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Relatively flat modules

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If (A', B'), (B', C') and (A, B), (B, C) are torsion-torsion free theories on ${}_R\mathrm{M}$ and M_R respectively which are generated by an idempotent ideal I of R, then $M \in {}_R\mathrm{M}$ is said to be relatively flat if $(\cdot) \otimes_R M$ preserves short exact sequences $0 \to L \to X \to N \to 0$ in M_R with $N \in B$. Several characterizations of relatively flat modules are given and it is shown that any module $M \in {}_R\mathrm{M}$ which is codivisible with respect to (A', B') is relatively flat. In addition, when (A', B') is hereditary, it is proven that $M \in {}_R\mathrm{M}$ is relatively flat if and only if M/IM is a flat R/I-module. Finally, a relatively flat dimension for $M \in {}_R\mathrm{M}$ and a left global relatively flat dimension for R are defined and it is shown, again when (A', B') is hereditary, that the left global relatively flat dimension of R coincides with the left global flat dimension of R/I.

1. Introduction

Throughout this paper R will denote an associative ring with identity and our attention will be confined to the categories ${}_R\mathrm{M}$ and M_R of unital left and right R-modules respectively. Our purpose is to study relatively flat modules in the setting of torsion-torsion free theories which are generated by an idempotent ideal of R. The reader is referred to [2], [5], and [7] for the general results and terminology on torsion theories.

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If (A, B) is a torsion theory on M_R, then a right R-module M is said to be divisible (codivisible) if given an exact sequence $0 \to L \to X \to N \to 0$ in M_R, where N is torsion (L is torsion free), the induced map $\hom_R(X, M) \to \hom_R(L, M)$ $\left(\hom_R(M, X) \to \hom_R(M, N)\right)$ is an epimorphism. By taking X to be projective (injective), we see that M is divisible (codivisible) if and only if $\operatorname{ext}_R^1(N, M) = 0$ for every torsion module N $\left\{\operatorname{ext}_R^1(M, L) = 0\right.$ for every torsion free module $\left.L\right\}$. Divisible modules are due to Lambek [5] and codivisible modules were introduced in [1].

In [4], Jans calls a class B of modules in M_R a torsion-torsion free class if B is closed under taking submodules, factors, extensions, direct products and direct sums. By saying that B is closed under extensions we mean that $M \in B$ whenever there is a submodule N of M such that N and M/N are in B. For such a torsion-torsion free class B there exist classes A and C of modules such that (A, B) and (B, C) are torsion theories on M_R. We shall refer to such a pair (A, B), (B, C) as a torsion-torsion free theory on M_R. Jans has also shown [4, Corollary 2.2] that there is a one-to-one correspondence $I \leftrightarrow B = \{M \mid MI = 0\}$ between idempotent ideals I of R and torsion-torsion free classes B in M_R. Thus it follows that:

(1) $A = \{M \mid MI = M\} = \{M \mid M \otimes_R I \cong M\}$;

(2) $B = \{M \mid MI = 0\} = \{M \mid M \otimes_R I = 0\}$;

(3) $C = \{M \mid \hom_R(B, M) = 0$ for all $B \in B\}$;

(4) $A(M) = MI$ for any $M \in M_R$ where $A(M)$ denotes the torsion submodule of M with respect to (A, B) ;

(5) the idempotent filter of right ideals of R associated with (B, C) is given by

$$F(R) = \{K \mid K \supseteq I,\ K \text{ a right ideal of } R\} .$$

Obviously, a given idempotent ideal I of R generates a torsion-torsion

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free theory (A', B'), (B', C') on ${}_R M$ and a torsion-torsion free theory (A, B), (B, C) on M_R. Notice that $A'(R) = A(R) = I$. Throughout the remainder of this paper we will suppose, unless stated otherwise, that I is an idempotent ideal of R and that (A', B'), (B', C') and (A, B), (B, C) are as above. For $M \in {}_R M$ or M_R, $M^* = \hom_Z(M, Q/Z)$ will denote the character module of M. Before beginning we record the fact that many of the following definitions and theorems reduce to definitions and theorems of "classical" homological algebra when $I = 0$.

