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REAL INTERPOLATION OF SOBOLEV SPACES ON SUBDOMAINS OF $\mathbf{R}^n$

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1. Introduction. The real interpolation method is a very convenient tool in the study of imbedding relationships among Sobolev spaces and some of their fractional order generalizations, (Besov spaces, Nikolskii spaces etc.) Central to the application of these methods is the *a priori* determination that a given Sobolev space $W^{k,p}(\Omega)$ belongs to an appropriate class of spaces intermediate between two other "extreme" spaces. Of special interest are interpolations involving only one of the parameters k and p; for interpolation on order of smoothness, k, we want to know that $W^{k,p}(\Omega)$ is "suitably intermediate" (see Section 3 for precise definition) between, say, $L^p(\Omega)$ and $W^{m,p}(\Omega)$ where $0 < k < m$, while for interpolation on order of summability, p, we want to know that $W^{k,p}(\Omega)$ is "suitably intermediate" between, say, $W^{k,p_1}(\Omega)$ and $W^{k,p_2}(\Omega)$ where $p_1 < p < p_2$.

Let us denote by $\|\cdot\|_p = \|\cdot\|_{p,\Omega}$ the norm in $L^p(\Omega)$, and by $\|\cdot\|_{k,p} = \|\cdot\|_{k,p,\Omega}$ the norm in $W^{k,p}(\Omega)$:

$$\|u\|_{k,p} = \left\{\sum_{|\alpha|\leqq k} \|D^\alpha u\|_p^{\,p}\right\}^{1/p}.$$

(See Adams [**1**] for details.) Involved in the matter of interpolation on order of smoothness is the following approximation question: does there exist a constant C, and for every function u in $W^{k,p}(\Omega)$ and every number $\epsilon > 0$ a function u_ϵ in $W^{m,p}(\Omega)$, such that

$$\|u - u_\epsilon\|_p \leqq C\,\epsilon^k\|u\|_{k,p}, \quad \text{and} \quad \|u_\epsilon\|_{m,p} \leqq C\,\epsilon^{k-m}\|u\|_{k,p}?$$

If $\Omega = \mathbf{R}^n$ the answer is fairly evidently "yes". In Section 2 below we provide an affirmative answer for a class of domains satisfying a "smooth cone property."

Involved in the matter of interpolation on order of summability is the following "lifting" question: given k does there exist a linear operator R defined on $\prod_{|\alpha|\leqq k} L_{\mathrm{loc}}^1(\Omega)$ into $L_{\mathrm{loc}}^1(\Omega)$ such that R is bounded on $\prod_{|\alpha|\leqq k} L^p(\Omega)$ into $W^{k,p}(\Omega)$ for $1 < p < \infty$, and such that $Rv = u$ if u belongs to $W^{k,p}(\Omega)$ and $v = \{D^\alpha u\}_{|\alpha|\leqq k}$? Again it is not difficult to construct such a lifting R for $\Omega = \mathbf{R}^n$, and in Section 6 we construct one for Ω with the smooth cone property. Unfortunately, this operator does not map $\prod_{|\alpha|\leqq k} L^1(\Omega)$ into $W^{k,1}(\Omega)$. However, if Ω is a homogeneous space with respect to its intrinsic metric (see Section 2 for details) we can conclude that Rv belongs to $W^{k,1}(\Omega)$ provided that v

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belongs to $\prod_{|\alpha|\leqq k} L^1(\Omega)$, and, in addition, v_α belongs to the Hardy space $H^1(\Omega)$ whenever $|\alpha| = k$.

Section 2 of this paper is devoted to a description of the smooth cone property and related regularity conditions on domains $\Omega \subset \mathbf{R}^n$, and to precise formulations of the approximation and lifting theorems. Section 3 presents a brief discussion of the real interpolation method and proceeds to application of the method in the theory of Sobolev, Besov, and Sobolev-Lorentz spaces defined over domains with the smooth cone property. Section 4 is concerned with applications of the interpolation theory yielding sharp imbeddings of these spaces into spaces of continuous functions satisfying fractional Lipschitz conditions (Hölder conditions) with respect to the intrinsic metric of the domain. The approximation and lifting theorems are proved in Sections 5 and 6 respectively. Some specific examples of domains with various properties are discussed in Section 7.

The present paper should be compared with the paper [**14**] by Peetre, and the series of papers [**11**; **12**; **13**] by Muramatu. We follow Peetre in using interpolation to derive imbedding theorems for Sobolev spaces from imbedding theorems for certain Besov spaces; in doing this we do not need the various intrinsic characterizations of Besov spaces. Like Muramatu, we consider Sobolev spaces on subdomains of $\mathbf{R}^n$; most of his work, however, is concerned with properties of Besov spaces on such domains, whereas our main goal is to obtain sharp imbedding theorems for Sobolev spaces. We feel that the methods we use in this paper are natural and direct; moreover we are able to deal with a wider class of domains than Muramatu or Peetre.

Note that throughout the paper C is used to denote various constants which change from line to line.

2. The smooth cone property. Throughout this work Ω shall denote a domain, that is, an open, not necessarily connected set in real Euclidean n-space, $\mathbf{R}^n$. We shall denote by $\mathscr{B}^\infty(\Omega)$ the class of all infinitely smooth vector fields Φ on Ω with values in $\mathbf{R}^n$ such that, for each multi-index α, $\sup_{x\in\Omega} |D^\alpha \Phi(x)|$ is finite. Given $\epsilon > 0$ and x in Ω we consider the finite "cone"

$$C_\epsilon(x; \Phi) = \bigcup_{0<\eta\leqq\epsilon} B_\eta(x + \eta\Phi(x)),$$

where $B_\eta(y)$ is the open ball of radius η centred at y. (If $|\Phi(x)| > 1$ then $C_\epsilon(x; \Phi)$ is conical; if $|\Phi(x)| \leqq 1$ it is just the ball $B_\epsilon(x + \epsilon\Phi(x))$.) Evidently $C_\epsilon(x; \Phi) \subset B_{\epsilon(1+||\Phi||)}(x)$, where $||\Phi|| = \sup_{x\in\Omega}|\Phi(x)|$.

We shall say that Ω has the *smooth cone property* if there exists Φ in $\mathscr{B}^\infty(\Omega)$ and $\epsilon > 0$ such that $C_\epsilon(x; \Phi) \subset \Omega$ for every x in Ω. We shall always assume that $\epsilon = 1$ since we can arrange this by dilating Ω.

It is useful to compare the smooth cone property with certain other cone properties to be found in the literature on Sobolev spaces. It is evidently stronger than the ordinary cone property which asserts that each x in Ω should be the vertex of a finite cone C_x of fixed dimensions contained in Ω. For the smooth cone property the cones C_x must vary smoothly from point to point. Muramatu [**13**, p. 328] uses a smooth cone condition in his approach to Sobolev and Besov spaces over domains Ω, but his version requires that the generating field Φ belong to $\mathscr{B}^\infty(\mathbf{R}^n)$ rather than just $\mathscr{B}^\infty(\Omega)$. This condition forces Ω to lie on only one side of its boundary. In contrast, a domain with the smooth cone property can lie on both sides of its boundary. We shall give an example of such a domain in Section 7. For all of our results below we could weaken the assumption on Φ so as to require the continuity and boundedness of derivatives $D^\alpha\Phi(x)$ for $|\alpha| \leqq m$ (i.e. $\Phi \in \mathscr{B}^m(\Omega)$) for suitably chosen m.

We remark here that there is a measure theoretic (nongeometric) version of the cone property which is slightly weaker than the ordinary cone property but is still sufficient to establish certain imbedding and interpolation results for Sobolev spaces. (See [**2**].) This *weak cone property* requires the existence of a positive number δ such that for every x in Ω the "cone" $\Gamma(x) = \{y \in B_1(x):$ segment $[x, y] \subset \Omega\}$ has measure not less than δ.

Because a domain with the smooth cone property may lie on both sides of an $(n-1)$-dimensional part of its boundary, the Euclidean metric in $\mathbf{R}^n$ is not appropriate for determining the closeness of points in Ω. We use the *intrinsic metric* ρ: if x and y are in Ω then $\rho(x, y)$ is the infimum of the lengths of piecewise smooth arcs in Ω joining x to y. (Of course $\rho(x, y) = +\infty$ if x and y do not lie in the same connected component of Ω.) An essential part of our proof of the lifting theorem is based on certain properties of the Hardy space $H^1(\Omega)$. (See Coifman and Weiss [**7**].) These properties are in turn obtained under the assumption that Ω is a space of homogeneous type with respect to the intrinsic metric. This homogeneity condition asserts that for every positive real number r and every point x in Ω,

$$(2.1) \qquad \mu(S_{2r}(x)) \leqq C\,\mu(S_r(x))$$

where μ is Lebesgue measure, $S_r(x)$ is the intrinsic ball $\{y \in \Omega : \rho(x, y) < r\}$, and C is a constant independent of r and x. If Ω has the ordinary cone property, then (2.1) holds for small r; so any such domain that is bounded relative to its intrinsic metric is of homogeneous type. A domain that is bounded relative to the Euclidean metric need not be bounded relative to its intrinsic metric, because it need not be connected, but any such domain with the cone property is of homogeneous type, because it is a union of finitely many intrinsically bounded components. Exterior domains (i.e., those with bounded complements) having the cone property are also of homogeneous type. Finally, if Ω is any domain with the cone property and Q is a bounded subset of Ω then there is a bounded domain Ω_1 with the cone property (and hence of homogeneous type) such that $Q \subset \Omega_1 \subset \Omega$. (Specifically, $\Omega_1 = \bigcup_{x\in Q} C_x$); this observation will play an important role in the proof of the lifting theorem.

We now give precise formulations of the approximation and lifting theorems.

THEOREM 1. (Approximation Theorem) *Let Ω be a domain in $\mathbf{R}^n$ having the smooth cone property. Let $1 \leqq p < \infty$ and let k and m be integers with $0 < k < m$. There exists a constant C depending on the numbers n, p, k, and m, and on the vector field Φ providing the smooth cone property, such that for each ϵ with $0 < \epsilon \leqq 1$ and each u in $W^{k,p}(\Omega)$ there exists u_ϵ in $W^{m,p}(\Omega)$ satisfying*

$$||u - u_\epsilon||_p \leqq C\, \epsilon^k ||u||_{k,p},$$

and

$$||u_\epsilon||_{m,p} \leqq C\, \epsilon^{k-m} ||u||_{k,p}.$$

Remark. In our proof of the approximation theorem for domains with the smooth cone property we will actually establish slightly stronger estimates for u_ϵ, namely

$$||u - u_\epsilon||_p \leqq C\, \epsilon^k |u|_{k,p},$$

and

$$|u_\epsilon|_{j,p} \leqq \begin{cases} C||u||_{k,p} & \text{if } 0 \leqq j \leqq k-1 \\ C\, \epsilon^{k-j}\, |u|_{k,p} & \text{if } k \leqq j \leqq m, \end{cases}$$

where $|u|_{j,p} = |u|_{j,p,\Omega}$ denotes the seminorm $\{\sum_{|\alpha|=j} ||D^\alpha u||_p{}^p\}^{1/p}$. The weaker inequalities, as stated in the theorem, are what we need for our applications, and in Section 7 we shall see an example of a domain not having the smooth cone property, but for which these inequalities still obtain.

