H^2(U) = 0$. So the homotopy classes of sections of $\dot{N}$ correspond bijectively with $H^1(U)$. Each homotopy class of sections corresponds to a homotopy class of fibre maps $\dot{N} \to S^1$. The only obstructions to extending the map $\dot{N} \to S^1$ over M lies in $H^2(M, Z)$. Now the coboundary and excision homomorphisms:

$$H^1(U) \xrightarrow{\delta} H^2(B^4, U) \xrightarrow{j^*} H^2(M, N)$$

are isomorphisms, and so we can choose the fibre mapping $\dot{N} \to S^1$ so that it extends over M. If the mapping is chosen to be smooth then the inverse image of a regular value gives us a 3-manifold, W_1 say, in M with $W_1 \cap \dot{N}$ being the image of a section of $\dot{N}$. We can extend W_1 to a 3-manifold W_2 with boundary $U \cup V_2$, where V_2 is a Seifert surface for L.

Of course V_2 need not be identical with the original Seifert surface V, but any two Seifert surfaces for an oriented link are cobordant, in the sense that there is a 3-manifold W_4 say, in $S^3 \times I$ with $\partial W_4 = V_2 \times \{0\} \cup V \times \{1\} \cup L \times I$. By attaching such a cobordism as a boundary collar of B^4 we can complete the construction of the manifold W required by the lemma. $\square$

Lemma 2. *Let W be a compact connected oriented 3-manifold with boundary $\partial W = U \cup V$, where U and V are compact connected surfaces with $\partial U = \partial V = U \cap V = L$. If $\beta_1(U) < \beta_1(V)$ then the kernel of the homomorphism $H_1(V, Q) \to H_1(W, Q)$ induced by inclusion, has dimension $\geqslant (\beta_1(V) - \beta_1(U))/2$.*

Proof of Lemma 2: Consider the diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_2(W) & \xrightarrow{j_*} & H_2(W,\partial W) & \xrightarrow{\partial} & H_1(\partial W) \\
 & & \uparrow{\scriptstyle \cap W} & & \uparrow{\scriptstyle \cap W} & & \uparrow{\scriptstyle \cap \partial W} \\
0 & \longrightarrow & H^2(W,\partial W) & \longrightarrow & H^1(W) & \longrightarrow & H^1(\partial W)
\end{array}$$

$$\begin{array}{ccccccc}
\xrightarrow{i_*} & H_1(W) & \xrightarrow{j_*} & H_1(W,\partial W) & \longrightarrow & 0 \\
 & \uparrow{\scriptstyle \cap W} & & \uparrow{\scriptstyle \cap W} & & \\
\longrightarrow & H^2(W,\partial W) & \longrightarrow & H^2(W) & \longrightarrow & 0
\end{array}$$

The horizontal sequences are the exact homology and cohomology sequences, the vertical arrows are the Poincare duality isomorphisms, and the diagram commutes up to sign.

It follows from the diagram that $\beta_1(\partial W) = 2(\beta_1(W) - \beta_2(W))$ and that the kernel of $i_*\colon H_1(\partial W) \to H_1(W)$ has dimension $(\beta_1(\partial W))/2$.

Now consider the inclusions $V \xrightarrow{v} \partial W \xrightarrow{i} W$ and the induced homomorphisms:

$$H_1(V) \xrightarrow{v_*} H_1(\partial W) \xrightarrow{i_*} H_1(W).$$

The Mayer-Vietoris for U and V gives the exact sequence:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_2(\partial W) & \longrightarrow & H_1(L) & \longrightarrow & H_1(U) \oplus H_1(V) \\
 & & \| & & \| & & \\
 & & Q & & kQ & &
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & H_1(\partial W) & \longrightarrow & H_0(L) & \longrightarrow 0 \\
 & & & \| & \\
 & & & (k-1)Q &
\end{array}$$

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where k is the number of components of L. It follows that $\beta_1(\partial W) = \beta_1(U) + \beta_1(V)$. From the exact sequence of the pair $(\partial W, V)$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(\partial W) & \xrightarrow{\cong} & H_2(\partial W, V) & \longrightarrow & H_1(V) & \xrightarrow{v_*} & H_1(\partial W) \\
 & & \| & & \| & & & & \\
 & & Q & & Q & & & &
\end{array}
$$

we see that $v_*\colon H_1(V) \to H_1(\partial W)$ is injective.

