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THE ADDITIVE GROUP OF AN f-RING

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The intent of this paper is to show that the additive l-group of an f-ring S determines the ring structure. This is why there are so many papers that simply extend known results for abelian l-groups to f-rings. Theorem 3.1 asserts that there is a one-to-one correspondence between the f-multiplications on S and a set of homomorphisms from the positive cone of the l-group S into the positive cone of the ring $\mathscr{P}(S)$ of polar preserving endomorphisms of the l-group S. In fact, each f-multiplication of S is determined by a homomorphism of S^+ into $\mathscr{P}(S)^+$. If S is archimedean then the ring has an identity if and only if the corresponding homomorphism is a bijection and in this case $S \cong \mathscr{P}(S)$ as an f-ring.

If S is an archimedean f-ring with identity and $\cdot$ is another f-multiplication of S then $a \cdot b = abp$ for all $a, b \in S$ and some fixed $0 \leqq p \in S$ and conversely (Theorem 2.2). For $0 \leqq p, q \in S$ define the ring multiplications

$$a \cdot b = abp \quad \text{and} \quad a \# b = abq.$$

Then the resulting f-rings are l-isomorphic if and only if there exists a group l-automorphism τ of S such that $p\tau = q$ (Theorem 2.3). The proof of the last result depends upon the fact that the group of all l-automorphisms of the additive group $(S, +)$ is a splitting extension of the polar preserving automorphisms of $(S, +)$ by the group of l-automorphisms of the ring S (Theorem 2.1).

In section 1 we show that if G is an archimedean l-group and $\{g_\gamma | \gamma \in \Gamma\}$ is a maximal disjoint subset of G, then there exists a minimal f-ring M containing $(G, +)$ as a large l-subgroup and with identity $\bigvee g_\gamma$. M is necessarily archimedean and if N is another such ring then there exists a unique ring l-isomorphism τ of M onto N such that $g\tau = g$ for all $g \in G$. If G^e is the essential closure of G then G is large in G^e and $u = \bigvee g_\gamma$ exists in G^e. Moreover, there is a unique multiplication on G^e so that it is an f-ring with identity u. Thus M is the l-subring of G^e that is generated by G and u.

By definition G is *large* in M or M is an *essential extension* of G if for each non-zero l-ideal L of M, $L \cap G \neq 0$ or equivalently if $0 < h \in M$ then $nh > g > 0$ for some $g \in G$ and some positive integer n.

We shall make frequent use of the following representation theory of Bernau [**1**]. Let G be an archimedean l-group and let $p(G)$ be the set of polars of G. Then $p(G)$ is a complete Boolean algebra [**10**] and so the associated

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Stone space X is extremely disconnected, Hausdorff and compact. Let $D(X)$ be the collection of almost finite continuous functions from X to $R \cup \{\pm\infty\}$ (i.e., $D(X) = \{f : X \rightarrow R \cup \{\pm\infty\} | f$ is continuous and $\{x \in X | f(x) \in R\}$ is dense$\}$). Then $D(X)$ is a complete vector lattice and an f-ring.

THEOREM [**1**]. *Let G be an archimedean l-group. Then there is an l-isomorphism σ of G into $D(X)$ which preserves all existing infima and suprema. G is large in $D(X)$. If $\{g_\gamma | \gamma \in \Gamma\}$ is a maximal disjoint subset of G then σ can be chosen so that each $g_\gamma \sigma$ is a characteristic function of a subset X_γ of X, where the family $\{X_\gamma | \gamma \in \Gamma\}$ is a collection of compact open subsets of X whose union is dense in X.*

Another way of describing $D(X)$ is that it is *the* essential closure G^e of G. That is, $D(X)$ is an essential extension of G and $D(X)$ admits no proper essential extensions in the category of archimedean l-groups (see [**6**]).

THEOREM [**1**]. *If α and β are l-isomorphisms of the l-group G onto large subgroups of $D(X)$ then there exists a homeomorphism τ of X and an element $0 < d \in D(X)$ with support all of X such that for all $g \in G$*

$$(x)g\alpha = (x\tau)g\beta \cdot (x)d$$

for all $x \in X$ for which the multiplication on the right is defined.

