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ANOTHER CHARACTERISATION OF PFAFFIAN BIPARTITE GRAPHS

CHARLES H. C. LITTLE

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Abstract

It has been known for over twenty years that every planar graph is Pfaffian. Recently a characterisation of planar graphs in terms of strict maximal odd rings has been discovered. This paper attempts to elucidate the connection between the Pfaffian property and planarity by characterising Pfaffian bipartite graphs in terms of maximal odd rings.

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1. Introduction

All graphs considered in this paper are finite. Let G^* be a directed graph with an even number of vertices, and let F be the set $\{f_1, \ldots, f_k\}$ of its 1-factors. For all i write $f_i = \{(u_{i1}, w_{i1}), (u_{i2}, w_{i2}), \ldots, (u_{in}, w_{in})\}$ where $n = \frac{1}{2}|V(G^*)|$ and, for all j, (u_{ij}, w_{ij}) denotes an edge directed from vertex u_{ij} to vertex w_{ij}. Associate with f_i a plus sign if $u_{i1}w_{i1}u_{i2}w_{i2} \cdots u_{in}w_{in}$ is an even permutation of $u_{11}w_{11}u_{12}w_{12} \cdots u_{1n}w_{1n}$, and a minus sign otherwise. If G is an undirected graph, we say that G is a *Pfaffian* graph if there exists a directed graph G^* with vertex set $V(G)$ and edge set $E(G)$ such that all the 1-factors of G^* have the same sign.

The idea of affixing signs to the 1-factors of a directed graph in this manner is due to Kasteleyn [2], who showed that every planar graph is Pfaffian. Pfaffian bipartite graphs have been characterised in [3]. In order to describe this characterisation, we need some more notation. Let G be a bipartite graph with

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bipartition $\{V, V'\}$. (In other words, every edge of $E(G)$ joins a vertex of V to one of V'.) Let f be a 1-factor of G. Then Gf denotes the directed graph obtained from G by orienting each edge of f away from the end in V and each edge of $E(G) - f$ toward the end in V. Furthermore, let H be the directed graph of Figure 1. Then the following theorem is the characterisation given in [3].

THEOREM 1. *A bipartite graph G is non-Pfaffian if and only if there exists a 1-factor f of G such that some subgraph of Gf is isomorphic to a subdivision of H.*

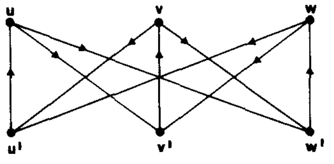

Figure 1

The connection between planarity and the Pfaffian property described above is perhaps rather unexpected. In this paper, we attempt to elucidate that connection by appealing to a recent characterisation of planarity. We turn now to a description of that characterisation.

If S is a set of circuits of G, then the circuits in S are *consistently orientable* if G can be oriented so that they are all directed circuits. A *ring* of circuits in G is a set S of consistently orientable circuits such that

(a) $|S| \geqslant 3$,

(b) there is a cyclic ordering $(C_0, C_1, \ldots, C_{n-1}, C_0)$ of the n circuits in S such that $E(C_i) \cap E(C_j) \neq \varnothing$ if and only if $j = i$, $j \equiv i - 1 \pmod n$ or $j \equiv i + 1 \pmod n$, and

(c) no edge of G belongs to more than two circuits of S.

A ring S is said to be *odd* if $|S|$ is odd. A ring $\{C_0, \ldots, C_{n-1}\}$ is said to be *maximum* if there does not exist a ring $\{C'_0, \ldots, C'_{m-1}\}$ for which $m > n$ and

$$\bigcup_{j=0}^{m-1} E(C'_j) \subseteq \bigcup_{j=0}^{n-1} E(C_j).$$

The ring $\{C_0, \ldots, C_{n-1}\}$ is *strict* if $|V(C_i) \cap V(C_j)| \leqslant 1$ whenever $E(C_i) \cap E(C_j) = \varnothing$.

The following characterisation of planar graphs has been proved by Chernyak [1].

THEOREM 2. *A graph is non-planar if and only if it contains a maximum strict odd ring.*

Chernyak's proof depends on work by Holton and Little that has not yet been published. See [4] for a self-contained proof for cubic graphs.

In the subsequent sections, we characterise Pfaffian bipartite graphs in terms of rings. The reader is referred to [4] for definitions and notation not explained here.

2. Some preliminary results

Our aim is to characterise Pfaffian bipartite graphs. We shall re-interpret the problem as one about cubic graphs, and shall solve the resulting problem concerning cubic graphs.