2. E-flat modules

If E is the class of all short exact sequences $0 \to L \to X \to N \to 0$ in M_R with $N \in B$, then $M \in {}_R M$ is said to be flat relative to E or simply E-flat if $(\cdot) \otimes_R M$ preserves short exact sequences in E. It is easy to see that M is E-flat if and only if $\mathrm{tor}_1^R(N, M) = 0$ for all $N \in B$.

THEOREM 2.1. *The following are equivalent for any* $M \in {}_R M$:

(1) M *is* E*-flat;*

(2) M^* *is divisible with respect to* (B, C) ;

(3) $K \otimes_R M \cong KM$ *canonically for each* $K \in F(R)$;

(4) *if* $M \cong E/N$ *where* E *is* E*-flat, then* $KE \cap N = KN$ *for each* $K \in F(R)$;

(5) $\mathrm{tor}_1^R(R/K, M) = 0$ *for all* $K \in F(R)$.

Proof. We will show (1) ⇒ (5) ⇒ (3) ⇒ (2) ⇒ (1) and then (1) ⇔ (4).

(1) ⇒ (5) is obvious since if $K \in F(R)$, then $R/K \in B$.

(5) ⇒ (3). If $K \in F(R)$, then $0 \to K \xrightarrow{j} R \to R/K \to 0$ is in E and so $0 = \mathrm{tor}_1^R(R/K, M) \to K \otimes_R M \xrightarrow{1 \otimes j} R \otimes_R M$ is exact. Thus if $\varphi : R \otimes_R M \to M$ is the canonical isomorphism, then $K \otimes_R M \cong \varphi \circ (1 \otimes j)(K \otimes_R M) = KM$.

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(3) $\Rightarrow$ (2). If $K \otimes_R M \cong KM$ canonically for each $K \in F(R)$, then

$K \otimes_R M \to M : \sum_{i=1}^{n} k_i \otimes m_i \to \sum_{i=1}^{n} k_i m_i$ is a monomorphism. Hence we have an epimorphism $\hom_R(R, M^*) \to \hom_R(K, M^*)$ since $(K \otimes_R M)^* \cong \hom_R(K, M^*)$ and $M^* \cong \hom_R(R, M^*)$. Thus, it follows from the Generalized Baer's Criterion [3, Proposition 3.2] that M^* is divisible with respect to $(\mathcal{B}, \mathcal{C})$.

(2) $\Rightarrow$ (1). If $0 \to L \to X \to N \to 0$ is in $\mathcal{E}$, then

$$0 \to \hom_R(N, M^*) \to \hom_R(X, M^*) \to \hom_R(L, M^*) \to 0$$

is exact. Thus $0 \to (N \otimes_R M)^* \to (X \otimes_R M)^* \to (L \otimes_R M)^* \to 0$ is exact and so $0 \to L \otimes_R M \to X \otimes_R M \to N \otimes_R M \to 0$ is exact.

Finally, let us show that (1) $\Leftrightarrow$ (4) .

(1) $\Rightarrow$ (4). For each $K \in (R)$, the exact sequence $N \xrightarrow{j} E \xrightarrow{\varphi} M \to 0$ where $N = \ker \varphi$ and j is the canonical injection yields a commutative diagram

$$\begin{array}{ccccc} K \otimes_R N & \xrightarrow{1 \otimes j} & K \otimes_R E & \xrightarrow{1 \otimes \varphi} & K \otimes_R M \to 0 \\ & & \downarrow \psi_1 & & \downarrow \psi_2 \\ & & KE & \xrightarrow{\theta} & KM \quad \to 0 , \end{array}$$

where ψ_1 and ψ_2 are the canonical isomorphisms given by (3) and

$\theta : KE \to KM : \sum_{i=1}^{n} k_i x_i \to \sum_{i=1}^{n} k_i \varphi(x_i)$. Hence

$$KN = \psi_1 \circ (1 \otimes j)(K \otimes_R N) = \psi_1(\ker(1 \otimes \varphi)) = \ker \theta = KE \cap N .$$