Thus $\alpha = \beta\bar{\tau}\bar{d}$, where $\bar{\tau}$ and $\bar{d}$ are the corresponding automorphisms of $D(X)$. That is, $(x)g\bar{\tau} = (x\tau)g$ and $(x)g\bar{d} = (x)g \cdot (x)d$ for all $g \in G$ and $x \in X$. In particular, $\bar{\tau}$ is a ring automorphism of $D(X)$.

Bernau establishes this result under the additional assumption that α and β preserve all joins and intersections that exist in G, but [**7**, Lemma 5.3] asserts that all joins and intersections in a large l-subgroup C of an abelian l-group A agree with those in A.

COROLLARY I. *If G is a large l-subgroup of an archimedean l-group H and α is an l-isomorphism of G onto a large subgroup of $D(X)$ then α is induced by an l-isomorphism of H into $D(X)$.*

Proof. Since $D(X)$ is the essential closure of G there exists an l-isomorphism β of H onto a large subgroup of $D(X)$. Since $G\beta$ is large in $D(X)$ we have $\alpha = \beta\bar{\tau}\bar{d}$ on G and so $\beta\bar{\tau}\bar{d}$ is an extension of α to H.

COROLLARY II. *An l-automorphism α of a large l-subgroup G of $D(X)$ is induced by an l-automorphism of $D(X)$.*

Proof. $\alpha = \bar{\tau}\bar{d}$ on G. Actually one can show that this is the unique extension of α to $D(X)$.

Finally, we wish to thank Simon Bernau for suggesting improvements of several of the proofs in this paper.

1. The f-ring hull of an archimedean l-group. This section is devoted to establishing the following result.

THEOREM 1.1. *If G is an archimedean l-group and $\{g_\gamma | \gamma \in \Gamma\}$ is a maximal disjoint subset of G then there exists a minimal f-ring H containing $(G, +)$ as a large l-subgroup and with identity $\bigvee g_\gamma$. H is necessarily archimedean and if K is another such ring then there exists a unique ring l-isomorphism τ of K onto H such that $g\tau = g$ for all $g \in G$.*

Remark. G is large in its essential closure G^e and $u = \bigvee g_\gamma$ exists in G^e. Moreover, there is a unique multiplication in G^e so that it is an f-ring with identity u. Thus H is the subring of G^e that is generated by G and u.

LEMMA 1.2. *An f-ring H that satisfies Theorem* 1.1 *is archimedean.*

Proof. Let $T = \{t \in H | t$ is a sum of products of positive elements from $G\}$ and for each $t \in T$ let $H(t)$ be the convex l-subgroup of $(H, +)$ that is generated by t. If $s, t \in T$ then $H(s) + H(t) \subseteq H(s + t)$ and so $\{H(t) | t \in T\}$ is directed by inclusion and hence $K = \bigcup H(t)$ is an l-subring of H that contains G.

Now suppose (by way of contradiction) that K is not archimedean. Then $a \gg b > 0$ for some $a, b \in K$. Since $a \in H(s)$ for some $s \in T$ we have $a < ns = t$ for some $n > 0$, and since G is large in K we may assume that $b \in G$. Thus $0 < g_\lambda \wedge b = g \in G$ for some $\lambda \in \Lambda$ and we may assume that $t \gg g > 0$, where $t \in T$ and $g_\lambda > g \in G$. Now $g^2 \leqq gg_\lambda = \bigvee (gg_\lambda) = g(\bigvee g_\lambda) = g$ and hence $g^k \leqq g$ for all $k > 0$.

$$t = a_{11}a_{12} \ldots a_{1n_1} + a_{21}a_{22} \ldots a_{2n_2} + \ldots + a_{s1}a_{s2} \ldots a_{sn_s}$$

where the $a_{ij} \in G^+$. Let a be the least upper bound of all the a_{ij} and let $n = \max \{n_1, n_2, \ldots, n_s\}$. Pick $m > 0$ so that $v = (mg - a)^+ > 0$. Then the polar v' of all the elements in H that are disjoint from v is an ideal in the ring H and so modulo v' we have $a < mg$. Thus $a_{j1}a_{j2} \ldots a_{jn_j} \leqq (mg)^n \leqq m^n g$ and hence $t \leqq sm^n g$ modulo v'. Therefore $(1 + sm^n)g \nleqq t$ in H, a contradiction. Thus $K = \bigcup H(t)$ is an archimedean f-ring. Let $H^L K = \bigcup H(t)$ is an archimedean f-ring. Let H^L be the lateral completion of H. Then H^L is an f-ring with identity $\bigvee g_\gamma$ and $G \subseteq K \subseteq H \subseteq H^L$. Thus since G is large in H^L, $G \subseteq K^L \subseteq H^L$. But K^L is an archimedean f-ring with identity $\bigvee g_\gamma$ (see [**7**]). Therefore $G \subseteq K^L \cap H$ an f-ring with identity $\bigvee g_\gamma$ and so by the minimality of H we have that $H = K^L \cap H$ is archimedean.