THEOREM 2. (Lifting Theorem) *Let Ω be a domain in $\mathbf{R}^n$ having the smooth cone property, and let k be a positive integer. There exists a linear operator R from $\prod_{|\alpha| \leqq k} L_{\mathrm{loc}}{}^1(\Omega)$ into $L_{\mathrm{loc}}{}^1(\Omega)$ such that $R(\{D^\alpha u\}_{|\alpha| \leqq k}) = u$ for every u in $C^\infty(\Omega)$, and such that for each real p with $1 < p < \infty$, R is bounded from $\prod_{|\alpha| \leqq k} L^p(\Omega)$ into $W^{k,p}(\Omega)$. If, in addition, Ω is of homogeneous type with respect to its intrinsic metric, then R is also bounded from $(\prod_{|\alpha| < k} L^1(\Omega)) \times (\prod_{|\alpha| = k} H^1(\Omega))$ into $W^{k,1}(\Omega)$.*

Remark. If D is the map $u \mapsto \{D^\alpha u\}_{|\alpha| \leqq k}$ the lifting theorem implies that RD is the identity map on $W^{k,p}(\Omega)$ for $1 < p < \infty$.

3. Real interpolation of Sobolev spaces. We begin by recalling the elements of the real interpolation method of Lions and Peetre. (A good reference for the fundamentals of this method is Butzer and Berens [**4**].)

Given a pair of Banach spaces B_0 and B_1 with respective norms $||\cdot||_0$ and $||\cdot||_1$, each continuously imbedded in the same topological vector space, the real interpolation methods associate with each pair of numbers θ and q satisfying $0 < \theta < 1$ and $1 \leqq q \leqq \infty$, a Banach space $(B_0, B_1)_{\theta,q}$ intermediate between B_0 and B_1 such that if C_0 and C_1 form another such pair, and T is a linear operator mapping B_i boundedly into C_i with norm M_i, $(i = 0, 1)$, then T maps $(B_0, B_1)_{\theta,q}$ boundedly into $(C_0, C_1)_{\theta,q}$ with norm at most $M_0{}^{1-\theta} M_1{}^\theta$. There are several methods for defining $(B_0, B_1)_{\theta,q}$, all leading to the same

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spaces with equivalent norms. Two of these (the K method and the J method) involve the following function norms: for $u \in B_0 + B_1$ and $t \in \mathbf{R}^+$,

$$K(t, u) = \inf \{||u_0||_0 + t||u_1||_1 : u = u_0 + u_1, u_0 \in B_0, u_1 \in B_1\};$$

and for $u \in B_0 \cap B_1$ and $t \in \mathbf{R}^+$,

$$J(t, u) = \max (||u||_0, t||u||_1).$$

The intermediate space $(B_0, B_1)_{\theta,q}$ consists of those $u \in B_0 + B_1$ for which the norm

$$(3.1) \qquad ||u||_{\theta,q} = \begin{cases} \sup\limits_{t>0} t^{-\theta}K(t, u) & \text{if } q = \infty \\ \left\{\displaystyle\int_0^\infty (t^{-\theta}K(t, u))^q \frac{dt}{t}\right\}^{1/q} & \text{if } q < \infty \end{cases}$$

is finite, or, equivalently, those $u \in B_0 + B_1$ representable in the form (Banach space valued integral)

$$(3.2) \qquad u = \int_0^\infty u(t) \frac{dt}{t}$$

with $u(t) \in B_0 \cap B_1$ for all $t > 0$, for which the norm

$$(3.3) \qquad ||u||_{\theta,q}^* = \inf \begin{cases} \sup\limits_{t>0} t^{-\theta}J(t, u(t)) & \text{if } q = \infty \\ \left\{\displaystyle\int_0^\infty (t^{-\theta}J(t, u(t)))^q \frac{dt}{t}\right\}^{1/q} & \text{if } q < \infty \end{cases}$$

is finite, the infimum being taken over all representations of u in the form (3.2). Then $(B_0, B_1)_{\theta,q}$ is a Banach space with respect to either of the equivalent norms (3.1) and (3.3). Moreover, $(B_0, B_1)_{\theta,q_1} \subset (B_0, B_1)_{\theta,q_2}$ if $1 \leqq q_1 \leqq q_2 \leqq \infty$. (We consistently use the symbol $\subset$ to denote continuous injection, that is, imbedding.)

A Banach space B is said to belong to the class $\mathscr{J}(\theta; B_0, B_1)$ if $(B_0, B_1)_{\theta,1} \subset B$; that is, if

$$(3.4) \qquad ||u||_B \leqq C\, t^{-\theta}J(t, u) \quad \text{for all } t \text{ in } \mathbf{R}^+ \text{ and } u \text{ in } B_0 \cap B_1.$$

B belongs to the class $\mathscr{K}(\theta; B_0, B_1)$ if $B \subset (B_0, B_1)_{\theta,\infty}$; that is, if

$$(3.5) \qquad K(t, u) \leqq C\, t^\theta ||u||_B \quad \text{for all } t \text{ in } \mathbf{R}^+ \text{ and } u \text{ in } B.$$

Let $\mathscr{H}(\theta; B_0, B_1) = \mathscr{J}(\theta; B_0, B_1) \cap \mathscr{K}(\theta; B_0, B_1)$. Thus B belongs to $\mathscr{H}(\theta; B_0, B_1)$ if and only if both (3.4) and (3.5) are satisfied. $\mathscr{H}(\theta; B_0, B_1)$ can also be defined when $\theta = 0$ or 1 (see [**4**, p. 175]; all we need to know here is that $B_0 \in \mathscr{H}(0; B_0, B_1)$ and $B_1 \in \mathscr{H}(1; B_0, B_1)$.

A key result in real interpolation theory is the *reiteration theorem* (Lions and Peetre [**9**]): if $0 \leqq \theta_0 < \theta_1 \leqq 1$, and if $C_i \in \mathscr{H}(\theta_i; B_0, B_1)$, $(i = 0, 1)$, then

$$(C_0, C_1)_{\theta,q} = (B_0, B_1)_{(1-\theta)\theta_0+\theta\theta_1,q}$$

holds (with equivalence of norms) for $0 < \theta < 1$ and $1 \leqq q \leqq \infty$.

We now consider the interpolation of Sobolev spaces with respect to order of smoothness. Let Ω be a domain in $\mathbf{R}^n$ and let $1 \leqq p < \infty$, and $0 < k < m$. A variant of the Ehrling-Nirenberg-Gagliardo inequality asserts that there is a constant C such that for all u in $W^{m,p}(\Omega)$,

$$\|u\|_{k,p} \leqq C\|u\|_p^{1-(k/m)}\|u\|_{m,p}^{k/m}$$

provided Ω has at least the weak cone property. (See [**2**, Theorem 2].) We invite the reader to verify that the above inequality implies that $W^{k,p}(\Omega) \in \mathscr{J}(k/m; L^p(\Omega), W^{m,p}(\Omega))$. We now show that $W^{k,p}(\Omega)$ belongs to $\mathscr{K}(k/m; L^p(\Omega), W^{m,p}(\Omega))$ if and only if the conclusion of the approximation theorem holds. If $t \geqq 1$ and $u \in W^{k,p}(\Omega)$ then $K(t, u) \leqq \|u\|_p + t\|0\|_{m,p} = \|u\|_p$. Hence $t^{-k/m}K(t, u) \leqq \|u\|_{k,p}$ for such t and u. If $t^{-k/m}K(t, u) \leqq C\|u\|_{k,p}$ for all $t \leqq 1$ as well, then we can choose u_0 in $L^p(\Omega)$ and u_1 in $W^{m,p}(\Omega)$ with $u = u_0 + u_1$ and $\|u_0\|_p + t\|u_1\|_{m,p} \leqq 2K(t, u)$. Thus

$$\|u - u_1\|_p = \|u_0\|_p \leqq 2C\, t^{k/m}\|u\|_{k,p},\ \|u_1\|_{m,p} \leqq 2C\, t^{(k/m)-1}\|u\|_{k,p}.$$

With $t = \epsilon^m$ we see that $u_1 = u_\epsilon$ is a solution to the approximation problem. Conversely, if the approximation problem can be solved when $\epsilon = t^{1/m} \leqq 1$, by u_ϵ say, then

$$t^{-k/m}K(t, u) \leqq t^{-k/m}(\|u - u_\epsilon\|_p + t\|u_\epsilon\|_{m,p}) \leqq C\|u\|_{k,p}$$

for all u in $W^{k,p}(\Omega)$. We have thus established the following corollary of the approximation theorem.

THEOREM 3. *If Ω is a domain in $\mathbf{R}^n$ with the smooth cone property, and if $1 \leqq p < \infty$ and $0 < k < m$, then $W^{k,p}(\Omega) \in \mathscr{K}(k/m; L^p(\Omega), W^{m,p}(\Omega))$.*

Now we wish to consider briefly the family of Besov spaces, $B^{s,p,q}(\Omega)$. These spaces are usually defined intrinsically (see [**16**, p. 150] or [**11**, p. 516]), and it is then shown that, for reasonable domains Ω,

$$B^{s,p,q}(\Omega) = (L^p(\Omega), W^{m,p}(\Omega))_{s/m,q}, \tag{3.6}$$

where m is the smallest integer exceeding s. Since we do not need the intrinsic characterizations of these spaces, we define them by formula (3.6). If Ω has the smooth cone property, then Theorem 3 and the reiteration theorem show that (3.6) holds (up to equivalence of norms) for any integer $m > s$, and if $s_1 > s$, then

$$B^{s,p,q}(\Omega) = (L^p(\Omega), B^{s_1,p,q}(\Omega))_{s/s_1,q}.$$

More generally, in this case we have for any integers k and m with $0 \leqq k < s < m$,

$$B^{s,p,q}(\Omega) = (W^{k,p}(\Omega), W^{m,p}(\Omega))_{\lambda,q},$$

where $s = (1 - \lambda)k + \lambda m$, and in fact if $0 < s_1 < s < s_2$ then for any numbers

q, q_1 and q_2 in the interval $[1, \infty]$,

$$B^{s,p,q}(\Omega) = (B^{s_1,p,q_1}(\Omega), B^{s_2,p,q_2}(\Omega))_{\lambda,q}$$

where $s = (1 - \lambda)s_1 + \lambda s_2$. We shall denote by $\|\cdot\|_{B(s,p,q)}$ the norm in $B^{s,p,q}(\Omega)$. We remark that Theorem 3 implies, for integer m, that $B^{m,p,1}(\Omega) \subset W^{m,p}(\Omega) \subset B^{m,p,\infty}(\Omega)$. In fact, it is known that if Ω is sufficiently regular, then

$$B^{m,p,p}(\Omega) \subset W^{m,p}(\Omega) \subset B^{m,p,2}(\Omega) \quad \text{for } 1 < p \leqq 2, \text{ and}$$
$$B^{m,p,2}(\Omega) \subset W^{m,p}(\Omega) \subset B^{m,p,p}(\Omega) \quad \text{for } 2 \leqq p < \infty.$$

(See [**16**, p. 155]. It is also asserted in [**3**, p. 301] that the first pair of inclusions also holds for $p = 1$.) The indices in these inclusions are best possible, even when $\Omega = \mathbf{R}^n$. We do not need these inclusions in the sequel.