(4) $\Rightarrow$ (1). The exact sequence $N \xrightarrow{j} E \xrightarrow{\varphi} M \to 0$ yields, for each $K \in F(R)$, a diagram

$$\begin{array}{ccccc} K \otimes_R N & \xrightarrow{1\otimes j} & K \otimes_R E & \xrightarrow{1\otimes\varphi} & K \otimes_R M \to 0 \\ & & \Big\downarrow \psi_1 & & \\ & & KE & \xrightarrow{\theta} & KM \to 0 \;, \end{array}$$

where ψ_1 and θ are as above. Since $KN = KE \cap N$, $KE/KN \cong KM$. Notice also that since $\psi_1 \circ (1 \otimes j)\left(K \otimes_R N\right) = KN$, ψ_1 induces an isomorphism $\frac{K\otimes_R E}{(1\otimes j)\left(K\otimes_R N\right)} \cong \frac{KE}{KN}$. But $\frac{K\otimes_R E}{(1\otimes j)\left(K\otimes_R N\right)} = \frac{K\otimes_R E}{\ker(1\otimes\varphi)} \cong K \otimes_R M$. Hence we have an isomorphism $\psi_2 : K \otimes_R M \to KM$ which can easily be shown to be the canonical isomorphism. Thus (3) holds and so M is E-flat.

In passing we note that if $\{M_a\}$ $(a \in I)$ is a family of modules in ${}_R M$, then $\oplus M_a$ $(a \in I)$ is E-flat if and only each M_a is E-flat. We now need the following

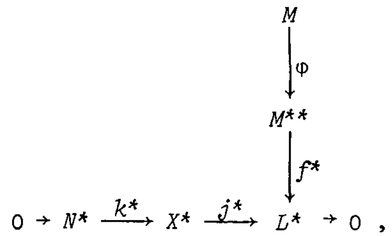
LEMMA 2.2. *If* $M \in M_R$, *then* $M \in B$ *if and only if* $M^* \in B'$.

Proof. If $M \in B$, then $MI = 0$ and so if $xf \in IM^*$, then $(xf)(m) = f(mx) = f(0) = 0$ for each $m \in M$. Thus $xf = 0$ and consequently, $IM^* = 0$. Therefore $M^* \in B'$. Conversely, suppose that $M^* \in B'$; then $(M^*)^* = M^{**} \in B$. But M embeds in M^{**} and so $M \in B$ since B is closed under submodules and isomorphic images.

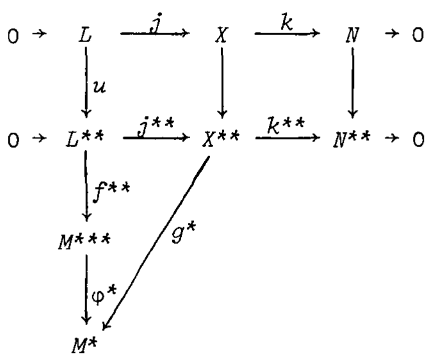
In the proof of the following theorem if $f : M \to N$ is an R-homomorphism, then f^* will denote the R-homomorphism $N^* \to M^* : g \to g \circ f$.