Proof of Theorem 1.1. We may assume that G is large in $G^e = D(X)$, each g_γ is a characteristic function, and $\bigvee g_\gamma$ is the identity u for the ring $D(X)$. The intersection H of all l-subrings of G^e that contain G and u satisfies the theorem.

Now suppose that K satisfies the Theorem. Then by the theory in [**1**] there exists a ring l-isomorphism β of K onto a large l-subring of $D(X)$. Thus $\bigvee(g_\gamma\beta) = (\bigvee g_\gamma)\beta = u$. By Bernau's Uniqueness Theorem,

$$(x)g_\gamma = (x\tau)g_\gamma\beta \cdot (x)d$$

and it follows that $d = u$ and that $\delta = \beta\bar{\tau}$ is the identity on G. Thus $G \subseteq K\delta \subseteq D(X)$ and so $K\delta = H$.

Thus if H_1 and H_2 satisfy the theorem then there exists an isomorphism σ of H_1 onto H_2 such that $g\sigma = g$ for each $g \in G$. If ρ is another such isomorphism then $\sigma\rho^{-1}$ is an l-automorphism of H_1 that induces the identity on G, but G generates H_1 as an f-ring and hence $\sigma\rho^{-1}$ is the identity on H_1. Therefore $\sigma = \rho$ is unique.

Let A and B be archimedean l-groups with maximal disjoint subsets $\{a_\gamma | \gamma \in \Gamma\}$ and $\{b_\gamma | \gamma \in \Gamma\}$ and let $\bar{A}$ and $\bar{B}$ be the corresponding f-rings given in Theorem 1.1.

COROLLARY. *If α is an l-isomorphism of A onto B such that $a_\gamma\alpha = b_\gamma$ for all $\gamma \in \Gamma$ then there exists a unique extension of α to a ring l-isomorphism β of $\bar{A}$ onto $\bar{B}$.*

Proof. Construct an f-ring $K \supseteq B$ and an isomorphism $\bar{\alpha}$ of $\bar{A}$ onto K that induces α. By the theorem there exists an isomorphism β of K onto $\bar{B}$ that induces the identity on B. Thus $\bar{\alpha}\beta$ is a ring l-isomorphism of $\bar{A}$ onto $\bar{B}$ that induces α on A.

If μ and ν are two such isomorphisms of $\bar{A}$ onto $\bar{B}$ then $\mu\nu^{-1}$ is an l-automorphism of $\bar{A}$ that induces the identity on A and so by the theorem must be the identity on $\bar{A}$. Therefore $\mu = \nu$.

2. The multiplications of an archimedean f-ring S with identity 1.

Let

$A(S)$ = group of all l-automorphisms of $(S, +)$,

$H(S)$ = group of all ring l-automorphisms of S,

$P(S)$ = group of all p-automorphisms of $(S, +)$.

In [**5**] it is shown that each p-endomorphism of $(S, +)$ is a multiplication by a fixed positive element in S. Hence $P(S)$ is isomorphic with the multiplicative group of positive units in the ring S.

THEOREM 2.1. *$A(S)$ is a splitting extension of $P(S)$ by $H(S)$.*

Proof. $P \cap H$ consists of the identity automorphism since the only multiplication of S that is a ring automorphism is the multiplication by 1. If $\gamma \in H$ and $\beta \in P$ then there exists $0 < p \in S$ such that $s\beta = ps$ for all $s \in S$ and so

$$(s\gamma)\beta\gamma^{-1} = (p(s\gamma))\gamma^{-1} = (p\gamma^{-1})s.$$

Thus $\gamma\beta\gamma^{-1}$ is a multiplication by $p\gamma^{-1}$ and so belongs to P. Therefore $P \triangleleft [P \cup H]$ and so it suffices to show that $A \subseteq HP$.

