

THE OTHER LAW OF THE ITERATED LOGARITHM FOR MARTINGALES

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A Skorokhod embedding approach is used to give the other law of the iterated logarithm for square integrable martingales.

1. INTRODUCTION

Let $\{S_n, \mathcal{F}_n: n \geq 1\}$ be a square integrable martingale, that is $ES_n^2 < \infty$ for all n , on a probability space (Ω, \mathcal{F}, P) and define $V_n^2 = \sum_{j=1}^n E(X_j^2 | \mathcal{F}_{j-1})$ where $X_n = S_n - S_{n-1}$. Let $M_n = \max_{1 \leq k \leq n} |S_k|$. We are concerned here with conditions under which the other law of the iterated logarithm, that is,

$$(1) \quad \liminf_{n \rightarrow \infty} \frac{M_n}{(V_n^2 / \log_2 V_n^2)^{\frac{1}{2}}} = \frac{\pi}{8^{\frac{1}{2}}} \text{ a.s.}$$

holds. Whilst some such conditions may be found using Theorem 6.1 of Jain, Jogdeo and Stout [2] these results depend on a sharper version of Strassen's almost sure invariance principle. A comparison with the result of Jain and Pruitt [3] when S_n is the sum of independent and identically distributed random variables motivates us to show that (1) holds under at most the conditions of the ordinary functional law of the iterated logarithm of, for example, Hall and Heyde [1]. Here we use a Skorokhod embedding approach to extend the methods of Jain and Pruitt [3] to the martingale case. We commence with a study of the time changed Brownian motion.

2. THE OTHER LAW OF THE ITERATED LOGARITHM FOR TIME CHANGED BROWNIAN MOTION

For a sequence $\{T_n: n \geq 1\}$ of positive random variables and a standard Brownian motion $\{B(t): t \geq 0\}$, all defined on the same probability space, let $S_n = B(T_n)$ and

$$M_n = \max_{1 \leq k \leq n} |S_k| = \max_{1 \leq k \leq n} |B(T_k)|.$$

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THEOREM 1. *If*

$$(2) \quad T_n \uparrow \infty \text{ a.s.}, T_n/T_{n+1} \rightarrow 1 \text{ a.s.}$$

then

$$(3) \quad \liminf_{n \rightarrow \infty} \frac{M_n}{(T_n/\log_2 T_n)^{\frac{1}{2}}} = \frac{\pi}{8^{\frac{1}{2}}} \text{ a.s.}$$

PROOF: We proceed via several lemmas.

LEMMA 1. *Under the conditions of the Theorem,*

$$\liminf_{n \rightarrow \infty} \frac{M_n}{(T_n/\log_2 T_n)^{\frac{1}{2}}} \leq \frac{\pi}{8^{\frac{1}{2}}} \text{ a.s.}$$

PROOF: Let $r_n = \max\{j : T_j \leq n^n\}$. From Lemma 1 of Jain and Pruitt [3] we have

$$\liminf_{n \rightarrow \infty} (n^n/\log_2 n^n)^{-\frac{1}{2}} \max_{0 \leq s \leq n^n} |B(s)| \leq \frac{\pi}{8^{\frac{1}{2}}} \text{ a.s.}$$

so that
$$\liminf_{n \rightarrow \infty} (n^n/\log_2 n^n)^{-\frac{1}{2}} \max_{0 \leq s \leq T_{r_n}} |B(s)| \leq \frac{\pi}{8^{\frac{1}{2}}} \text{ a.s.}$$

and hence
$$\liminf_{n \rightarrow \infty} (n^n/\log_2 n^n)^{-\frac{1}{2}} M_{r_n} \leq \frac{\pi}{8^{\frac{1}{2}}} \text{ a.s.}$$

which yields the lemma once it is observed that $n^{-n}T_{r_n} \rightarrow 1$ a.s. □

Now define for $\alpha > 1$,

$$p_n = \max\{j : T_j \leq \alpha^n\}$$

LEMMA 2. *Under the condition of the Theorem, for $c < (\pi/8^{1/2})$,*

$$P\left(M_{p_n} < c\left[\frac{\alpha^n}{\log_2 \alpha^n}\right]^{\frac{1}{2}} \text{ i.o.}\right) = 0.$$

PROOF: As in Jain and Pruitt [3], choose

$$\xi \in (c, \pi/8^{1/2}), \eta \in (1, \pi^2/8\xi^2)$$

and then take β so large that

$$2^{1/\beta} < e^{-\eta+\pi^2/8\xi^2}.$$

Let $\nu = [\beta^n/\log_2 \alpha^n]$ and $N = [\log_1 \alpha^n/\beta]$.

Now

$$\begin{aligned}
 & \left\{ M_{p_n} < c \left(\frac{\alpha^n}{\log_2 \alpha^n} \right)^{\frac{1}{2}} \right\} \\
 &= \left\{ \max_{1 \leq k \leq p_n} |B(T_k)| < c \left(\frac{\alpha^n}{\log_2 \alpha^n} \right)^{\frac{1}{2}} \right\} \\
 &\subset \left\{ \sup_{0 \leq s \leq T_{p_n}} |B(s)| < c \left(\frac{\alpha^n}{\log_2 \alpha^n} \right)^{\frac{1}{2}} \right\} \\
 &\subset \left\{ \sup_{0 \leq s \leq \alpha^n} |B(s)| < c \left(\frac{\alpha^n}{\log_2 \alpha^n} \right)^{\frac{1}{2}} \right\} \\
 &\subset \bigcap_{k=1}^N \left\{ \sup_{(k-1)\nu < s \leq k\nu} |B(s)| < c \left(\frac{\alpha^n}{\log_2 \alpha^n} \right)^{\frac{1}{2}} \right\} \\
 &\subset \bigcap_{k=1}^N \left\{ B(s) - B((k-1)\nu) \in \left(-\frac{\xi}{\beta^{\frac{1}{2}}} \nu^{\frac{1}{2}}, \frac{\xi}{\beta^{\frac{1}{2}}} \nu^{\frac{1}{2}} \right) - B((k-1)\nu), \right. \\
 &\quad \left. \text{for all } (k-1)\nu < s \leq k\nu \right\}
 \end{aligned}$$