The following standard imbedding theorem obtains for the Besov spaces, and requires only the weak cone property. Here $CB(\Omega)$ denotes the space $L^\infty(\Omega) \cap C(\Omega)$ with the norm $\|\cdot\|_\infty$, and $L^{r,q}(\Omega)$ is, for $1 < r < \infty$ the Lorentz space consisting of those measurable functions u on Ω for which the equimeasurable decreasing rearrangement u^* of $|u|$ satisfies (see [**17**, Section 5.3]);

$$\int_0^\infty (t^{1/p}u^*(t))^q \frac{dt}{t} < \infty \quad \text{if } 1 \leqq q < \infty$$

$$\operatorname*{ess\,sup}_{t>0} t^{1/p}u^*(t) < \infty \quad \text{if } q = \infty.$$

Note that $L^{p,p}(\Omega) = L^p(\Omega)$.

THEOREM 4. *Let Ω be a domain in $\mathbf{R}^n$ having the weak cone property.*
(a) *If $sp < n$ then $B^{s,p,q}(\Omega) \subset L^{r,q}(\Omega)$ for $r = np(n - sp)^{-1}$, $1 \leqq q \leqq \infty$.*
(b) *If $sp = n$ then $B^{s,p,1}(\Omega) \subset CB(\Omega)$.*
(c) *If $sp > n$ then $B^{s,p,q}(\Omega) \subset CB(\Omega)$ for $1 \leqq q \leqq \infty$.*

Proof. We require, and so prove, only (b). Under the smooth cone condition, (a) follows from (b) and the fact that $L^{r,q}(\Omega) = (L^{p_1}(\Omega), L^{p_2}(\Omega))_{\lambda,q}$ whenever $1 \leqq p_1 < r < p_2 \leqq \infty$ and $1/r = (1 - \lambda)/p_1 + \lambda/p_2$.

Let m be the smallest integer exceeding n/p. Let u belong to $B^{n/p,p,1}(\Omega) = (L^p(\Omega), W^{m,p}(\Omega))_{n/mp,1}$. For each t in $\mathbf{R}^+$ there exists $u(t)$ in $W^{m,p}(\Omega)$ such that $u = \int_0^\infty u(t)\, dt/t$ and

$$\int_0^\infty t^{-n/mp}\, J(t, u(t)) \frac{dt}{t} \leqq C\|u\|_{B(n/p,p,1)},$$

where $J(t, u(t)) = \max \{\|u(t)\|_r, t\|u(t)\|_{m,p}\}$. Since $mp > n$ and Ω has the weak cone property there exists (see [**2**, Theorem 3]) a constant C independent of v in $W^{m,p}(\Omega)$ such that

$$\|v\|_\infty \leqq C\|v\|_p{}^{1-(n/mp)}\|v\|_{m,p}{}^{n/mp}.$$

Thus

$$\begin{aligned}||u||_\infty &\leqq \int_0^\infty ||u(t)||_\infty \frac{dt}{t}\\ &\leqq C\int_0^\infty ||u(t)||_p{}^{1-(n/mp)}\, ||u(t)||_{m,p}{}^{n/mp}\frac{dt}{t}\\ &\leqq C\int_0^\infty t^{-n/mp}\, J(t, u(t))\frac{dt}{t}\\ &\leqq 2C\,||u||_{B(n/p,p,1)}.\end{aligned}$$

Hence $B^{n/p,p,1}(\Omega) \subset L^\infty(\Omega)$. Now $C(\Omega) \cap B^{n/p,p,1}(\Omega)$ is readily seen to be dense in $B^{n/p,p,1}(\Omega)$ and a uniform convergence argument implies this latter space must imbed into $CB(\Omega)$.

Remark. It will be apparent from Theorem 7 below that if $kp < n$ and $p > 1$ then $W^{k,p}(\Omega) \subset L^{r,p}(\Omega)$ for $r = np(n - kp)^{-1}$, provided Ω has the weak cone property. This slightly improves the result $W^{k,p}(\Omega) \subset L^r(\Omega)$ (which, however, also holds for $p = 1$) given in [**2**] for such domains. It is a weakness of the technique of interpolation on order of smoothness alone, that neither of these imbeddings can be obtained directly from Theorems 3 and 4 by interpolation, even for domains with the smooth cone property. The best we can do *a priori* is $W^{k,p}(\Omega) \subset L^{r,\infty}(\Omega)$. The best Lorentz target space for imbeddings of $W^{k,1}(\Omega)$ is still in question. We postpone to the next section the problem of refining conclusion (c) of Theorem 4 to yield fractional order Lipschitz imbeddings.

We now turn our attention to the interpolation of Sobolev spaces with respect to order of summability. We require the following lemma.

LEMMA 1. *Let Ω be a space of homogeneous type. Let $1 < p < r \leqq \infty$ and $1 \leqq q \leqq \infty$. Then $(H^1(\Omega), L^r(\Omega))_{\theta,q} = L^{p,q}(\Omega)$ where $1/p = 1 - \theta + \theta/r$. In particular, $(H^1(\Omega), L^\infty(\Omega))_{1-(1/p),q} = L^{p,q}(\Omega)$.*

Proof. We prove the case where $r = \infty$; the general case follows by reiteration. Let $\theta = 1 - (1/p)$. Since $H^1(\Omega) \subset L^1(\Omega)$ we have that $(H^1(\Omega), L^\infty(\Omega))_{\theta,q} \subset L^{p,q}(\Omega)$. In particular $(H^1(\Omega), L^\infty(\Omega))_{\theta,1} \subset L^p(\Omega)$, and we will be able to conclude that $L^p(\Omega) \in \mathscr{H}(\theta; H^1(\Omega), L^\infty(\Omega))$ if we can show that $L^p(\Omega) \subset (H^1(\Omega), L^\infty(\Omega))_{\theta,\infty}$.

Arguments in the proof of Theorem D of [**7**] yield a constant C such that for any $r > 0$, any u in $L^p(\Omega)$ can be expressed as a sum $u = v_r + w_r$ where $v_r \in H^1(\Omega)$ and $w_r \in L^\infty(\Omega)$ satisfy

$$||v_r||_{H^1(\Omega)} \leqq C\, r^{1-p}||u||_p, \quad \text{and} \quad ||w_r||_\infty \leqq C\, r||u||_p.$$

For $t > 0$ let $r = t^{-1/p} = t^{\theta-1}$. Then

$$K(t, u) \leqq ||v_r||_{H^1(\Omega)} + t||w_r||_\infty \leqq 2C\, t^\theta||u||_p.$$

Thus $L^p(\Omega) \subset (H^1(\Omega), L^\infty(\Omega))_{\theta,\infty}$.

Now pick s such that $1 < s < p$ and let $\lambda = 1 - (s/p)$. Since $L^s(\Omega) \in \mathscr{H}(1 - (1/s); H^1(\Omega), L^\infty(\Omega))$ we have by reiteration that $L^{p,q}(\Omega) = (L^s(\Omega), L^\infty(\Omega))_{\lambda,q} = (H^1(\Omega), L^\infty(\Omega))_{\theta,q}$.

Remark. Using the duality between $H^1(\Omega)$ and the space $BMO(\Omega)$ of functions of bounded mean oscillation on Ω, we could obtain, under the same hypotheses, that

$$L^{p,q}(\Omega) = (L^1(\Omega), BMO(\Omega))_{\theta,q} = (H^1(\Omega), BMO(\Omega))_{\theta,q}.$$

(See Riviere and Sagher [**15**] for the case where $\Omega = \mathbf{R}^n$.)

THEOREM 5. *Let Ω be a domain in $\mathbf{R}^n$ having the smooth cone property. If $1 < p_1 < p < p_2 < \infty$, then*

$$(3.7) \qquad (W^{k,p_1}(\Omega), W^{k,p_2}(\Omega))_{\theta,p} = W^{k,p}(\Omega), \quad 1/p = (1-\theta)/p_1 + \theta/p_2.$$

If, in addition, Ω is a space of homogeneous type then also

$$(3.8) \qquad (W^{k,1}(\Omega), W^{k,p_2}(\Omega))_{\theta,p} = W^{k,p}(\Omega), \quad 1/p = 1 - \theta + \theta/p_2.$$

Proof. The operator $D : u \mapsto \{D^\alpha u\}_{|\alpha| \leqq k}$ maps $W^{k,p_i}(\Omega)$ boundedly into $\prod_{|\alpha|\leqq k} L^{p_i}(\Omega)$, $(i = 1, 2)$, and therefore it maps $(W^{k,p_1}(\Omega), W^{k,p_2}(\Omega))_{\theta,p}$ boundedly into $(\prod_{|\alpha|\leqq k} L^{p_1}(\Omega), \prod_{|\alpha|\leqq k} L^{p_2}(\Omega))_{\theta,p} = \prod_{|\alpha|\leqq k} L^p(\Omega)$ for $1/p = (1-\theta)/p_1 + \theta/p_2$. Thus $(W^{k,p_1}(\Omega), W^{k,p_2}(\Omega))_{\theta,p} \subset W^{k,p}(\Omega)$. Conversely, the lifting operator R, (see Theorem 2), maps $\prod_{|\alpha|\leqq k} L^{p_i}(\Omega)$ boundedly into $W^{k,p_i}(\Omega)$, so

$$(3.9) \qquad \begin{aligned} W^{k,p}(\Omega) &= RD(W^{k,p}(\Omega)) = R(\textstyle\prod_{|\alpha|\leqq k} L^p(\Omega)) \\ &\subset R((\textstyle\prod_{|\alpha|\leqq k} L^{p_1}(\Omega), \prod_{|\alpha|\leqq k} L^{p_2}(\Omega))_{\theta,p}) \\ &\subset (W^{k,p_1}(\Omega), W^{k,p_2}(\Omega))_{\theta,p}. \end{aligned}$$

This completes the proof of the first assertion, (3.7).

The proof of (3.8) is identical to that of (3.7), except that in (3.9) above $\prod_{|\alpha|\leqq k} L^{p_1}(\Omega)$ is replaced by $(\prod_{|\alpha|<k} L^1(\Omega)) \times (\prod_{|\alpha|=k} H^1(\Omega))$. The first inclusion in (3.9) then requires Lemma 1 and the second requires the second assertion of the lifting theorem. The homogeneity of Ω is required for both of these steps.

Remark. The analogue of Theorem 5 involving the complex interpolation method (see Calderón [**5**]) also holds, with the same proof. We do not need this fact in the sequel.