We may assume that S is a large l-subring of $D(X)$ that contains the identity u of $D(X)$, where X is the associated Stone space of S. If $\alpha \in A$ then by Corollary II of the uniqueness theorem $\alpha = \bar{\tau}\bar{d}$. In particular $u\alpha = u\bar{\tau}\bar{d} = u\bar{d} = d$ and so $d \in S$. Let $q = u\alpha^{-1} \in S$. Then $u = q\alpha = q(\bar{\tau}\bar{d}) = q\bar{\tau} \cdot d$ and

$$q^2 = q^2(\bar{\tau}\bar{d}) = (q\bar{\tau})^2 \cdot d = d^{-1} \cdot [q\bar{\tau} \cdot d]^2 = d^{-1}.$$

Thus $d^{-1} \in S$ and so $\bar{d} \in P$ and $\bar{\tau} = \alpha\bar{d}^{-1}$ on D. Thus $\bar{\tau}$ restricted to S belongs to H.

COROLLARY. *If $\alpha \in A(S)$ and $1\alpha = 1$, then $\alpha \in H(S)$.*

Proof. $u = u\alpha = u\bar{\tau}\bar{d} = u\bar{d} = d$, so $\alpha = \bar{\tau} \in H(S)$.

THEOREM 2.2. *Let $(S, +, \cdot, \leqq)$ be an archimedean f-ring with identity* 1. *If $\circ$ is another multiplication of S so that it is an f-ring then there exists $0 < p \in S$ such that*

$$a \circ b = abp \quad \text{for all } a, b \in S,$$

and conversely.

Proof. Pick $0 < a \in S$. Then the map $s \to s \circ a$ for all $s \in S$ is a p-endomorphism of the l-group $(S, +)$ and hence there exists $0 \leqq \bar{a} \in S$ such that

$$s \circ a = s\bar{a} \quad \text{for all } s \in S.$$

Thus, we have a map $a \to \bar{a}$ of S^+ into itself. Moreover

$$b\bar{a} = b \circ a = a \circ b = a\bar{b} \quad \text{for} \quad a, b \in S^+.$$

Let $p = \bar{1}$. Then $\bar{a} = 1\bar{a} = a\bar{1} = ap$. If $u, v \in S$ then $v = a - b$ where $a, b \in S^+$ and hence

$$u \circ v = u \circ (a - b) = u \circ a - u \circ b = u\bar{a} - u\bar{b} = uap - ubp$$
$$= u(a - b)p = uvp.$$

Remarks. (1) The multiplications agree if and only if $p = 1$.

(2) The ring $(S, \circ)$ has an identity if and only if $p^{-1} \in S$ and in this case p^{-1} is the identity.

(3) If $(S, \circ)$ has an identity then

$$s \xrightarrow{\tau} sp \quad \text{for all } s \in S$$

is a ring l-isomorphism of $(S, \circ)$ onto $(S, \cdot)$ and, of course, both rings are l-isomorphic to the ring $\mathscr{P}(S)$ of all p-endomorphisms of $(S, +)$.

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Proof. For $a, b \in S$ we have

$$(a \circ b)\tau = (a \circ b)p = (abp)p = apbp = a\tau b\tau,$$

$$(a + b)\tau = (a + b)p = ap + bp = a\tau + b\tau, \text{ and}$$

$$0 = a\tau = ap \rightarrow 0 = app^{-1} = a.$$

(4) The given multiplication on S is the unique multiplication so that S is an f-ring with identity if and only if 1 is the only positive element with a multiplicative inverse.

(5) $(S, \circ)$ has no non-zero nilpotents if and only if p is an order unit.

Proof. Consider $0 < a \in S$. Then $a \circ a = a^2 p = 0$ if and only if $a^2 \wedge p = 0$.

($\Leftarrow$) If p is an order unit then $a \circ a \neq 0$ for each $0 < a \in S$.

($\Rightarrow$) If p is not an order unit then $a \wedge p = 0$ for some $0 < a \in S$ and hence $a^2 \wedge p = 0$. Thus $a \circ a = 0$.

(6) If the principal polar p'' is a cardinal summand of S,

$$S = p'' \mid + \mid p'$$

then $(p', \circ)$ is a zero ring and $(p'', \circ)$ is an f-ring with no non zero nilpotents.