THEOREM 2.3. *If* $M \in {}_R M$ *is codivisible with respect to* (A', B'), *then* M *is* E*-flat.*

Proof. Let $0 \to L \xrightarrow{j} X \xrightarrow{k} N \to 0$ be in E and suppose that $f : L \to M^*$ is R-linear. By taking character modules we obtain a diagram



where φ is the canonical embedding $M \to M^{**} : m \to \varphi_m$ and $\varphi_m : M^* \to Q/Z : h \to h(m)$. Since $N^* \in \mathcal{B}'$ (Lemma 2.2) and M is codivisible with respect to $(\mathcal{A}', \mathcal{B}')$ this diagram can be completed commutatively by an R-homomorphism $g : M \to X^*$. Thus if u, v , and w are the canonical embeddings shown below, then we have a commutative diagram



But if $x \in L$, then

$$(\varphi^* \circ f^{**} \circ u)(x) = \varphi^*\left(f^{**}(u_x)\right) = \varphi^*\left(u_x \circ f^*\right) = u_x \circ f^* \circ \varphi ,$$

and so for $m \in M$,

$$(u_x \circ f^* \circ \varphi)(m) = u_x(\varphi_m \circ f) = (\varphi_m \circ f)(x) = f(x)(m) .$$

Thus $\varphi^* \circ f^{**} \circ u = f$ and therefore we have shown that M^* is divisible with respect to $(\mathcal{B}, \mathcal{C})$. Hence, by Theorem 2.1, M is $\mathcal{E}$-flat.

The following theorem relates $\mathcal{E}$-flat modules to flat R/I-modules.

THEOREM 2.4. *If* $(\mathcal{A}', \mathcal{B}')$ *is hereditary, then* $M \in {}_R\mathrm{M}$ *is* $\mathcal{E}$*-flat if and only if* M/IM *is a flat* R/I*-module.*

Proof. Suppose that M is $\mathcal{E}$-flat and let $\varphi : C \to M$ be an

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R-epimorphism where C is codivisible with respect to (A', B'). If $\bar{\varphi} : C/IC \to M/IM : x + IC \to \varphi(x) + IM$ is the induced R/I-epimorphism and $N = \ker \varphi$, then $\ker \bar{\varphi} = (N+IC)/IC$. Hence, since C/IC is a projective R/I-module [6, Theorem 8], to show that M/IM is a flat R/I-module it suffices to show that $(N+IC)/IC \cap KC/IC \subseteq (KN+IC)/IC$ for any $K \in F(R)$. Let $x = n + IC \in (N+IC)/IC$, $n \in N$, and $x = m + IC \in KC/IC$, $m \in KC$; then $n - m \in IC \subseteq KC$ and so $n \in KC$. But, by Theorem 2.3, C is E-flat and therefore, by Theorem 2.1, $KC \cap N = KN$. Hence $n \in KN$ and therefore $x \in (KN+IC)/IC$.

Conversely, suppose that M/IM is a flat R/I-module and let $\varphi : C \to M$ and N be as above. Let $K \in F(R)$ and suppose that $x \in KC \cap N$. Since $x \in KC$ and $x \in N$,

$$x + IC \in (N+IC)/IC \cap KC/IC = (KN+IC)/IC .$$

If $x + IC = y + IC$, $y \in KN \subseteq N$, then $x - y \in IC \cap N$. But (A', B') is hereditary and so $IC \cap N = IN \subseteq KN$. Thus it follows that $x \in KN$ and, by Theorem 2.1, that M is E-flat.

COROLLARY 2.5. *If (A', B') is hereditary, then:*

(1) *every module in A' is E-flat;*

(2) *if $M \in B'$, then M is flat if and only if M is E-flat;*

(3) *every $M \in {}_R M$ is E-flat if and only if R/I is a (von Neumann) regular ring.*

3. E-flat dimension

If $M \in {}_R M$, then we can build a codivisible resolution

$$(*) \qquad \dots \to C_n \xrightarrow{\alpha_n} \dots \to C_0 \xrightarrow{\alpha_0} M \to 0$$

of M where each C_i is codivisible with respect to (A', B') and $\ker \alpha_i \in B'$ for $i \geq 0$. (Notice that $C_i \in B'$ for $i \geq 1$ since B' is closed under extensions.) Indeed, if $\varphi : F \to M$ is the free module on M and $K = \ker \varphi$, then F/IK is codivisible with respect to (A', B') and $K/IK \in B'$. Thus we need only set $C_0 = F/IK$ and let α_0 be the induced mapping. Hence the result follows by induction. Consequently, for any