A useful generalization of Theorem 5 obtains for Sobolev spaces modelled on Lorentz instead of Lebesgue spaces. We term these latter *Sobolev-Lorentz spaces* and denote them by $W^{k,p,q}(\Omega)$:

$$W^{k,p,q}(\Omega) = \{u \in L^{p,q}(\Omega) : D^\alpha u \in L^{p,q}(\Omega) \text{ for } |\alpha| \leqq k\}.$$

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These are Banach spaces with respect to an appropriate norm, say

$$||u||_{W(k,p,q)} = \begin{cases} \left\{ \sum_{|\alpha| \leqq k} ||D^\alpha u||^q_{L^{p,q}(\Omega)} \right\}^{1/q} & \text{if } 1 \leqq q < \infty \\ \max_{|\alpha| \leqq k} ||D^\alpha u||_{L^{p,\infty}(\Omega)} & \text{if } q = \infty. \end{cases}$$

Evidently $W^{k,p,p}(\Omega) = W^{k,p}(\Omega)$.

THEOREM 6. *Let Ω be a domain in $\mathbf{R}^n$ having the smooth cone property. If $1 < p_1 < p < p_2 < \infty$, $1 \leqq q_1 \leqq \infty$, $1 \leqq q_2 \leqq \infty$, $1 \leqq q \leqq \infty$ and $1/p = (1 - \theta)/p_1 + \theta/p_2$ then*

$$(W^{k,p_1,q_1}(\Omega), W^{k,p_2,q_2}(\Omega))_{\theta,q} = W^{k,p,q}(\Omega).$$

In particular

$$(W^{k,p_1,q_1}(\Omega), W^{k,p_2,q_2}(\Omega))_{\theta,p} = W^{k,p}(\Omega),$$

and

$$(W^{k,p_1}(\Omega), W^{k,p_2}(\Omega))_{\theta,q} = W^{k,p,q}(\Omega).$$

Proof. The operator D maps $W^{k,p_i,q_i}(\Omega)$ boundedly into $\prod_{|\alpha| \leqq k} L^{p_i,q_i}(\Omega)$, $(i = 1, 2)$, and by interpolation $(W^{k,p_1,q_1}(\Omega), W^{k,p_2,q_2}(\Omega))_{\theta,q}$ into $\prod_{|\alpha| \leqq k} L^{p,q}(\Omega)$. Thus $(W^{k,p_1,q_1}(\Omega), W^{k,p_2,q_2}(\Omega))_{\theta,q} \subset W^{k,p,q}(\Omega)$. Now DR is bounded from $\prod_{|\alpha| \leqq k} L^{p_i}(\Omega)$ to $\prod_{|\alpha| \leqq k} L^{p_i}(\Omega)$, $(i = 1, 2)$, and hence from $\prod_{|\alpha| \leqq k} L^{p,q}(\Omega)$ to $\prod_{|\alpha| \leqq k} L^{p,q}(\Omega)$. Therefore R is bounded from $\prod_{|\alpha| \leqq k} L^{p,q}(\Omega)$ to $W^{k,p,q}(\Omega)$ and RD is the identity map on $W^{k,p,q}(\Omega)$. Thus $W^{k,p,q}(\Omega) \subset R(\prod_{|\alpha| \leqq k} L^{p,q}(\Omega))$. Since R maps $\prod_{|\alpha| \leqq k} L^{p_i,q_i}(\Omega)$ boundedly into $W^{k,p_i,q_i}(\Omega)$, $(i = 1, 2)$, it is also bounded from $\prod_{|\alpha| \leqq k} L^{p,q}(\Omega)$ into $(W^{k,p_1,q_1}(\Omega), W^{k,p_2,q_2}(\Omega))_{\theta,q}$. Thus $W^{k,p,q}(\Omega) \subset (W^{k,p_1,q_1}(\Omega), W^{k,p_2,q_2}(\Omega))_{\theta,q}$ as required.

Analagous to the results obtained for Besov spaces in Theorem 4 above, we have the following imbedding theorem for Sobolev-Lorentz spaces. Observe that, in this theorem also, we assume only that Ω has the weak cone property.

THEOREM 7. *Let Ω be a domain in $\mathbf{R}^n$ having the weak cone property.*

(a) *If $p > 1$ and $kp < n$ then $W^{k,p,q}(\Omega) \subset L^{r,q}(\Omega)$ for $r = np(n - mp)^{-1}$ and $1 \leqq q \leqq \infty$.*

(b) *If $kp = n$ then $W^{k,p,1}(\Omega) \subset CB(\Omega)$.*

(c) *If $kp > n$ then $W^{k,p,q}(\Omega) \subset CB(\Omega)$ for $1 \leqq q \leqq \infty$.*

Proof. We prove only (b); the other parts can be established by suitably generalizing arguments given in [**2**]. (In any event, they follow directly from (b) by interpolation if Ω has the smooth cone property.)

The result (b) is known for $p = 1$. (See [**2**, Theorem 1].) For $p > 1$ we have, (by [**2**, Lemma 2]), that for all u in $C^\infty(\Omega)$ and x in Ω,

$$(3.10) \quad |u(x)| \leqq C\left(\sum_{|\alpha| \leqq k-1} 1_B * |D^\alpha u|(x) + \sum_{|\alpha| = k} \omega_k * |D^\alpha u|(x) \right)$$

where 1_B is the characteristic function of the unit ball in $\mathbf{R}^n$, and $\omega_k(x)$ is the Riesz potential $|x|^{k-n}$. In (3.10) all $D^\alpha u$ are considered to be extended to $\mathbf{R}^n$, vanishing identically outside Ω. It suffices, therefore, to show that for any v in $L^{p,1}(\mathbf{R}^n)$ and x in $\mathbf{R}^n$,

$$(3.11) \quad \int_{\mathbf{R}^n} |v(y)||x-y|^{k-n}dy \leqq C \int_0^\infty t^{k/n}v^*(t)\frac{dt}{t},$$

where v^* is the (scalar) equimeasurable decreasing rearrangement of $|v|$. In turn, it is sufficient to verify (3.11) for $x = 0$ and v radially symmetric and decreasing. In this case, the radius r of the ball on which v exceeds λ and the length t of the interval on which v^* exceeds λ are related by $Cr^n = t$. Thus

$$\int_{\mathbf{R}^n} |v(y)||y|^{k-n}dy = C\int_0^\infty v(r)\, r^{k-1}dt = C\int_0^\infty t^{k/n}v^*(t)\frac{dt}{t}$$

and (3.11) follows.

4. Intrinsic Lipschitz spaces. Let $0 < \lambda \leqq 1$. Denote by $\mathrm{Lip}_\lambda(\Omega)$ the space of all functions u in $CB(\Omega)$ with finite norm

$$(4.1) \qquad ||u||_{\mathrm{Lip}_\lambda} = ||u||_\infty + \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\lambda}.$$

$\mathrm{Lip}_\lambda(\Omega)$ is a Banach space under the norm (4.1). It is well known (see, for example, [**1**, p. 98]) that if Ω is sufficiently regular, and if $kp > n$, then $W^{k,p}(\Omega)$ imbeds into $\mathrm{Lip}_\lambda(\Omega)$ for certain values of λ. If Ω lies on both sides of some $(n-1)$-dimensional part of its boundary, however, then no such imbedding can occur, because there will exist elements of $W^{k,p}(\Omega)$ that are not essentially uniformly continuous, while every element of $\mathrm{Lip}_\lambda(\Omega)$ must be uniformly continuous. To avoid this problem we replace the Euclidean distance $|x-y|$ by the intrinsic distance $\rho(x, y)$.

Let $\omega(t, u)$ be the modulus of continuity of $u \in C(\Omega)$, taken with respect to the intrinsic metric on Ω:

$$\omega(t, u) = \sup_{\substack{x,y\in\Omega\\ \rho(x,y)\leqq t}} |u(x) - u(y)|.$$

For $0 < \lambda \leqq 1$ and $1 \leqq q \leqq \infty$ the intrinsic Lipschitz space $\mathrm{ILip}_{\lambda,q}(\Omega)$ consists of those functions u in $CB(\Omega)$ for which $t^{-\lambda}\omega(t, u)$ belongs to $L^q(0, \infty)$ with respect to the measure dt/t. The norm in $\mathrm{ILip}_{\lambda,q}(\Omega)$ for $1 \leqq q < \infty$ is

$$||u||_{(\lambda,q)} = ||u||_\infty + \left\{\int_0^\infty (t^{-\lambda}\omega(t, u))^q\frac{dt}{t}\right\}^{1/q}.$$

We also denote by $\mathrm{ILip}_\lambda(\Omega)$ the space $\mathrm{ILip}_{\lambda,\infty}(\Omega)$ with norm

$$||u||_{(\lambda)} = ||u||_\infty + \sup_{t>0} t^{-\lambda}\omega(t, u).$$

In order to identify $\mathrm{ILip}_{\lambda,q}(\Omega)$ as an interpolation space, we require the following approximation result analagous to Theorem 1. Let $CB^k(\Omega)$ denote the space $W^{k,\infty}(\Omega) \cap C^k(\Omega)$ with norm $\|\cdot\|_{k,\infty}$.

LEMMA 2. *Let Ω be a domain in $\mathbf{R}^n$ having the smooth cone property. There exist constants C and C' such that for each u in $CB(\Omega)$ and each $\epsilon > 0$ there exists u_ϵ in $CB^1(\Omega)$ satisfying*

$$\|u - u_\epsilon\|_\infty \leqq \omega(C\epsilon, u)$$

$$\|u_\epsilon\|_{1,\infty} \leqq C'\left(\frac{\omega(C\epsilon, u)}{C\epsilon} + \|u\|_\infty\right).$$

Proof. Let $Q \in C_0^\infty(B_1(0))$ be such that $Q(y) \geqq 0$ and $\int_{B_1(0)} Q(y)\, dy = 1$. Let Φ be the vector field determining the smooth cone property for Ω. For u in $CB(\Omega)$ let

$$u_\epsilon(x) = \int_{B_1(0)} Q(y)\, u(x + \epsilon(\Phi(x) + y))\, dy$$

$$= \int_\Omega u(z)\, Q\left(\frac{z - x}{\epsilon} - \Phi(x)\right) \epsilon^{-n}\, dz.$$

Clearly $\|u_\epsilon\|_\infty \leqq \|u\|_\infty$ and

$$\|u - u_\epsilon\|_\infty \leqq \sup_x \int_{B_1(0)} Q(y)|u(x) - u(x + \epsilon(\Phi(x) + y))|dy$$

$$\leqq \omega((1 + \|\Phi\|)\epsilon, u).$$

Finally,

$$\left|\frac{\partial}{\partial x_j} u_\epsilon(x)\right| \leqq C'\|\nabla\Phi\|_\infty \|u\|_\infty$$

$$+ \frac{1}{\epsilon}\left|\int_\Omega (u(z) - u(x))(D_jQ)\left(\frac{z - x}{\epsilon} - \Phi(x)\right)\epsilon^{-n} dz\right|$$

$$\leqq C\left(\|u\|_\infty + \frac{\omega((1 + \|\Phi\|)\epsilon, u)}{\epsilon}\right)$$

which completes the proof.