Now $i_*\colon H_1(\partial W) \to H_1(W)$ has kernel of dimension

$$(\beta_1(\partial W))/2 = (\beta_1(U) + \beta_1(V))/2.$$

It follows from elementary linear algebra that $(iv)_*\colon H_1(V) \to H_1(W)$ has a kernel of dimension $\geqslant (\beta_1(V) - \beta_1(U))/2$. $\square$

Lemma 3. *Let L be an oriented link bounding compact connected oriented surfaces U in B^4 and V in S^3. Let M be the Seifert linking matrix associated with the Seifert surface V. If $\beta_U < \beta_V$ then M is congruent (over Q) to a matrix of the form:*

$$\begin{bmatrix} O_k & A \\ B & C \end{bmatrix}$$

where O_k is a $k \times k$ matrix of zeros and $k \geqslant (\beta_V - \beta_U)/2$.

Proof of Lemma 3: A tubular neighbourhood of V in S^3 is a product $V \times [-1, +1]$. This enables us to define the 'push' mapping $p\colon V \to V \times \{1\}$ in $S^3 - V$.

The Seifert linking matrix represents the linking form: $\theta\colon H_1(V) \times H_1(V) \to Q$ defined by $\theta(\alpha, \beta) =$ the linking number of α with $p_*(\beta)$ in S^3.

Now the push mapping p can be extended over $W \to B^4 - W$. If $\alpha, \beta \in$ kernel $[(iv)_*\colon H_1(V) \to H_1(W)]$ then α, β bound chains α_1, β_1 say in W and $p_*(\beta)$ bounds $p_*(\beta_1)$. So α and $p_*(\beta)$ bound disjoint chains in B^4. Hence $\theta(\alpha, \beta) = 0$.

If k is the dimension of the kernel of $(iv)_*$, then we can choose a basis $e_1, e_2, \ldots, e_n$ of $H_1(V)$ such that $e_1, e_2, \ldots, e_k$ span the kernel of the homomorphism $(iv)_*\colon H_1(V) \to H_1(\partial W)$. Then $\theta(e_i, e_j) = 0$ for all $i, j \leqslant k$. So M is congruent to a matrix with a $k \times k$ matrix of zeros in the top left corner. $\square$

Lemma 4. *Let M be an $n \times n$ matrix congruent over the rationals to a matrix of the form:*

$$\begin{bmatrix} O_k & A \\ B & C \end{bmatrix}.$$

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Let $H = \alpha M + \overline{\alpha} M^T$ for some complex number α. If H has signature σ and rank r then $k \leqslant n - (r + |\sigma|)/2$.

PROOF OF LEMMA 4: If H is a Hermitian form on an n-dimensional space V, then we can write $V = P \oplus M \oplus Z$ where H is positive definite on P, negative definite on M and zero on Z. If $H(x, x) = 0$ write $x = p + m + z$ where $p \in P$, $m \in M$ and $z \in Z$. Now $H(x, x) = H(p, p) + H(m, m)$. If $p \neq 0$ then $m \neq 0$ and conversely. If U is a subspace on which H vanishes, then the projections $U \to P \oplus Z$ and $U \to M \oplus Z$ must both be injective. So $\dim U \leqslant n + \min(p, m)$. But $p = (r + \sigma)/2$ and $m = (r - \sigma)/2$, with $r =$ rank(H). So $\dim U \leqslant v + (r - |\sigma|)/2 = n - (r + |\sigma|)/2$. ∎

PROOF OF THEOREM 1: By Lemmas 1, 2, 3 and 4, M is congruent (over Q) to an $n \times n$ matrix of the form

$$\begin{bmatrix} O_k & A \\ B & C \end{bmatrix}$$

with $(\beta_1(V) - \beta_1(U))/2 \leqslant k \leqslant n - (r + |\sigma|)/2$. Now $\beta_1(V) = n$ and so $\beta_1(U) \geqslant |\sigma| + r - n = |\sigma| - \nu$. ∎

PROOF OF THEOREM 2:

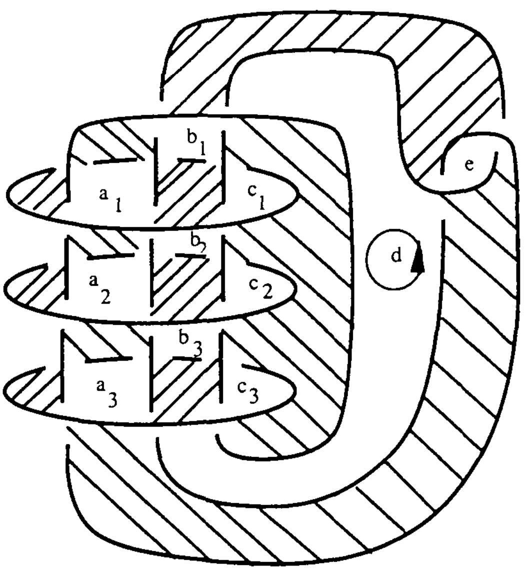

Figure 2. The Seifert surface and its homology generators. Each generator is a cycle in the surface going anti-clockwise around the corresponding letter.

Figure 2 shows a Seifert surface for the repeated Mazur link (with $k = 3$). It shows how we can select a set of homology generators: $\{a_i, b_i, c_i : i = 1, 2, \ldots, k\} \cup \{d, e\}$.

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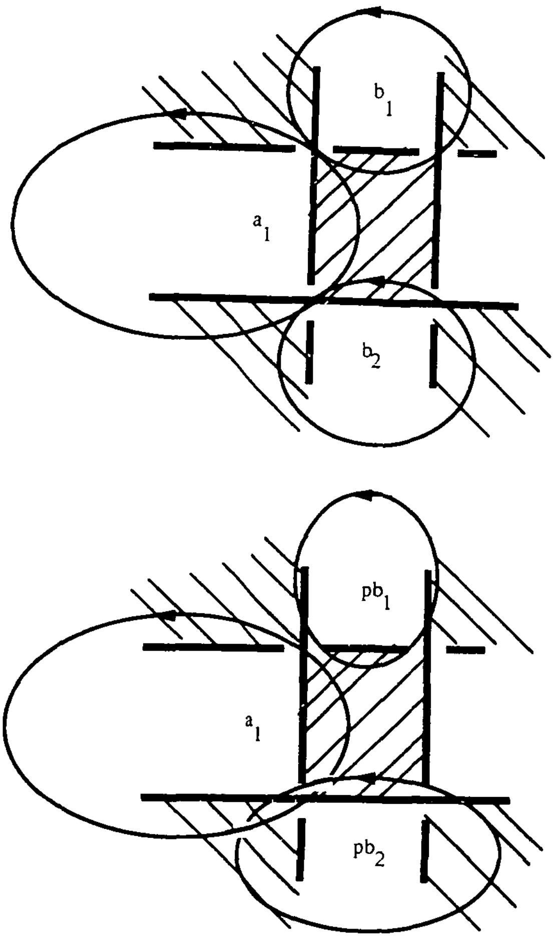

Figure 3 - a detail from Figure 2 showing the homology generators, with some of them pushed off the surface.

From the diagram we can read off the Seifert linking numbers. Figure 3 shows a detail. We may assume that the areas shaded with lines bottom left to top right represent the 'front' of the Seifert surface, and that the push p is towards us in these regions and away from us in the other regions. Examining the cross-overs carefully we see that for instance, a_1 and pb_1 do not link, whereas a_1 and pb_2 link once in a negative (left-handed) sense.

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All the non-zero linking numbers are given in the following table:

$$
\begin{array}{ll}
lk(a_i,\, b_{i+1}) = -1 & \text{for } 1 \leqslant i < k, \\
lk(a_k,\, d) = -1 & \\
lk(b_i,\, a_i) = -1 & \text{for } 1 \leqslant i \leqslant k, \\
lk(b_i,\, c_i) = +1 & \text{for } 1 \leqslant i \leqslant k, \\
lk(c_i,\, b_{i+1}) = +1 & \text{for } 1 \leqslant i < k, \\
lk(c_i,\, c_i) = -1 & \text{for } 1 \leqslant i \leqslant k, \\
lk(c_k,\, d) = -1 & \\
lk(d,\, b_1) = +1 & \\
lk(d,\, d) = -1 & \\
lk(e,\, d) = +1 & \\
lk(e,\, e) = -1. &
\end{array}
$$

We can write this in the block matrix form:

	$\{a\}$	$\{b\}$	$\{c\}$	d	e
$\{a\}$	0	$-E$	0	$-B$	0
$\{b\}$	$-I$	0	I	0	0
$\{c\}$	0	E	$-I$	B	0
d	0	C	0	-1	0
e	0	0	0	1	-1

where E has 1's immediately above the diagonal and 0's elsewhere, $B = (0,\, 0,\, \ldots,\, 0,\, 1)^T$ and $C = (1,\, 0,\, 0,\, \ldots,\, 0)$.