The elements $0 \leqq p, q \in S$ determine two f-ring multiplications for S, namely

$$a \circ b = abp \quad \text{and} \quad a \# b = abq.$$

THEOREM 2.3. *The following are equivalent.*

(a) *There exists a ring l-isomorphism* δ *of* $(S, \circ)$ *onto* $(S, \#)$.

(b) *There exists a ring l-automorphism* α *of* $(S, \cdot)$ *and an element* $x \in S^+$ *such that* $x^{-1} \in S^+$ *and* $p\alpha = qx$.

(c) *There exists a group l-isomorphism* β *of* $(S, +)$ *such that* $p\beta = q$.

Proof. (a) $\Rightarrow$ (b): Clearly δ is an l-automorphism of $(S, +)$ and so by Theorem 2.1 $\delta = \alpha\gamma$, where α is a ring l-automorphism of $(S, \cdot)$ and γ is a multiplication by $x \in S^+$ and $x^{-1} \in S^+$.

$$(p\alpha)x = (p\alpha)\gamma = p(\alpha\gamma) = p\delta = (1 \circ 1)\delta = 1\delta \# 1\delta = 1\alpha\gamma \# 1\alpha\gamma$$
$$= 1\gamma \# 1\gamma = x \# x = x^2 q.$$

Thus $p\alpha = xq$.

(b) $\Rightarrow$ (a): Define $s\delta = (s\alpha)x$ for all $s \in S$. Then for $s, t \in S$, $(s + t)\delta = ((s + t)\alpha)x = (s\alpha + t\alpha)x = (s\alpha)x + (t\alpha)x = s\delta + t\delta$.

$$(s \circ t)\delta = (stp)\delta = ((stp)\alpha)x = (s\alpha)(t\alpha)(p\alpha)x = (s\alpha)(t\alpha)qx^2$$
$$= (s\alpha)x(t\alpha)xq = (s\alpha)x \# (t\alpha)x = s\delta \# t\delta.$$

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$$(sx^{-1}\alpha^{-1})\delta = ((sx^{-1}\alpha^{-1})\alpha)x = s.$$

$$s\delta = t\delta \Rightarrow (s\alpha)x = (t\alpha)x \Rightarrow s\alpha = t\alpha \Rightarrow s = t.$$

Therefore δ satisfies (a).

(b) $\Rightarrow$ (c): Let $\beta = \alpha$ followed by the multiplication by x^{-1}.

(c) $\Rightarrow$ (b): By Theorem 2.1, $\beta = \alpha\gamma$ where α is a ring l-automorphism of $(S, \cdot)$ and γ is a multiplication by an element in S, say x^{-1}. Thus $q = p\beta = p\alpha x^{-1}$ so $p\alpha = qx$.

3. In this section we show that the multiplication on an f-ring S is essentially determined by the additive structure. For each $s \in S^+$ define

$x\bar{s} = sx$ for all $x \in S$.

Then $s \to \bar{s}$ is an additive homomorphism of S^+ into $\mathscr{P}(S)^+$ such that for $x \in S$ and $a, s, t \in S^+$,

(1) $x(s - t) = x\bar{s} - x\bar{t}$,

(2) $s \wedge t = 0 \Rightarrow a\bar{s} \wedge t = 0$,

(3) $\overline{st} = s\bar{t}$.

Moreover, S is commutative if and only if

(4) $s\bar{t} = t\bar{s}$.

THEOREM 3.1. *Suppose that* $(S, +, \leqq)$ *is an archimedean l-group and* $s \to \bar{s}$ *is a homomorphism of* S^+ *into* $\mathscr{P}(S)^+$ *that satisfies* (2) *or* (4). *For* $x \in S$ *and* $s, t \in S^+$ *define*

$$x(s - t) = x\bar{s} - x\bar{t}.$$

Then $(S, +, \cdot, \leqq)$ *is an f-ring. Thus there is a one-to-one correspondence between the elements in* $\mathrm{Hom}(S^+, \mathscr{P}(S)^+)$ *that satisfy* (2) *or* (4) *and the multiplications on S so that it is an f-ring.*

Remark. If we drop the hypothesis that S is archimedean then there is a one-to-one correspondence between the elements of $\mathrm{Hom}(S^+, \mathscr{P}(S)^+)$ that satisfy (2) and (3) and the multiplications on S so that it is an f-ring.