Throughout the following, G is a bipartite graph and f is a 1-factor of G. We denote by G^*f the cubic directed graph constructed from Gf in the following way. Delete every vertex of valency 1 (and the edge incident on it). Let u be any vertex of valency 2, and let u_1 and u_2 be the vertices adjacent to u. Since no vertex of Gf of valency greater than 1 is a source or sink, we may assume that $(u_1, u) \in E(Gf)$ and $(u, u_2) \in E(Gf)$. Then replace u, (u_1, u) and (u, u_2) by the single edge (u_1, u_2). In this way, we remove all vertices of valency 2. Finally, let v be a vertex of valency $j > 3$. Let $v_1, \ldots, v_j$ be the vertices adjacent to v, and let (v_1, v) be the unique edge of Gf directed toward v. Replace (v_1, v), (v, v_2), $(v, v_3), \ldots, (v, v_j)$ and v by the edges (v_1, w_2), (w_2, v_2), (w_2, w_3), (w_3, v_3), (w_3, w_4), (w_4, v_4), $(w_4, w_5), \ldots, (w_{j-2}, v_{j-2})$, (w_{j-2}, w_{j-1}), (w_{j-1}, v_{j-1}), (w_{j-1}, v_j), where $w_2, w_3, \ldots, w_{j-1}$ are new vertices not in $V(Gf)$.

We use the corresponding procedure if there is a unique edge of Gf directed away from v. In this way all vertices of degree greater than 3 are elminated.

The following two lemmas are easily proved.

LEMMA 1. *G^*f contains a subdivision of H if and only if Gf does.*

LEMMA 2. *G^*f contains a maximum odd ring if and only if Gf does.*

We conclude this section with a lemma that is proved in [3].

LEMMA 3. *Let f and f' be distinct 1-factors of a bipartite graph G. Then Gf' is obtained from Gf by reversing the orientation of every edge of $f \oplus f'$.*

Here $f \oplus f'$ denotes the symmetric difference of f and f'. Note that it is the union of vertex-disjoint circuits.

3. The main theorem

We can now state and prove our main theorem.

THEOREM 2. *A bipartite graph G is Pfaffian if and only if, for every* 1*-factor f, Gf does not contain a maximum odd ring.*

REMARK. Clearly any ring in Gf is strict.

PROOF. From Theorem 1 and Lemmas 1 and 2, it suffices to prove that a necessary and sufficient condition for the existence of a 1-factor f such that G^*f contains a maximum odd ring is the existence of a 1-factor f' such that G^*f' contains a subdivision of H.

Suppose first that f' is a 1-factor of G such that G^*f' contains a subdivision of H. Then G^*f' contains a maximum odd ring by the argument at the beginning of the proof of the main theorem of [4].

Now let f be a 1-factor of G such that G^*f contains a maximum odd ring. Let S be a maximum odd ring $\{C_0, \ldots, C_{n-1}\}$ in G^*f such that $|\bigcup_{j=0}^{n-1} E(C_j)|$ is minimal. Without loss of generality, we assume that $G = \bigcup_{j=0}^{n-1} C_j$, for any edge not in a circuit of S is irrelevant. It follows that for any maximum odd ring S' of G^*f, any edge of G^*f belongs to some circuit of S'. Furthermore, let $C_0, C_1, \ldots, C_{n-1}, C_0$ be the cyclic ordering of the elements of S that satisfies condition (b) of the definition of a ring.

As each circuit of S is a directed circuit, the argument now follows closely the proof of the main theorem of [4]. The condition that G is planar is replaced by the condition that G^*f contains a subdivision of H. If G^*f does not contain a subdivision of H, then it can be shown as in [4], with only very minor modifications, that every $\overline{C}_i C_{i+1}$-chord contains edges in common with C_{i+2} and every $\overline{C}_i C_{i-1}$-chord contains edges in common with C_{i-2}. The reorientations of edges that occur in the proof can be justified by appealing to Lemma 3 to show that they merely amount to choosing a new 1-factor for G. In case $IA(2)$, the graph

$$\bigcup_{j=0}^{k} P_j \cup \bigcup_{j=1}^{k-2} C_i(u_{j+2}, w_j) \cup C_i(w_1, w_{k-1})$$

becomes a subdivision of H upon the reorientation of every edge of every circuit $P_j(u_j, w_j)C_i(w_j, u_j)$ for all odd j such that $j > 1$.

As in [4], we now distinguish two cases, but the discussion of them is more involved.

Case I. Suppose $|S| \geqslant 5$. Since no edge belongs to more than two circuits of S, it follows that $C_i \neq C_{i+1}$ because $E(C_{i+1}) \cap E(C_{i+2}) \neq \varnothing$. Hence for any value of i there must be a $\bar{C}_i C_{i+1}$-chord. We now introduce two subcases.