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$X \in M_R$, we have a complex

$$\cdots \to X \otimes {}_R C_n \to \cdots \to X \otimes {}_R C_0 \to 0 .$$

If $\operatorname{tor}_n^E(X, M)$ denotes the n-th homology group of this sequence, then it is easy to show that $\operatorname{tor}_n^E(X, M)$ is independent of the particular codivisible resolution selected for M and that $\operatorname{tor}_0^E(X, M) \cong X \otimes {}_R M$. Therefore if $0 \to L \to X \to N \to 0$ is in E , then by tensoring this into (*) and taking homology we obtain an exact sequence

$$\cdots \to \operatorname{tor}_n^E(L, M) \to \operatorname{tor}_n^E(X, M) \to \operatorname{tor}_n^E(N, M) \to \cdots \to \operatorname{tor}_1^E(L, M)$$
$$\to \operatorname{tor}_1^E(X, M) \to \operatorname{tor}_1^E(N, M) \to L \otimes {}_R M \to X \otimes {}_R M \to N \otimes {}_R M \to 0 .$$

Thus if $\operatorname{tor}_n^E(\cdot, M) = 0$ for all $n \geq 1$, then M is E-flat. To show the converse we need the following lemmas.

LEMMA 3.1. *If* (A', B') *is hereditary and* $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *is exact in* ${}_R M$ *, then the induced sequence* $0 \to L/IL \xrightarrow{\bar{f}} M/IM \xrightarrow{\bar{g}} N/IN \to 0$ *is exact in* ${}_{R/I}M$.

Proof. If $\bar{f}(x+IL) = f(x) + IM = 0$, then $f(x) \in IM$. Hence $f(x) \in IM \cap f(L)$. But (A', B') is hereditary and so $IM \cap f(L) = If(L)$. If $f(x) = \sum_{i=1}^{n} k_i f(y_i) \in If(L)$, then $x - \sum_{i=1}^{n} k_i y_i \in \ker f = 0$. Thus $x \in IL$ and so $\bar{f}$ is a monomorphism. The proofs when $\operatorname{im} \bar{f} = \ker \bar{g}$ and $\bar{g}$ is an epimorphism are similar and will therefore be omitted.

LEMMA 3.2. *If* (A', B') *is hereditary and* $0 \to L \to M \to N \to 0$ *is exact in* ${}_R M$ *with* N *E-flat, then* M *is E-flat if and only if* L *is E-flat.*

Proof. By the lemma above the induced sequence $0 \to L/IL \to M/IM \to N/IN \to 0$ is exact. Now, by Theorem 2.4, N/IN is a flat R/I-module and so, as is well known, M/IM is a flat R/I-module if and only if L/IL is a flat R/I-module. Hence the result, again by

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Theorem 2.4.

LEMMA 3.3. *If* (A', B') *is hereditary and* $0 \to L \to M \to N \to 0$ *is exact in* ${}_R\mathrm{M}$ *with* N *E-flat and* $L \in B'$ *, then* $0 \to X \otimes_R L \to X \otimes_R M \to X \otimes_R N \to 0$ *is exact for any* $X \in \mathrm{M}_R$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & XI \otimes_R L & & & & & & \\
 & & \downarrow & & & & & & \\
 & & X \otimes_R L & \to & X \otimes_R M & \to & X \otimes_R N & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X/XI \otimes_{R/I} L & \to & X/XI \otimes_{R/I} M/IM & \to & X/XI \otimes_{R/I} N/IN & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array},$$

where the columns and rows are exact, the maps being the obvious ones. Notice that the bottom row is exact since N/IN is a flat R/I-module. Now $L \in B'$ and so $XI \otimes_R L = 0$. Hence $X \otimes_R L \to X/XI \otimes_{R/I} L$ is an isomorphism and from this one can see, by chasing around the diagram, that $X \otimes_R L \to X \otimes_R M$ is an injection.