Proof of theorem. If $s - t = u - v$, where $s, t, u, v \in S^+$, then

$$s + v = u + t \Rightarrow \bar{s} + \bar{v} = \bar{u} + \bar{t} \Rightarrow x\bar{s} + x\bar{v} = x\bar{u} + x\bar{t} \Rightarrow$$

$$x\bar{s} - x\bar{t} = x\bar{u} - x\bar{v}$$

so our definition of multiplication is single valued.

For $a, b, c \in S$ we have

$$\begin{aligned} a(b + c) &= a(b^+ + c^+ - (b^- + c^-)) = a\overline{b^+ + c^+} - a\overline{b^- + c^-} \\ &= a\overline{b^+} + a\overline{c^+} - a\overline{b^-} - a\overline{c^-} \\ &= a(b^+ - b^-) + a(c^+ - c^-) = ab + ac; \end{aligned}$$

$$(b + c)a = (b + c)(a^+ - a^-) = (b + c)\overline{a^+} - (b + c)\overline{a^-}$$

$$= b\overline{a^+} + c\overline{a^+} - b\overline{a^-} - c\overline{a^-}$$

$$= b(a^+ - a^-) + c(a^+ - a^-) = ba + ca.$$

If $s \wedge t = 0$ and $a > 0$ then since $\bar{a} \in \mathscr{P}(S)^+$

$$0 = s\bar{a} \wedge t = sa \wedge t.$$

Thus if (4) holds then $0 = s\bar{a} \wedge t = a\bar{s} \wedge t = as \wedge t$; otherwise by (2) $0 = a\bar{s} \wedge t = as \wedge t$. Thus we have an archimedean f-ring and so both the commutative and associative laws for multiplication hold.

COROLLARY 1. *The element $s \rightarrow \bar{s}$ in* $\mathrm{Hom}(S^+, \mathscr{P}(S)^+)$ *satisfies* (2) *if and only if it satisfies* (4). *If the map satisfies* (2) *then it also satisfies* (3) *and it is an l-homomorphism of S^+ into $\mathscr{P}(S)^+$ and so determines a ring l-homomorphism of $(S, +, \cdot, \leqq)$ into $\mathscr{P}(S)$.*

Proof. If $x, s, t \in S^+$ then [**5**, p. 229]

$$x(\bar{s} \vee \bar{t}) = x\bar{s} \vee x\bar{t} = xs \vee xt$$
$$= x(s \vee t) = x(\overline{s \vee t}).$$

Now define $\overline{s - t} = \bar{s} - \bar{t}$; then this is a ring l-isomorphism of $(S, +, \cdot, \leqq)$ into $\mathscr{P}(S)$. For, if $s \wedge t = 0$ then $0 = x\theta = x\overline{s \wedge t} = x(\bar{s} \wedge \bar{t}) = x\bar{s} \wedge x\bar{t} = xs \wedge xt = x(s \wedge t) = x\theta = 0$ so $\bar{s} \wedge \bar{t} = \theta$.

An f-ring F has no non-zero nilpotents if and only if for each $a \in F^+$

$$a^2 = 0 \Rightarrow a = 0.$$

COROLLARY 2. *For the ring S the following are equivalent:*
(1) *S has no non-zero nilpotent elements;*
(2) *$a\bar{a} = 0 \Rightarrow a = 0$ for all $a \in S^+$;*
(3) *$\bar{a} = \theta \Rightarrow a = 0$;*
(4) *The map $s \rightarrow \bar{s}$ is one-to-one.*

Proof. Since $a^2 = a\bar{a}$, (1) and (2) are equivalent.
(2) $\Rightarrow$ (3): $\bar{a} = \theta \Rightarrow a\bar{a} = 0 \Rightarrow a = 0$.
(3) $\Rightarrow$ (2): $a\bar{a} = 0 \Rightarrow \bar{a}^2 = \overline{a\bar{a}} = \theta \Rightarrow \bar{a} = \theta \Rightarrow a = 0$. Here we use the fact that $\mathscr{P}(S)$ has no non-zero nilpotents.
(4) $\Rightarrow$ (3): This is trivial.
(3) $\Rightarrow$ (4): We can extend $s \rightarrow \bar{s}$ to an l-homomorphism of $(S, +)$ into $(\mathscr{P}(S), +)$, but by (3) the kernel is zero and so the map is one-to-one.