THEOREM 8. *Let Ω be a domain in $\mathbf{R}^n$ having the smooth cone property. If $0 < \theta < 1$ and $1 \leqq q \leqq \infty$ then*

$$(4.2) \qquad (CB(\Omega), CB^1(\Omega))_{\theta,q} = \mathrm{ILip}_{\theta,q}(\Omega).$$

Proof. Let u belong to $CB(\Omega)$, and let $t > 0$. We may choose $u_0 \in CB(\Omega)$ and $u_1 \in CB^1(\Omega)$ such that $u = u_0 + u_1$ and $\|u_0\|_\infty + t\|u_1\|_{1,\infty} \leqq 2K(t, u)$. Let x and y belong to Ω and satisfy $\rho(x, y) \leqq t$. Let ϕ be a piecewise smooth arc

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in Ω with $\phi(0) = x$, $\phi(1) = y$, and $\int_0^1 |\phi'(\tau)| d\tau \leqq 2t$. Then

$$|u_1(y) - u_1(x)| \leqq \int_0^1 \left| \frac{d}{d\tau} u_1(\phi(\tau)) \right| d\tau$$
$$\leqq \int_0^1 |\nabla u_1(\phi(\tau))| |\phi'(\tau)| d\tau \leqq Ct ||u_1||_{1,\infty}.$$

Hence

$$|u(y) - u(x)| \leqq |u_0(y)| + |u_0(x)| + |u_1(y) - u_1(x)|$$
$$\leqq 2||u_0||_\infty + Ct||u_1||_{1,\infty} \leqq C\,K(t, u),$$

that is,

(4.3) $\quad \omega(t, u) \leqq C\,K(t, u).$

On the other hand, $K(t, u) \leqq ||u||_\infty + t||0||_{1,\infty} = ||u||_\infty$. Suppose now that $0 < t < 1$, and let $\epsilon = t/C$. Write $u = (u - u_\epsilon) + u_\epsilon$, where u_ϵ is the function given by Lemma 2. Then

$$K(t, u) \leqq ||u - u_\epsilon||_\infty + t||u_\epsilon||_{1,\infty} \leqq C'(\omega(t, u) + t||u||_\infty).$$

Thus, for all $t > 0$ we have that

(4.4) $\quad K(t, u) \leqq C'\,(\omega(t, u) + \min\{1, t\}||u||_\infty).$

It follows immediately from inequalities (4.3) and (4.4) that $t^{-\theta}K(t, u)$ belongs to the space $L^q(0, \infty)$ with respect to the measure dt/t if and only if $t^{-\theta}\omega(t, u)$ belongs to the same space. Hence (4.2) holds.

We now come to our main theorem concerning imbeddings of Sobolev spaces into intrinsic Lipschitz spaces. In the case where $\Omega = \mathbf{R}^n$ this result is due to Morrey [**10**].

THEOREM 9. *Let Ω be a domain in $\mathbf{R}^n$ having the smooth cone property. Let $1 < p < \infty$, let $(k - 1)p < n < kp$, and let $\lambda = k - (n/p)$. Then*

(4.5) $\quad W^{k,p}(\Omega) \subset \mathrm{ILip}_{\lambda,p}(\Omega).$

Proof. We first show that, under these hypotheses, $W^{k,p}(\Omega) \subset \mathrm{ILip}_\lambda(\Omega)$. To do this we interpolate with respect to smoothness. Let $s = n/p$; then $B^{s,p,1}(\Omega) \subset CB(\Omega)$ by Theorem 4. Now every element of $B^{s+1,p,1}(\Omega)$ has the property that its first-order partial derivatives belong to $B^{s,p,1}(\Omega)$; therefore $B^{s+1,p,1}(\Omega) \subset CB^1(\Omega)$. Next observe that $k = (1 - \lambda)s + \lambda(s + 1)$. Therefore

$$W^{k,p}(\Omega) \subset B^{k,p,\infty}(\Omega) \subset (B^{s,p,1}(\Omega), B^{s+1,p,1}(\Omega))_{\lambda,\infty}$$
$$\subset (CB(\Omega), CB^1(\Omega))_{\lambda,\infty} = \mathrm{ILip}_{\lambda,\infty}(\Omega) = \mathrm{ILip}_\lambda(\Omega).$$

To prove (4.5) we now interpolate with respect to summability. Choose indices p_1 and p_2 with $p_1 < p < p_2$ such that the hypotheses of the theorem still hold if p is replaced by p_1 or p_2. Let $\lambda_i = k - (n/p_i)$ for $i = 1$ or 2. Choose θ so that $1/p = (1 - \theta)/p_1 + \theta/p_2$. Then, by Theorem 5, $W^{k,p}(\Omega) \subset$

$(\mathrm{ILip}_{\lambda_1}(\Omega), \mathrm{ILip}_{\lambda_2}(\Omega))_{\theta,p}$. By Theorem 8 and the reiteration theorem, this latter space is just $\mathrm{ILip}_{\lambda,p}(\Omega)$.

We remark that, under the hypotheses of the theorem, the inclusion $W^{k,p}(\Omega) \subset \mathrm{ILip}_{\mu,r}(\Omega)$ holds if and only if $\mu < \lambda$, or $\mu = \lambda$ and $r \geqq p$. It is clear that such inclusions hold for these values of μ and r, because $\mathrm{ILip}_{\lambda,p}(\Omega) \subset \mathrm{ILip}_{\mu,r}(\Omega)$. To see that the imbeddings fail if $\mu > \lambda$, or if $\mu = \lambda$ and $r < p$, consider functions of the form $x \mapsto |x - x_0|^s(\log |x - x_0|)^t$ for fixed x_0 in Ω. It is also easy to see that the hypotheses of Theorem 9 imply that $W^{k+m,p}(\Omega)$ imbeds into the space of all functions in $CB^m(\Omega)$ all of whose derivatives of order m belong to $\mathrm{ILip}_{\lambda,p}(\Omega)$.

Now suppose that Ω has the smooth cone property, that $1 < p < \infty$, and that $(k - 1)p = n$. Then $W^{k,p}(\Omega) \subset \mathrm{ILip}_{\mu,r}(\Omega)$ for all r and all $\mu < 1$; indeed $B^{k,p,\infty}(\Omega) \subset \mathrm{ILip}_{\mu,r}(\Omega)$ for such μ and r. To get a sharp imbedding in this case, we have to consider second differences. Call a pair of points x and z in $\mathbf{R}^n$ *admissible* if the segment joining x to $x + 2z$ lies entirely in Ω. Given a function u on Ω, and a number $t > 0$, let $\omega^*(t, u)$ be the supremum, over all admissible pairs x and z, with $|z| \leqq t$, of $|u(x) - 2u(x + z) + u(x + 2z)|$. Then for $0 < \lambda < 2$, and $1 \leqq q \leqq \infty$, let $\mathrm{ILip}_{\lambda,q}{}^*(\Omega)$ be the space of functions u on Ω for which $t^{-\lambda}\omega^*(t, u)$ belongs to $L^q(0, \infty)$ with respect to the measure dt/t. Then, under the above hypotheses, $W^{k,p}(\Omega) \subset \mathrm{ILip}_{1,p}{}^*(\Omega)$; we omit the proof of this fact.

The situation is much simpler when $p = 1$. If Ω merely has the weak cone property, then $W^{n+m,1}(\Omega) \subset CB^m(\Omega)$, for all nonnegative integers m. (See [**2**, Theorem 1].)

The hypotheses of Theorem 9 also imply that for all q

$$W^{k,p,q}(\Omega) \subset \mathrm{ILip}_{\lambda,q}(\Omega).$$

Finally, if Ω has the smooth cone property, if $1 \leqq p < \infty$, if $(s - 1)p < n < sp$, and if $\lambda = s - (n/p)$, then

$$B^{s,p,q}(\Omega) \subset \mathrm{ILip}_{\lambda,q}(\Omega).$$

We omit the proofs of these imbeddings.

The weakest geometric property of a domain Ω that is known to imply that $W^{k,p}(\Omega) \subset \mathrm{Lip}_\lambda(\Omega)$, for suitable indices k, p, and λ, is the strong local Lipschitz property [**1**, p. 66]. The significance of this property will be discussed in Section 7. We mention here, however, that if Ω has this property, then the various intrinsic Lipschitz spaces $\mathrm{ILip}_{\lambda,q}(\Omega)$ coincide (up to equivalence of norms) with their counterparts $\mathrm{Lip}_{\lambda,q}(\Omega)$ that arise when the Euclidean distance is used instead of the intrinsic distance.

5. Proof of the approximation theorem (Theorem 1). Let Φ be a smooth vector field determining the smooth cone property for Ω. Let Q be a nonnegative, infinitely differentiable function on $\mathbf{R}^n$ having support in the unit ball

$B_1(0)$ and satisfying $\int_{B_1(0)} Q(y)\,dy = 1$. Given u in $C^\infty(\Omega)$ and ϵ satisfying $0 < \epsilon \leqq 1$, we may, noting that for x in Ω the segment from x to $x + \epsilon(\Phi(x) + y)$ belongs to Ω for any y in $B_1(0)$, write Taylor's formula for $u(x)$ in the form

$$\begin{aligned} u(x) = &\sum_{|\alpha|\leqq k-1} \frac{(-1)^{|\alpha|}}{\alpha!} \epsilon^{|\alpha|} \\ &\times \int_{B_1(0)} Q(y) D^\alpha u(x + \epsilon(\Phi(x) + y))(\Phi(x) + y)^\alpha dy \\ &+ (-1)^k \epsilon^k \sum_{|\alpha|=k} \frac{k}{\alpha!} \int_{B_1(0)} Q(y)(\Phi(x) + y)^\alpha dy \\ &\qquad \times \int_0^1 D^\alpha u(x + t\epsilon(\Phi(x) + y)) t^{k-1} dt. \end{aligned}$$

We let $u_\epsilon(x)$ be the first sum above and so obtain

$$\begin{aligned} u(x) &- u_\epsilon(x) \\ &= (-1)^k \epsilon^k \sum_{|\alpha|=k} \frac{k}{\alpha!} \int_0^1 t^{k-1} dt \\ &\qquad \times \int_{B_1(0)} Q(y)(\Phi(x) + y)^\alpha D^\alpha u(x + t\epsilon(\Phi(x) + y)) dy \\ &= (-1)^k \sum_{|\alpha|=k} \frac{k}{\alpha!} \int_0^1 \frac{dt}{t} \int_\Omega Q\left(\frac{z - x}{t\epsilon} - \Phi(x)\right) \\ &\qquad \times (z - x)^\alpha D^\alpha u(z)(t\epsilon)^{-n} dz \\ &= \sum_{|\alpha|=k} \int_0^\epsilon \frac{dt}{t} \int_\Omega K_\alpha(t; x, z) D^\alpha u(z) dz \end{aligned}$$

where

$$K_\alpha(t; x, z) = \frac{(-1)^k k}{\alpha!} t^{-n} Q\left(\frac{z - x}{t} - \Phi(x)\right)(z - x)^\alpha, \quad |\alpha| = k. \tag{5.1}$$