Now choose a complex number α with $|\alpha| = 1$, and let $H = \alpha M + \overline{\alpha} M^T$. Then H is:

	$\{a\}$	$\{b\}$	$\{c\}$	d	e
$\{a\}$	0	$-K^*$	0	$-\beta$	0
$\{b\}$	$-K$	0	K	γ^*	0
$\{c\}$	0	K^*	$-xI$	β	0
d	$-\beta^*$	γ	β^*	$-x$	$\bar{\alpha}$
e	0	0	0	α	$-x$

where $K = \alpha I + \overline{\alpha} E^T$, $\beta = \alpha B$, $\gamma = \alpha C$ and $x = \alpha + \overline{\alpha}$.

We shall now reduce H to standard form by a series of matching row and column operations.

STEP 1. Add rows in block 1 to the corresponding rows in block 3, and similarly for columns.

STEP 2. Pre-multiply the rows in block 1 by $-(K^*)^{-1}$ and post-multiply the corresponding columns by $-K^{-1}$.

These steps yield the matrix:

$$\begin{array}{c|c|c|c|c|c|}
 & \{a\} & \{b\} & \{c\} & d & e \\ \hline
\{a\} & 0 & I & 0 & K^{*-1}\beta & 0 \\ \hline
\{b\} & I & 0 & 0 & \gamma^* & 0 \\ \hline
\{c\} & 0 & 0 & -xI & 0 & 0 \\ \hline
d & \beta^* K^{-1} & \gamma & 0 & -x & \bar{\alpha} \\ \hline
e & 0 & 0 & 0 & \alpha & -x \\ \hline
\end{array}$$

STEP 4. Use the second block of rows to eliminate $\beta^* K^{-1}$ and the second block of columns to eliminate $(K^*)^{-1}\beta$.

STEP 5. Use the bottom right-hand element to eliminate α and $\overline{\alpha}$.

These steps yield the matrix:

$$\begin{array}{c|c|c|c|c|c|}
 & \{a\} & \{b\} & \{c\} & d & e \\ \hline
\{a\} & 0 & I & 0 & 0 & 0 \\ \hline
\{b\} & I & 0 & 0 & \gamma^* & 0 \\ \hline
\{c\} & 0 & 0 & -xI & 0 & 0 \\ \hline
d & 0 & \gamma & 0 & \delta & 0 \\ \hline
e & 0 & 0 & 0 & 0 & -x \\ \hline
\end{array}$$

where $\delta = -x + (1/x) - \beta^* K^{-1}\gamma^* - \gamma K^{*-1}\beta$.

Now $K = \alpha I + \overline{\alpha} E^T$, $|\alpha| = 1$ and $E^k = 0$. So $K^{-1} = \overline{\alpha} I - \overline{\alpha}^3 F + \overline{\alpha}^5 F^2 - (-1)^{k-1}\overline{\alpha}^{(2k-1)} F^{k-1}$ where $F = E^T$. But $\beta = (0, 0, \ldots, 0, \alpha)^T$ and $\gamma = (\alpha, 0, 0, \ldots, 0)$. So $\beta^* K^{-1}\gamma^* = (-1)^{k-1}\overline{\alpha}^{(2k+1)}$.

The complex conjugate of the transpose gives: $\gamma K^{*-1}\beta = (-1)^{k-1}\alpha^{(2k+1)}$. So $\delta = -x + (1/x) + (-1)^k\left(\alpha^{(2k+1)} + \overline{\alpha}^{(2k+1)}\right)$.

CASE 1. k odd. Put $\alpha = 1$. Then $\delta = -2 + 1/2 - 2 < 0$.

CASE 2. k even. Choose $\theta = \pi/(2k+1)$ and $\alpha = e^{i\theta}$. Then $\delta = -2\cos\theta + 1/(2\cos\theta) - 2$. Now $\theta \leqslant \pi/5$, so $\cos\theta > 1/2$ and thus $\delta < 0$.

In both cases H is non-singular with signature $-(k+2)$ and nullity $2k$.

Now the link L has $(k+1)$ components, and so a punctured disk spanning L must have $\beta_1 = k$. This would contradict the Theorem 1. $\square$

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