COROLLARY 3. *The following are equivalent:*
(1) *$(S, +, \cdot, \leqq)$ has an identity;*
(2) *$\bar{s}$ is the identity automorphism for some $s \in S^+$;*
(3) *$s \rightarrow \bar{s}$ is an isomorphism of S^+ onto $\mathscr{P}(S)^+$.*
In this case $S \cong \mathscr{P}(S)$.

Proof. (3) $\Rightarrow$ (2): This is clear.

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(2) $\Rightarrow$ (1): $x = x\bar{s} = xs$ all $x \in S$ so s is an identity for S since S is commutative.

(1) $\Rightarrow$ (3): Each p-endomorphism α of S is a multiplication by a positive element $s \in S^+$. Therefore, $x\alpha = xs = x\bar{s}$ for all $x \in S$ and so the map is epimorphic. If $\bar{s} = \bar{t}$ then $s = 1s = 1\bar{s} = 1\bar{t} = 1t = t$, so the map is one-to-one.

COROLLARY 4. *An archimedean l-group S admits a multiplication so that it is an f-ring with identity if and only if $S^+ \cong \mathscr{P}(S)^+$, where the map satisfies* (2). *If this is the case then the ring is l-isomorphic to $\mathscr{P}(S)$.*

4. The relationship between G^u and the various other hulls of G. Let G be an archimedean l-group with order unit u and let G^u be the minimal f-ring with u as an identity in which G is large. Let (see [**7**])

G^d = divisible closure of G,
G^c = Dedekind-MacNeille completion of G,
G^e = essential closure of G,
G^v = vector lattice hull of G,
G^P = projectable hull of G,
G^{SP} = strongly projectable hull of G,
G^L = lateral completion of G, and
G^O = orthocompletion of G.

Let $w = d, c, e, v, P, SP, L$, or O. Then G^w is archimedean and G is large in G^w. In fact, if H is a w-group in which G is large, then G^w is the intersection of all l-subgroups of H that are w-groups. Here we use the fact that an essentially closed group is by definition archimedean.

PROPOSITION 4.1. $(G^w)^u \subseteq (G^u)^w$ *the unique minimal f-ring with identity u that is a w-group and in which G is large. In particular* $(G^w)^u = (G^u)^w$ *if and only if* $(G^w)^u$ *is a w-group.*

Proof. Since G is large in $(G^u)^w$, $G^w \subseteq (G^u)^w$ and since $(G^u)^w$ is an f-ring with identity u, $(G^w)^u \subseteq (G^u)^w$.

If K is a minimal f-ring with identity u that is a w-group and in which G is large then

$$G \subseteq G^u \subseteq K \Rightarrow G \subseteq (G^u)^w \subseteq K \Rightarrow (G^u)^w = K.$$

Note, for example, that $(G^u)^v$ is the minimal f-algebra with identity u in which G is large.

PROPOSITION 4.2. $(G^w)^u$ *is a w-group for $w = d, v, e$ or SP. The statement does not hold for $w = P$ or c and is open for $w = L$ or O.*

Proof. We may assume that

$$G \subseteq G^w \subseteq (G^w)^u \subseteq G^e = D(X)$$

where X is the associated Stone space of G and u is the identity for D. Thus if $w = e$ then $(G^e)^u = D$ and so is essentially closed.

Since $Ru \subseteq G^v$ it follows that $(G^v)^u$ is a vector lattice and since $Qu \subseteq G^d$, $(G^d)^u$ is divisible.

In order to prove that $(G^{SP})^u$ is an SP-group we need:

LEMMA. *If* $G = A \,|+|\, B$ *and* $u = a + b$ *with* $a \in A$ *and* $b \in B$ *then* $G^u = A^a \,|+|\, B^b$.

Proof. Clearly $G^u \subseteq A^a \,|+|\, B^b$. Now $A \subseteq G^u \cap A^a \subseteq A^a$ and so by the minimality of A^a we have $G^u \cap A^a = A^a$. Thus $G^u \supseteq A^a \cup B^b$ so $G^u \supseteq A^a \,|+|\, B^b$.