Since $Q((z - x)/t - \Phi(x))$ vanishes for $|z - x| \geqq t|\Phi(x)|$ we have

$$\begin{cases} \sup_x \int_0^\epsilon \frac{dt}{t} \int_\Omega |K_\alpha(t; x, z)| dz \leqq C\epsilon^k \\ \sup_z \int_0^\epsilon \frac{dt}{t} \int_\Omega |K_\alpha(t; x, z)| dx \leqq C\epsilon^k \end{cases} \tag{5.2}$$

It follows from these two estimates that

$$\|u - u_\epsilon\|_p \leqq C\,\epsilon^k |u|_{k,p}, \quad 1 \leqq p \leqq \infty.$$

In order to estimate its derivatives, it is useful to write u_ϵ in the form

$$u_\epsilon(x) = \epsilon^{-n} \int_\Omega Q\left(\frac{z - x}{\epsilon} - \Phi(x)\right) P_{k-1}(u; x, z) dz$$

where $P_j(u; x, z)$ is the Taylor polynomial of degree j of $u(x)$ in powers of $x - z$:

$$P_j(u; x, z) = \sum_{i=0}^{j} T_i(u; x, z)$$

$$T_j(u; x, z) = \sum_{|\alpha|=j} \frac{1}{\alpha!} D^\alpha u(z)(x-z)^\alpha.$$

Straightforward calculation shows that

$$\frac{\partial}{\partial x_i} T_j(u; x, z) = \begin{cases} T_{j-1}(D_i u; x, z) & \text{if } j > 0 \\ 0 \quad \text{if } j = 0 \end{cases}$$

$$\frac{\partial}{\partial x_i} P_j(u; x, z) = \begin{cases} P_{j-1}(D_i u; x, z) & \text{if } j > 0 \\ 0 \quad \text{if } j = 0 \end{cases}$$

$$\frac{\partial}{\partial z_i} P_j(u; x, z) = T_j(D_i u; x, z) \quad \text{for } j \geqq 0.$$

Since

$$\frac{\partial}{\partial x_i} Q\left(\frac{z-x}{\epsilon} - \Phi(x)\right) = -\frac{\partial}{\partial z_i} Q\left(\frac{z-x}{\epsilon} - \Phi(x)\right) - \epsilon \sum_{r=1}^{n} D_i\Phi_r(x) \frac{\partial}{\partial z_r} Q\left(\frac{z-x}{\epsilon} - \Phi(x)\right)$$

we may readily compute

$$D_i u_\epsilon(x) = \epsilon^{-n} \int_\Omega Q\left(\frac{z-x}{\epsilon} - \Phi(x)\right) P_{k-2}(D_i u; x, z)dz$$
$$+ \epsilon^{-n} \int_\Omega Q\left(\frac{z-x}{\epsilon} - \Phi(x)\right) T_{k-1}(D_i u; x, z)dz$$
$$+ \epsilon^{1-n} \sum_{r=1}^{n} D_i\Phi_r(x) \int_\Omega Q\left(\frac{z-x}{\epsilon} - \Phi(x)\right) T_{k-1}(D_r u; x, z)dz.$$

More generally, it can be verified by induction that $D^\alpha u_\epsilon(x)$ can be written as a sum of finitely many terms of one or both of the following types:

$$(5.3) \quad \epsilon^{-n} \int_\Omega Q\left(\frac{z-x}{\epsilon} - \Phi(x)\right) P_{k-1-|\alpha|}(D^\alpha u; x, z)dz = \sum_{|\delta| \leqq k-1-|\alpha|} \int_\Omega U_{\alpha\delta}(\epsilon; x, z) D^{\alpha+\delta}u(z)dz$$

$$(5.4) \quad \epsilon^{-n-|\alpha|+|\gamma|} \pi_{\alpha\beta\gamma}(\epsilon; \Phi; x) \int_\Omega (D^\beta Q)\left(\frac{z-x}{\epsilon} - \Phi(x)\right) T_{k-|\gamma|}(D^\gamma u; x, z)dz = \sum_{|\delta|=k-|\gamma|} \int_\Omega V_{\alpha\beta\gamma\delta}(\epsilon; x, z)\, D^{\delta+\gamma}u(z)dz$$

where $|\beta| \leqq |\gamma| \leqq \min\{k, |\alpha|\}$, $\pi_{\alpha\beta\gamma}(\epsilon; \Phi; x)$ is a polynomial of degree at most $|\alpha|$ in the variables $\epsilon^{|\lambda|}D^\lambda\Phi(x)$ for $|\lambda| \leqq |\alpha|$, and

$$U_{\alpha\delta}(\epsilon; x, z) = \frac{\epsilon^{-n}}{\delta!} Q\left(\frac{z-x}{\epsilon} - \Phi(x)\right) (x-z)^\delta,$$

$$V_{\alpha\beta\gamma\delta}(\epsilon; x, z) = \frac{\epsilon^{-n-|\alpha|+|\gamma|}}{\delta!} \pi_{\alpha\beta\gamma}(\epsilon; \Phi; x)\, (D^\beta Q)\left(\frac{z-x}{\epsilon} - \Phi(x)\right) (x-z)^\delta.$$

If $|\alpha| \geqq k$ no terms of type (5.3) are present in the expression for $D^\alpha u_\epsilon(x)$.

Since $Q((z-x)/\epsilon - \Phi(x)) = 0$ if $|z-x| \geqq \epsilon|\Phi(x)|$ we have the bounds

$$\sup_x \int_\Omega |U_{\alpha\delta}(\epsilon; x, z)|dz \leqq C\epsilon^{|\delta|}$$

$$\sup_z \int_\Omega |U_{\alpha\delta}(\epsilon; x, z)|dx \leqq C\epsilon^{|\delta|},$$

and, since $|\delta| = k - |\gamma|$ in (5.4),

$$\sup_x \int_\Omega |V_{\alpha\beta\gamma\delta}(\epsilon; x, z)|dz \leqq C\epsilon^{k-|\alpha|}\, ||\pi_{\alpha\beta\delta}(\epsilon; \Phi; .)||_\infty$$

$$\sup_z \int_\Omega |V_{\alpha\beta\gamma\delta}(\epsilon; x, z)|dx \leqq C\epsilon^{k-|\alpha|}\, ||\pi_{\alpha\beta\gamma}(\epsilon; \Phi; .)||_\infty.$$

Hence, for $1 \leqq p \leqq \infty$, all terms of type (5.3) in $D^\alpha u_\epsilon(x)$ are bounded in $L^p(\Omega)$ by $C||u||_{k-1,p}$, and all terms of type (5.4) are bounded in $L^p(\Omega)$ by $C\epsilon^{k-|\alpha|}\,|u|_{k,p}$ (where this latter constant C involves a sum of constants $||\pi_{\alpha\beta\gamma}(\epsilon; \Phi; \cdot)||_\infty$ for $|\beta| \leqq |\gamma| \leqq \min\{k, |\alpha|\}$), that is

$$|u_\epsilon|_{j,p} \leqq \begin{cases} C\,||u||_{k,p} & \text{if } 0 \leqq j \leqq k-1 \\ C\epsilon^{k-j}\,|u|_{k,p} & \text{if } k \leqq j. \end{cases}$$

This conclusion follows for $j \leqq m$ and any u in $W^{m,p}(\Omega)$, $(1 \leqq p < \infty)$, since $C^\infty(\Omega)$ is dense in $W^{m,p}(\Omega)$.

6. Proof of the lifting theorem (Theorem 2). We make use of the mollifier Q and the notations introduced at the beginning of Section 5. Given $v = \{v_\alpha\}_{|\alpha|\leqq k}$ in $\prod_{|\alpha|\leqq k} L_{\text{loc}}{}^1(\Omega)$ let Rv be defined by

$$Rv(x) = \sum_{|\alpha|\leqq k-1} \frac{(-1)^{|\alpha|}}{\alpha!} \int_{B_1(0)} Q(y) v_\alpha(x + \Phi(x) + y)(\Phi(x) + y)^\alpha dy$$
$$+ (-1)^k \sum_{|\alpha|=k} \frac{k}{\alpha!} \int_{B_1(0)} Q(y)(\Phi(x) + y)^\alpha dy$$
$$\times \int_0^1 t^{k-1} v_\alpha(x + t(\Phi(x) + y))dt.$$

It can be verified that $Rv \in L_{\text{loc}}{}^1(\Omega)$, but we do not need this fact. By Taylor's theorem, $Rv = u$ if $v = \{D^\alpha u\}_{|\alpha|\leqq k}$ and $u \in C^\infty(\Omega)$. We rewrite $Rv(x)$ in the form

$$Rv(x) = \sum_{|\alpha|\leqq k-1} \int_\Omega K_\alpha(x, z)v_\alpha(z)dz + \sum_{|\alpha|=k} \int_0^1 \frac{dt}{t} \int_\Omega K_\alpha(t; x, z)v_\alpha(z)dz$$

where

$$K_\alpha(x, z) = \frac{(-1)^{|\alpha|}}{\alpha!} Q(z - x - \Phi(x))(z - x)^\alpha, \quad |\alpha| < k$$

and $K_\alpha(t; x, z)$ is given by (5.1) above if $|\alpha| = k$. In view of the estimates

$$\sup_x \int_\Omega |K_\alpha(x, z)|dz < \infty$$

$$\sup_z \int_\Omega |K_\alpha(x, z)|dx < \infty$$

and the corresponding estimates (5.2) for $K_\alpha(t; x, z)$, we have that

$$||Rv||_p \leqq C \sum_{|\alpha|\leqq k} ||v_\alpha||_p, \quad 1 \leqq p \leqq \infty.$$

We require similar estimates for $||D^\beta Rv||_p$ for $|\beta| \leqq k$ and $1 < p < \infty$.