so that using the Markov property and well-known estimates, following Jain and Pruitt [3],

$$\begin{aligned}
 & P \left(M_{p_n} < c \left(\frac{\alpha^N}{\log_2 \alpha^N} \right)^{\frac{1}{2}} \right) \\
 &\leq \left(2e^{-\pi^2 \beta / 8 \xi^2} \right)^N \\
 &< K n^{-\eta}, \text{ for some constant } K.
 \end{aligned}$$

The Borel-Cantelli lemma now completes the proof of Lemma 2. □

To prove the Theorem observe that from Lemma 2 we have, with probability one,

$$M_{p_n} \geq c \left(\frac{\alpha^n}{\log_2 \alpha^n} \right)^{\frac{1}{2}}$$

for all n sufficiently large. Then if $p_n < k \leq p_{n+1}$, $M_k \geq M_{p_n} \geq c((\alpha^n)/(\log_2 \alpha^n))^{1/2} \geq c/\alpha((T_k)/(\log_2 T_k))^{1/2}$ as $T_{p_n} \leq \alpha^n \leq T_k \leq \alpha^{n+1}$. As c/α may be made arbitrarily close to $\pi/8^{1/2}$, this and Lemma 1 prove the Theorem. □

3. THE OTHER LAW OF THE ITERATED LOGARITHM FOR MARTINGALES

We now return to the martingale case discussed in the introduction.

THEOREM 2. *If $V_n^2 \rightarrow \infty$ a.s.,*

$$(4) \quad V_{n+1}^2 / V_n^2 \rightarrow 1 \text{ a.s.,}$$

and for some $\delta > 0$,

$$(5) \quad \sum_{n=1}^{\infty} V_n^{-2(1+\delta)} E(|X_n|^{2(1+\delta)} | \mathcal{F}_{n-1}) < \infty$$

then
$$\liminf_{n \rightarrow \infty} \frac{M_n}{(V_n^2 / \log_2 V_n^2)^{\frac{1}{2}}} = \frac{\pi}{8^{\frac{1}{2}}} \text{ a.s.}$$

PROOF: Using the Skorokhod embedding of, for example, Scott and Huggins [4], there exists a sequence of stopping times $\{T_n; n \geq 1\}$ and a standard Brownian motion $\{B(t); t \geq 0\}$ such that

$$B(T_n) = S_n \text{ a.s.,}$$

and, where $t_n = T_n - T_{n-1}$, for some increasing family of σ -fields \mathcal{G}_n with $T_n \in \mathcal{G}_n$, $\mathcal{G}_n \supseteq \mathcal{F}_n$,

$$E(t_n | \mathcal{G}_{n-1}) = E(X_n^2 | \mathcal{F}_{n-1}) \text{ a.s.}$$

Further, for $1 < p < \infty$, there exists a constant C_p depending only on p such that

$$E(t_n^{p/2} | \mathcal{G}_{n-1}) \leq C_p E(|X_n|^p | \mathcal{F}_{n-1}) \text{ a.s.}$$

Thus Theorem 2 will follow from Theorem 1 once we show that

$$(6) \quad V_n^{-2} T_n \rightarrow 1 \text{ a.s.}$$

To see that (6) does hold let

$$Y_n = V_n^{-2}(t_n - E(t_n | \mathcal{G}_{n-1}))$$

so that $\{Y_n; n \geq 1\}$ forms a martingale difference sequence. Then

$$\begin{aligned} & \sum_{n=1}^{\infty} V_n^{-2(1+\delta)} E(|t_n - E(t_n | \mathcal{G}_{n-1})|^{1+\delta} | \mathcal{G}_{n-1}) \\ & \leq 2^{(2+\delta)} \sum_{n=1}^{\infty} V_n^{-2(1+\delta)} E(|t_n|^{1+\delta} | \mathcal{G}_{n-1}) \\ & \leq w^{(2+\delta)} C_p \sum_{n=1}^{\infty} V_n^{-2(1+\delta)} E(|X_n|^{2(1+\delta)} | \mathcal{F}_{n-1}) \text{ a.s.} \\ & < \infty \text{ a.s., using (5).} \end{aligned}$$

Thus by Corollary 2.8.5 of Stout [5] $\sum_{n=1}^{\infty} V_n^{-2}(t_n - E(t_n | \mathcal{G}_{n-1}))$ converges and the Kronecker lemma yields (6). □

REMARKS. 1. Whilst the application of Theorem 2 to the case when the X_n are independently and identically distributed random variables does not give the Theorem of Jain and Pruitt [3], their result can of course be obtained directly from Theorem 1 using the law of large numbers to verify (2).

2. As the conditions of our Theorem 1 are contained in the conditions of Theorem A of Hall and Heyde [1] our results will hold under the various conditions for the law of the iterated logarithm for martingales given in their paper.

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