Now suppose that G is a SP-group and M is a polar in G^u. We shall denote the polar operation in G and G^u by $'$ and $*$. Since G is large in G^u, $M \cap G$ is a polar in G so

$$G = (M \cap G) \,|+|\, B \quad \text{and} \quad u = u_1 + u_2.$$

Thus by the Lemma

$$G^u = (M \cap G)^{u_1} \,|+|\, B^{u_2}.$$

Since u_1 is an order unit in $M \cap G$, $u_1'' = M \cap G$ and $u_1^{**} = (M \cap G)^{u_1}$. Also

$$u_1^{**} \cap G = u_1'' = M \cap G$$

and so $(M \cap G)^{u_1} = u_1^{**} = M$. Therefore M is a cardinal summand of G^u and hence G^u is an SP-group.

Examples 5.6 and 5.7 complete the proof of Proposition 4.2.

5. Examples and open questions.

Example 5.1. Let S be the cardinal sum $R \,|+|\, R$. Then $\mathscr{P}(S)$ is the ring $R + R$. An additive l-isomorphism of $(S, +)$ onto $(\mathscr{P}(S), +)$ need not satisfy property (2) in section 3.

For $(x, y) \in S^+$ let $\overline{(x, y)}$ be the multiplication by (y, x). Then $(1, 0) \wedge (0, 1) = (0, 0)$ and $(1, 1) > (0, 0)$ but

$$(1, 1)\overline{(1, 0)} \wedge (0, 1) = (0, 1),$$

so (2) is not satisfied and clearly $(x, y) \rightarrow \overline{(x, y)}$ is an l-isomorphism of $(S, +)$ onto $(\mathscr{P}(S), +)$.

Example 5.2. Let H be the ring $R \oplus R$ and define (a, b) positive if $a > 0$ or $a = 0$ and $b > 0$. Let G be the subgroup of H generated by $u = (1, 1)$ and $a = (\sqrt{2}, 1)$. Then G is archimedean and o-isomorphic to the subgroup of R generated by 1 and $\sqrt{2}$, but the subring K of H generated by G is not archimedean and of course G is not large in K.

Examples 5.3. Consider $a = (1, 2, 3, \ldots) \in \prod_{i+1}^{\infty} Z_i$. Thus $[a] \cong Z$ but the l-subring of $\prod Z_i$ generated by a is not totally ordered and of course is not an essential extension of $[a]$ nor does it have an identity.

Example 5.4. Let G be the l-subgroup of $\prod_{i=1}^{\infty} R_t$ generated by $a = (1, 1, 1, \ldots)$ and $b = (1, 1/2, 1/3, \ldots)$. Then

$$G^a \not\cong G^b$$

because the identity a in G^a is a strong order unit but the identity b in G^b is not.

Example 5.5. Let $G = [1/8] \subseteq Q$, $u = 1/2$ and $v = 1/4$. Then $G^u \cong G^v \cong \{m/2^n | m, n \in Z\}$ but there does not exist an l-automorphism of G that maps u onto v. Thus the converse to the corollary of Theorem 1.1 does not hold.

Example 5.6. Let G be the cyclic subgroup of Q generated by $1/2$ and let u = 1. Then G^u is the ring of all rationals with denominators a power of 2. Thus G is complete but G^u is not.

Example 5.7. A P-group G such that G^u is not a P-group: Let

$$u = (1, 1, 1, \ldots)$$

$$a = (1, 1/2, 1/3, \ldots)$$

$$b = (1, 1/5, 1/9, 1/17, 1/25, 1/37, 1/49, \ldots)$$

$$G = \sum_{i=1}^{\infty} Q_i \oplus [u] \oplus [a] \oplus [b] \subseteq \prod_{i=1}^{\infty} Q_i = H.$$

Then G is an l-subgroup of H and if $g \in G$ has an infinite number of non-zero components then all but a finite number of components of G are non-zero. Thus clearly G is a P-group but not an SP-group.

Now $a^2 - b = (0, 1/4 - 1/5, 0, 1/16 - 1/17, 0\ 1/36 - 1/37, \ldots)$ and $(a^2 - b)^{**}$ is not a summand of G^u since $(0, 1, 0, 1, 0, \mathbf{1}, \ldots) \notin G^u$.

Questions. Let G be an archimedean l-group with order unit u.

(1) If H is a minimal archimedean f-ring with identity u that contains G then is $H = G^u$?

(2) If π is an l-homomorphism of G onto an l-group K then can π be extended to a ring l-homomorphism of G^u onto $K^{u\pi}$?

(3) If G is an L-group (O-group) then is G^u an L-group (O-group)?

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