Suppose, for the moment, that each v_α belongs to $C^\infty(\Omega)$. Computation of $D^\beta Rv(x)$ yields a shower of terms, the "worst" of which are of the following type (for $|\beta| = k$):

$$(6.1) \quad \frac{k}{\alpha!} \int_{B_1(0)} Q(y)(\Phi(x) + y)^\alpha dy \int_0^1 (D^\beta v_\alpha)(x + t(\Phi(x) + y))t^{k-1}dt$$

$$= \frac{k}{\alpha!} \int_0^1 \frac{dt}{t} \int_{B_1(0)} Q(y)(\Phi(x) + y)^\alpha D_y^\beta v_\alpha(x + t(\Phi(x) + y))dy$$

$$(6.2) \quad = \int_0^1 \frac{dt}{t} \int_\Omega K_{\alpha\beta}(t; x, z)v_\alpha(z)dz$$

where, for $|\alpha| = |\beta| = k$,

$$K_{\alpha\beta}(t; x, z) = \frac{(-1)^k k}{\alpha!} t^{-n} D_y^\beta(Q(y)(\Phi(x) + y)^\alpha), \quad z = x + t(\Phi(x) + y).$$

Since $F(y, x) \equiv D_y^\beta(Q(y)(\Phi(x) + y)^\alpha)$ and $\nabla_y F(y, x)$ are uniformly bounded, the kernels $K_{\alpha\beta}$ have the following properties:

$$(6.3) \quad K_{\alpha\beta}(t; x, z) = 0 \quad \text{if } |x - z| \geqq t(1 + ||\Phi||),$$

$$(6.4) \quad \int_\Omega K_{\alpha\beta}(t; x, z)dz = 0 \quad \text{for all } x \text{ and } t,$$

$$(6.5) \quad |K_{\alpha\beta}(t; x, z)| \leqq C\,t^{-n} \quad \text{for all } x, z, t,$$

$$(6.6) \quad |\nabla_z K_{\alpha\beta}(t; x, z)| \leqq C\,t^{-n-1} \quad \text{for all } x, z, t.$$

Since v_α is smooth the integral (6.1) is absolutely convergent; so the change of order of integration is justified. The integration by parts is also justified for each nonzero t, so that (6.2) should be interpreted in the principal value sense:

$$(6.7) \qquad \int_0^1 \frac{dt}{t} \int_\Omega K_{\alpha\beta}(t; x, z) v_\alpha(z) dz = \lim_{N\to\infty} S_N v_\alpha(x) = S v_\alpha(x),$$

where

$$S_N v(x) = \int_{2^{-N}}^1 \frac{dt}{t} \int_\Omega K_{\alpha\beta}(t; x, z) v(z) dz.$$

We shall show that S is a bounded operator on $L^p(\Omega)$ for $1 < p < \infty$.

For $j = 1, 2, \ldots$ let

$$T_j v(x) = \int_{2^{-j}}^{2^{-j+1}} \frac{dt}{t} \int_\Omega K_{\alpha\beta}(t; x, z) v(z) dz.$$

By (6.3) and (6.5), T_j is bounded on $L^p(\Omega)$ for $1 \leqq p \leqq \infty$, with bound independent of j. Now the adjoint of T_j is given by

$$T_j^* v(x) = \int_{2^{-j}}^{2^{-j+1}} \frac{dt}{t} \int_\Omega K_{\alpha\beta}(t; z, x) v(z) dz.$$

Denote the norm of any operator T on $L^2(\Omega)$ by $||T||$. We shall use a lemma of Cotlar (see Fefferman [**8**, pp. 102-103]) to derive estimates for $||S_N||$ from estimates on $||T_i T_j^*||$ and $||T_i^* T_j||$. Now $T_i T_j^* v(x) = \int_\Omega H_{ij}(x, y)\, v(y)\, dy$, where

$$H_{ij}(x, y) = \int_{2^{-i}}^{2^{-i+1}} \frac{dt}{t} \int_{2^{-j}}^{2^{-j+1}} G(t, s; x, y) \frac{ds}{s}$$

$$G(t, s; x, y) = \int_\Omega K_{\alpha\beta}(t; x, z) K_{\alpha\beta}(s; y, z) dz.$$

By (6.3) the line segment from x to z lies in Ω provided $K_{\alpha\beta}(t; x, z) \neq 0$. Hence using (6.3) – (6.6) we obtain, for $t \leqq s$,

$$|G(t, s; x, y)| = \left| \int_\Omega K_{\alpha\beta}(t; x, z)[K_{\alpha\beta}(s; y, z) - K_{\alpha\beta}(s; y, x)] dz \right|$$
$$\leqq C s^{-n-1} \int_\Omega |K_{\alpha\beta}(t; x, z)||x - z| dz \leqq C s^{-n-1} t.$$

Moreover, by (6.3) again, $G(t, s; x, y) = 0$ unless $|x - y| \leqq 2s(1 + ||\Phi||)$; so

$$\sup_x \int_\Omega |G(t, s; x, y)| dy \leqq C \frac{t}{s},$$

$$\sup_y \int_\Omega |G(t, s; x, y)| dx \leqq C \frac{t}{s}.$$

If $s \leqq t$ we can obtain similar estimates with s/t in place of t/s. It follows that

$$\sup_x \int_\Omega |H_{ij}(x, y)| dy \leqq C\, 2^{-|i-j|},$$

$$\sup_y \int_\Omega |H_{ij}(x, y)| dx \leqq C\, 2^{-|i-j|},$$

and $||T_i T_j^*|| \leqq C\, 2^{-|i-j|}$. We require a similar estimate for $T_i^* T_j$ but cannot obtain it in the same way, since, in general, $\int_\Omega K_{\alpha\beta}(t; x, z)\, dx \neq 0$. Recalling that

$$K_{\alpha\beta}(t; x, z) = \frac{(-1)^k k}{\alpha!} t^{-n} D_z^\beta \left(Q\left(\frac{z-x}{t} - \Phi(x)\right)(z-x)^\alpha \right), \qquad |\alpha| = |\beta| = k,$$

we let

$$\tilde{K}_{\alpha\beta}(t; x, z) = \frac{k}{\alpha!} t^{-n} D_x \left(Q\left(\frac{z-x}{t} - \Phi(x)\right)(z-x)^\alpha \right).$$

Let $\tilde{T}_j$ be the operator associated with the kernel $\tilde{K}_{\alpha\beta}$ in the same way that T_j was associated with $K_{\alpha\beta}$. Now $|K_{\alpha\beta}(t; x, z) - \tilde{K}_{\alpha\beta}(t; x, z)| \leqq C\, t^{-n+1}$, so $||T_j - \tilde{T}_j|| \leqq C\, 2^{-j}$, and $||T_i^* T_j - T_i^* \tilde{T}_j|| \leqq C\, 2^{-j}$. Suppose that $i \leqq j$; the fact that

$$\int_\Omega \tilde{K}_{\alpha\beta}(t; x, z) dx = 0 \quad \text{for all } t \text{ and } z$$

implies, as above, that $||T_i^* \tilde{T}_j|| \leqq C\, 2^{i-j}$, whence $||T_i^* T_j|| \leqq C\, 2^{i-j}$. If $i > j$, then the estimates $||T_i^* T_j - \tilde{T}_i^* T_j|| \leqq 2^{-i}$, and $||\tilde{T}_i^* T_j|| \leqq C\, 2^{j-i}$ imply that $||T_i^* T_j|| \leqq C\, 2^{j-i}$. Thus $||T_i T_j^*|| \leqq C\, 2^{-|i-j|}$, and $||T_i^* T_j|| \leqq C\, 2^{-|i-j|}$, in any case. By Cotlar's lemma, there is a constant C so that $||S_N|| \leqq C$ for all N.

Suppose, for the moment, that Ω is a space of homogeneous type with respect to its intrinsic metric. We show that

$$(6.8) \qquad ||S_N v||_1 \leqq C\, ||v||_{H^1(\Omega)},$$

with constant independent of N, for all v in the Hardy space $H^1(\Omega)$. Now S_N is an integral operator with kernel

$$K_N(x, z) = \int_{2^{-N}}^1 K_{\alpha\beta}(t; x, z) \frac{dt}{t}.$$

Since Ω is of homogeneous type it suffices (see Coifman and Weiss [**7**, formula (2.14)]) to prove that

$$(6.9) \qquad \int_{\rho(x,z) > C\rho(x,y)} |K_N(x, y) - K_N(x, z)| dx \leqq C$$

for some constant C independent of N. (Recall that ρ is the intrinsic metric on Ω). Let $C > 4(1 + ||\Phi||)$. Fix y and z in Ω and let $\epsilon = \rho(y, z)$. If $\rho(x, z) > C\epsilon$ then $\rho(x, y) > (C - 1)\epsilon$. If $K_{\alpha\beta}(t; x, y) \neq 0$ then $\rho(x, y) = |x - y|$. Similarly, if $K_{\alpha\beta}(t; x, z) \neq 0$ then $\rho(x, z) = |x - z|$. Thus the left hand side of (6.9) is less than the sum of the integrals

$$\int_{|x-z|>C\epsilon} |K_N(x, y) - K_N(x, z)|dx \quad \text{and}$$

$$\int_{|x-y|>(C-1)\epsilon} |K_N(x, y) - K_N(x, z)|dx.$$

If suffices to estimate either of these integrals. The first is dominated by

$$(6.10) \qquad \int_{|x-z|>C\epsilon} dx \int_0^1 |K_{\alpha\beta}(t; x, y) - K_{\alpha\beta}(t; x, z)| \frac{dt}{t}.$$

Both terms in the integrand of (6.10) vanish if $|x - y| > t(1 + ||\Phi||)$ and $|x - z| > t(1 + ||\Phi||)$, both of which conditions are satisfied if either $|x - z| > 1 + ||\Phi|| + \epsilon$, or $t < \min\{|x - y|, |x - z|\}/(1 + ||\Phi||)$. Note also that $|x - y| \geqq |x - z| - |y - z| \geqq |x - z|/2$. By (6.6) $|K_{\alpha\beta}(t; x, y) - K_{\alpha\beta}(t; x, z)| \leqq C\, t^{-n-1}\rho(y, z) = C\, t^{-n-1}\epsilon$. Thus (6.10) is dominated by

$$C\epsilon \int_{|x-z|>C\epsilon} dx \int_{C|x-z|}^{\infty} t^{-n-2}dt \leqq C\epsilon \int_{|x-z|>C\epsilon} |x - z|^{-n-1}dx = C.$$

Hence (6.9) holds and S_N is bounded on $H^1(\Omega)$ into $L^1(\Omega)$ independently of N. The same is true of S_N^*. By interpolation (see [**7**, Theorem D]) S_N and S_N^* are bounded (independently of N) on $L^p(\Omega)$ for $1 < p \leqq 2$, and then by duality for $2 \leqq p < \infty$. We now show that these latter assertions hold even if the homogeneity assumption on Ω is dropped. Thus we may assume that Ω is unbounded.