Now for the theorem.

THEOREM 3.4. *If* (A', B') *is hereditary, then* $\mathrm{tor}_n^E(\cdot, M) = 0$ *for all* $n \geq 1$ *if and only if* M *is E-flat.*

Proof. We have already seen that if $\mathrm{tor}_n^E(\cdot, M) = 0$ for all $n \geq 1$, then M is E-flat. Conversely, suppose that M is E-flat and let $\ldots \to C_n \to \ldots \to C_0 \to M \to 0$ be a codivisible resolution of M with respect to (A', B') . If $X \in \mathrm{M}_R$, then by applying Lemma 3.2, Lemma 3.3, and Theorem 2.3, it follows that $\ldots \to X \otimes_R C_n \to \ldots \to X \otimes_R C_0 \to X \otimes_R M \to 0$ is exact. Consequently, the result.

Suppose next that $0 \to L \to X \to N \to 0$ is exact in ${}_R\mathrm{M}$ and let $L \in B'$; then, by using standard arguments, we can find a short exact

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sequence $0 \to \{C'_n\} \to \{C_n\} \to \{C''_n\} \to 0$ of codivisible resolutions of L, M, and N respectively. Thus for any $M \in \mathsf{M}_R$ we have a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & & 0 & \\
 & & \downarrow & & & \downarrow & \\
\cdots \to & & M \otimes_R C'_n & \to \cdots \to & & M \otimes_R C'_0 & \to 0 \\
 & & \downarrow & & & \downarrow & \\
\cdots \to & & M \otimes_R C_n & \to \cdots \to & & M \otimes_R C_0 & \to 0 \\
 & & \downarrow & & & \downarrow & \\
\cdots \to & & M \otimes_R C''_n & \to \cdots \to & & M \otimes_R C''_0 & \to 0 \\
 & & \downarrow & & & \downarrow & \\
 & & 0 & & & 0 & ,
\end{array}$$

Notice that the columns are exact since each C''_i is E-flat and $C'_i \in B'$ for $i \geq 0$. Taking homology we obtain an exact sequence

$$\cdots \to \mathrm{tor}_n^E(M, L) \to \mathrm{tor}_n^E(M, X) \to \mathrm{tor}_n^E(M, N) \to \cdots \to \mathrm{tor}_1^E(M, L)$$
$$\to \mathrm{tor}_1^E(M, X) \to \mathrm{tor}_1^E(M, N) \to M \otimes_R L \to M \otimes_R X \to M \otimes_R N \to 0 .$$

From this it follows that $M \otimes_R (\cdot)$ preserves short exact sequences $0 \to L \to X \to N \to 0$ in ${}_R\mathsf{M}$ with $L \in B'$ if and only if $\mathrm{tor}_n^E(M, \cdot) = 0$ for all $n \geq 1$. Hence we have

THEOREM 3.5. *If* (A', B') *is hereditary, then the following are equivalent:*

(1) $\mathrm{tor}_n^E = 0$ *for all* $n \geq 1$ *;*

(2) *every* $M \in {}_R\mathsf{M}$ *is* E*-flat;*

(3) *if* $M \in \mathsf{M}_R$ *, then* $M \otimes_R (\cdot)$ *preserves short exact sequences* $0 \to L \to X \to N \to 0$ *in* ${}_R\mathsf{M}$ *with* $L \in B'$.

If $M \in {}_R\mathsf{M}$, then we define the E-flat dimension of M to be the smallest integer such that $\mathrm{tor}_{n+1}^E(\cdot, M) = 0$. If $E\text{-fd}(M)$ denotes the

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E-flat dimension of M, the left global E-flat dimension of R is $\text{l.gl.}E\text{-fd}(R) = \sup\{E\text{-fd}(M) \mid M \in {}_R\text{M}\}$. The following theorem relates the E-flat dimension of $M \in {}_R\text{M}$ to the flat dimension of the R/I-module M/IM. Let $\text{fd}(M/IM)$ denote the flat dimension of M/IM as an R/I-module.