Let $\{\psi_\nu : \nu \in Z^n\}$ be a C^∞ partition of unity for $\mathbf{R}^n$ subordinate to the cover of $\mathbf{R}^n$ by balls $B_n(\nu)$ of radius n with centres ν in the integer lattice Z^n. For each ν in Z^n let $\Omega_\nu = \bigcup_{x\in\Omega \cap B(\nu)} C_x$ where $B(\nu) = B_{n+1+||\Phi||}(\nu)$ and C_x is the cone $C_1(x; \Phi) = \bigcup_{0<\eta\leqq 1} B_\eta(x + \eta\Phi(x))$. Clearly $\Omega_\nu \subset \Omega$. The domain Ω_ν is bounded and has the ordinary cone property, and so is a space of homogeneous type with homogeneity constant (C in (2.1)) that can be chosen independent of ν. Since S_N and S_N^* are bounded on $L^2(\Omega)$, their restrictions to each Ω_ν are similarly bounded on $L^2(\Omega_\nu)$. In addition they are bounded from $H^1(\Omega_\nu)$ to $L^1(\Omega_\nu)$, independently on N and ν. To prove this assertion for S_N, for instance, it suffices (see [**7**, Section 2]) to show that $||S_N u||_1 \leqq C$ for all $(1, \infty)$-atoms u on the space Ω_ν; and such estimates follow easily from inequality (6.9), the fact that the operators S_N are uniformly bounded on $L^2(\Omega_\nu)$, and the boundedness of the domains Ω_ν. It then follows by interpolation and duality that the operators S_N and S_N^* are uniformly bounded on $L^p(\Omega_\nu)$ for any fixed index p in the interval $(1, \infty)$. Applied to functions with support in $B_n(\nu)$ these operators yield functions with support in Ω_ν, which can then be extended to vanish identically on $\Omega \sim \Omega_\nu$. If $v = \{v_\alpha\}_{|\alpha| \leqq k}$ belongs to $\prod_{|\alpha| \leqq k} L^p(\Omega)$ then $\psi_\nu v = \{\psi_\nu v_\alpha\}$ belongs to $\prod_{|\alpha| \leqq k} L^p(B_n(\nu))$ and $S_N(\psi_\nu v_\alpha)$ belongs to $L^p(\Omega)$ and has support in Ω_ν. There is an integer M such that any x in Ω belongs to at most M of the domains Ω_ν. Hence

$$||S_N v_\alpha||_{p,\Omega}^p = \left\| \sum_{\nu \in Z^n} S_N(\psi_\nu v_\alpha) \right\|_{p,\Omega_\nu}^p$$

$$\leqq M^p \sum_{\nu \in Z^n} ||S_N(\psi_\nu v_\alpha)||_{p,\Omega_\nu}^p \leqq C||v_\alpha||_{p,\Omega}^p.$$

We return to formula (6.7). If v_α belongs to $C^\infty(\Omega) \cap L^p(\Omega)$ then by Fatou's lemma

$$||Sv_\alpha||_p \leqq \liminf_{N \to \infty} ||S_N v_\alpha||_p \leqq C||v_\alpha||_p.$$

It follows that if such holds for all α then Rv belongs to $W^{k,p}(\Omega)$ and

$$(6.11) \quad ||Rv||_{k,p} \leqq C \sum_{|\alpha| \leqq k} ||v_\alpha||_p.$$

Since $C^\infty(\Omega) \cap L^p(\Omega)$ is dense in $L^p(\Omega)$, inequality (6.11) holds for all v in $\prod_{|\alpha| \leqq k} L^p(\Omega)$, and the first assertion of the theorem is proved.

In order to obtain the boundedness of R from $(\prod_{|\alpha|<k} L^1(\Omega)) \times (\prod_{|\alpha|=k} H^1(\Omega))$ into $W^{k,1}(\Omega)$, where Ω is a space of homogeneous type, we first note from (6.1) and (6.7) that S_N converges to S strongly on $L^2(\Omega)$, and therefore the expression for $D^\beta Rv$ is valid for v in $\prod_{|\alpha| \leqq k} L^2(\Omega)$. Since $H^1(\Omega) \cap L^2(\Omega)$ is dense in $H^1(\Omega)$, and since the operators S_N are uniformly bounded on $H^1(\Omega)$ into $L^1(\Omega)$, so is S and the proof is complete.

Remark. We could avoid the use of the H^1 theory in the proof of the part of the lifting theorem where $1 < p < \infty$ by replacing inequality (6.8) by weak-type $(1, 1)$ estimates for the operators S_N, uniform in N. It is shown in Coifman and Weiss [**6**], that such estimates also follow from (6.9).

7. Geometric and analytic properties of domains. Given a map $\Psi : \Omega \to \Omega'$ between domains in $\mathbf{R}^n$, and a function u on Ω', let $\Psi^* u$ be the function on Ω defined by $(\Psi^* u)(x) = u(\Psi(x))$ for all x in Ω. It can be verified (see [**1**, p. 63]) that if Ψ belongs to $\mathscr{B}^\infty(\Omega)$, then, for all indices k and p, the map Ψ^* is a bounded linear operator from $W^{k,p}(\Omega')$ to $W^{k,p}(\Omega)$. We will call Ω and Ω' *isomorphic* if there is a bijection $\Psi : \Omega \to \Omega'$ such that $\Psi \in \mathscr{B}^\infty(\Omega)$ and $\Psi^{-1} \in \mathscr{B}^\infty(\Omega')$; in this case the map $\Psi^* : W^{k,p}(\Omega') \to W^{k,p}(\Omega)$ is an isomorphism between these Banach spaces.

We call a property of domains *intrinsic* if it is preserved by isomorphisms. The cone property and the smooth cone property are examples of intrinsic geometric properties. Let us say that Ω has the *approximation property* if the conclusion of Theorem 1 holds for all indices k, m, and p; it is easy to see that the approximation property is also intrinsic. A domain Ω is said to have the *extension property* if there is a linear operator $E : L_{\text{loc}}^1(\Omega) \to L_{\text{loc}}^1(\mathbf{R}^n)$ such that, for functions $u \in L_{\text{loc}}^1(\Omega)$, the restriction of Eu to Ω coincides *a.e.* with u, and such that, for all positive integers k, and all indices p, the restriction of E to $W^{k,p}(\Omega)$ is a bounded operator from $W^{k,p}(\Omega)$ to $W^{k,p}(\mathbf{R}^n)$. The weakest geometric property of a domain that is known (see Stein [**16**, p. 180]) to imply the extension property is the strong local Lipschitz property [**1**, p. 66]).

Let Ω_1 be the domain in $\mathbf{R}^2$ obtained from $\mathbf{R}^2$ by deleting the negative x_1-axis and the closed ball of unit radius centred at the origin; let Ω_2 be the intersection of Ω_1 with the upper half-plane, $x_2 > 0$. It is easy to see that Ω_1 and Ω_2 are isomorphic, and that both of them have the smooth cone property, and hence the approximation property. In addition, Ω_2 has the strong local Lipschitz property, and hence the extension property, and it satisfies Muramatu's cone condition. In contrast, Ω_1 has none of the latter three properties, so these properties are not intrinsic, and the smooth cone property does not imply any of them. Clearly Muramatu's cone condition implies the smooth cone property; it also implies the strong local Lipschitz property, and hence the extension property. Conversely, if a domain has the smooth cone property and the extension property, then it satisfies Muramatu's cone condition.

Next consider the domain Ω obtained from $\mathbf{R}^2$ by deleting the non-positive x_1-axis. This domain has the cone property, but not the smooth cone property. Nevertheless, it has the approximation property; we now outline a proof of this fact. For each number δ in the interval $(0, 1)$, let Ω_δ be the domain obtained from Ω by deleting the closed ball of radius δ centred at the origin. The domains Ω_δ all have the smooth cone property; indeed we can specify a smooth vector field Φ on Ω, depending only on the polar angle θ, such that, for each δ, the restriction of Φ to Ω_δ determines the smooth cone property for Ω_δ, and such that for all integers j,

$$(7.1) \qquad \max_{|\alpha|=j} \sup_{x\in\Omega_\delta} |D^\alpha\Phi(x)| \leqq C\,\delta^{-j}.$$

Let $K = 3 + ||\Phi||$. Fix integers k and m, with $0 < k < m$, and an index p. Let u belong to $W^{k,p}(\Omega)$, and let $0 < \delta < 1$. The first step in the proof is to obtain a function v_δ in $W^{k,p}(\Omega)$ such that

(i) v_δ coincides with a polynomial of degree $k - 1$ in the ball $B_{K\delta}(0)$,

(ii) $||v_\delta||_{k,p} \leqq C||u||_{k,p}$, and

(iii) $||v_\delta - u||_p \leqq C\,\delta^k||u||_{k,p}$.

To obtain the function v_δ, we first find a polynomial P of degree $k - 1$ such that

$$||P||_{k,p,B_{2K\delta}(0)} \leqq C||u||_{k,p}, \quad \text{and}$$
$$||P - u||_{p,B_{2K\delta}(0)} \leqq C\,\delta^k||u||_{k,p};$$

then we let $v_\delta = g_\delta \cdot (u - P) + P$, where g_δ is an appropriate, smooth radial function that vanishes on $B_{K\delta}(0)$ and is equal to unity outside $B_{2K\delta}(0)$. Next we temporarily regard v_δ as a function on Ω_δ, and consider the proof of the approximation theorem, for this domain, with $\epsilon = \delta$. The estimates (7.1) above imply that the polynomials $\pi_{\alpha\beta\gamma}(\delta;\ \Phi;\ x)$ appearing in formula (5.4) are uniformly bounded on Ω_δ by a constant independent of δ. The approximation procedure therefore yields a function u_δ in $W^{m,p}(\Omega_\delta)$ such that

$$||u_\delta - v_\delta||_{p,\Omega_\delta} \leqq C\,\delta^k ||u||_{k,p,\Omega}, \quad \text{and}$$
$$||u_\delta||_{m,p,\Omega_\delta} \leqq C\,\delta^{k-m} ||u||_{k,p,\Omega}.$$

Recall that v_δ coincides with a polynomial of degree $k - 1$ in the ball $B_{K\delta}(0)$; a glance at the proof of the approximation theorem shows that u_δ also coincides with this polynomial in $\Omega_\delta \cap B_{2\delta}(0)$. We extend u_δ so that it coincides with this polynomial in $\Omega \cap B_{2\delta}(0)$. Then

$$||u - u_\delta||_{p,\Omega} \leqq C\,\delta^k ||u||_{k,p,\Omega}, \quad \text{and}$$
$$||u_\delta||_{m,p,\Omega} \leqq C\,\delta^{m-k} ||u||_{k,p,\Omega}.$$

Since the domain Ω has the approximation property and the weak cone property, we could use the first part of the proof of Theorem 9 to show that, under the hypotheses on k and p of that theorem, $W^{k,p}(\Omega) \subset \mathrm{ILip}_\lambda(\Omega)$. This imbedding holds, however, for all domains Ω with the (ordinary) cone property. We omit the proof of this fact, except to mention that it depends on Gagliardo's observation (see [**1**, p. 68]) that a domain with the cone property can be expressed as the union of a locally bounded collection of subdomains each with the strong local Lipschitz property.

One strategy for proving theorems about Sobolev spaces is to first prove these for $\mathbf{R}^n$, and then to argue that they hold for all domains with the extension property. For instance, all domains with the extension property have the approximation property, because $\mathbf{R}^n$ has this property. Since the approximation property is intrinsic, it is possessed by all domains that are merely isomorphic to some domain with that property. This is an alternate proof that the domain Ω_1 considered above has the approximation property.

It is not clear whether the strategy described above can be used to prove the lifting theorem for subdomains of $\mathbf{R}^n$, because it is not clear whether the extension property implies the conclusions of this theorem. (Also, we do not know whether the extension property implies the smooth cone property.) Furthermore, we do not know whether every domain with the smooth cone property is isomorphic to some domain with the extension property, nor even whether the conclusions of the lifting theorem are intrinsic. The first conclusion of Theorem 5 does hold, however, for all domains with the extension property. Finally, the conclusions of Theorem 9 hold, even with $\mathrm{Lip}_{\lambda,q}(\Omega)$ in place of $\mathrm{ILip}_{\lambda,q}(\Omega)$, for all such domains.

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