THEOREM 3.6. *If* (A', B') *is hereditary, then for any* $M \in {}_R\text{M}$, $E\text{-fd}(M) = \text{fd}(M/IM)$.

Proof. If $\ldots \to C_n \xrightarrow{\alpha_n} \ldots \to C_1 \xrightarrow{\alpha_1} C_0 \xrightarrow{\alpha_0} M \to 0$ is a codivisible resolution of M with respect to (A', B'), then the induced sequence

$$\ldots \to C_n \xrightarrow{\bar{\alpha}_n} \ldots C_1 \xrightarrow{\bar{\alpha}_1} C_0/IC_0 \xrightarrow{\bar{\alpha}_0} M/IM \to 0 ,$$

where $\bar{\alpha}_i = \alpha_i$ for $i \geq 2$ is an R/I-projective resolution of M/IM [6, Theorem 8]. Hence if $E\text{-fd}(M) = k$, then k is the smallest integer such that $\text{tor}^E_{k+1}(\cdot, M) = 0$. Now it is easy to show that $\text{tor}^E_{k+1}(X, M) \cong \text{tor}^E_1(X, \text{im}\,\alpha_k)$ for any $X \in \text{M}_R$ and so k is the smallest integer such that $\text{im}\,\alpha_k$ is E-flat. But (A', B') is hereditary and therefore $\text{im}\,\alpha_k \cap IC_{k-1} = I\,\text{im}\,\alpha_k$. Hence

$$\text{im}\,\bar{\alpha}_k = (\text{im}\,\alpha_k + IC_{k-1})/IC_{k-1} \cong \text{im}\,\alpha_k/(\text{im}\,\alpha_k \cap IC_{k-1}) = \text{im}\,\alpha_k/I\,\text{im}\,\alpha_k$$

and so, by Theorem 2.4, $\text{im}\,\bar{\alpha}_k$ is a flat R/I-module. In fact, one can show that k is the smallest integer such that $\text{im}\,\bar{\alpha}_k$ is a flat R/I-module. Therefore $E\text{-fd}(M) = \text{fd}(M/IM)$.

COROLLARY 3.7. *If* (A', B') *is hereditary, then* $\text{l.gl.}E\text{-fd}(R) = \text{l.gl.fd}(R/I)$.

In [6], Rangaswamy has defined a codivisible dimension for modules in ${}_R\text{M}$ and a left global codivisible dimension for R with respect to *any* hereditary torsion (A', B') on ${}_R\text{M}$. Briefly, if

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$\cdots \to C_n \xrightarrow{\alpha_n} \cdots \to C_0 \xrightarrow{\alpha_0} M \to 0$ is a codivisible resolution of M with respect to (A', B'), then the codivisible dimension of M is the smallest integer n such that im α_n is codivisible. The l.gl.cod(R) (the left global codivisible dimension of R) is then defined in the obvious way. Rangaswamy has shown [6, Theorem 14] that the left global codivisible dimension of R equals the left global homological dimension of $R/A'(R)$ where $A'(R)$ is the torsion ideal of R with respect to (A', B'). If E' is the class of all short exact sequences $0 \to L \to X \to N \to 0$ in ${}_R M$ with $N \in B'$ and (A', B') and (A, B) are both hereditary, when viewing the obvious symmetry of our work we see that l.gl.E-fd(R) = r.gl.E-fd(R). Since the left global flat dimension of a left noetherian ring coincides with its left global homological dimension, we conclude with the following observation.

THEOREM 3.8. *If* (A', B') *is hereditary and the left ideals in* $F'(R) = \{K \mid K \supseteq I,\ K$ *a left ideal of* $R\}$ *satisfy the ascending chain condition, then* l.gl.E-fd(R) = l.gl.cod(R).

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