Case IA. Suppose that for some value of i there exist a $\bar{C}_i C_{i+1}$-chord P_1 with origin u and terminus y and a $\bar{C}_i C_{i+1}$-chord P_2 with origin w and terminus x, and that there vertices occur on C_i in the cyclic order x, v, w, u, x. We have $E(P_1) \cap E(C_{i+2}) \neq \varnothing$ since P_1 is a $\bar{C}_i C_{i+1}$-chord. Let y, z be two internal vertices of P_1 such that $P_1(y, z)$ is a $C_{i+1}C_{i+2}$-chord.

Suppose the choice of y and z is not unique. Clearly z is determined if y is given and vice versa (since $C_{i+1}C_{i+2}$-chords are by definition of maximal length); therefore let us suppose that there are y', z', where $\{y', z'\} \cap \{y, z\} = \varnothing$, such that $P_1(y', z')$ is a $C_{i+1}C_{i+2}$-chord. By the definition of a $C_{i+1}C_{i+2}$-chord, these vertices must occur on P_1 in the order y', z', y, z or y, z, y', z'. Without loss of generality we choose the former order. Therefore $P_1(z', y)$ is a $\bar{C}_{i+2}C_{i+1}$-chord and hence must contain edges in common with C_i. It does not do so, however, because it is a subpath of a $\bar{C}_i C_{i+1}$-chord. This contradiction shows that the choice of y and z is unique.

Therefore $E(C_{i+2}) \cap E(P_1) = E[P_1(y, z)]$. Since $|S| \geqslant 5$, $E(C_{i+2}) \cap E(C_i) = \varnothing$. However $E(C_{i+2}) \cap E(P_2) \neq \varnothing$ since P_2 is a $\bar{C}_i C_{i+1}$-chord. Therefore P_2 contains vertices a, b such that $P_2(a, b)$ is a $C_{i+1}C_{i+2}$-chord. As before, a and b are chosen uniquely. In summary, paths $C_{i+2}(z, a)$ and $C_{i+2}(b, y)$ have no edges in common with P_1, P_2 or C_i. Hence the graph

$$C_i(x, u) \cup C_{i+1}(w, a) \cup C_{i+1}(u, y) \cup C_{i+1}(z, v) \cup C_{i+1}(b, x) \cup C_{i+2}$$

is a subdivision of H.

Case IB. The only other possibility is the case where, for every value of i and every $\bar{C}_i C_{i+1}$-chord P_1 with origin u_1 and terminus v_1, there does not exist a $\bar{C}_i C_{i+1}$-chord with origin in $V[C_i(v_1, u_1)]$ and terminus in $V[C_i(u_1, v_1)]$. Clearly there is no $\bar{C}_i C_{i+1}$-chord with origin in $V[C_i(u_1, v_1)]$ either, for if there were, there would then have to be a $\bar{C}_i C_{i+1}$-chord with origin in $V[C_i(v_1, u_1)]$ and terminus in $V[C_i(u_1, v_1)]$, since C_{i+1} is a circuit. This property will be referred to as the non-interlocking property of $\bar{C}_i C_{i+1}$-chords.

Without loss of generality, let $i = 0$ so that P_1 is a $\bar{C}_0 C_1$-chord. We have $E(P_1) \cap E(C_2) \neq \varnothing$, so that P_1 contains a C_1C_2-chord P_2 with origin u_2 and terminus v_2. As before, there is only one choice for P_2. There must be a $\bar{C}_2 C_3$-chord since $C_2 \neq C_3$. In particular, since no two $\bar{C}_2 C_3$-chords can interlock, and since $E(C_1) \cap E(C_3) = \varnothing$, there must be a $\bar{C}_2 C_3$-chord P_3 with origin u_3 and terminus v_3 in $C_2(v_2, u_3)$. Again since no two $\bar{C}_2C_3$-chords can interlock, the path $C_2(u_3, v_3)$ is a $\bar{C}_3C_2$-chord, and therefore contains a unique subpath, namely P_2, which is a C_1C_2-chord. Since P_3 is a $\bar{C}_2C_3$-chord, it contains a C_3C_4-chord P_4 with origin u_4 and terminus v_4. Since $C_5 \neq C_4$, there must be a $\bar{C}_4C_5$-chord. Since no two $\bar{C}_4C_5$-paths can interlock, and since $E(C_3) \cap E(C_5) = \varnothing$, there must be a $\bar{C}_4C_5$-chord P_5 with origin u_5 and terminus v_5 in $C_4(v_4, u_5)$. We proceed in this manner until a path P_k is found, where $k = |S| - 1$. Since k is even, P_k must be a $C_{k-1}C_k$-chord.

Now we consider C_k. Let u_0 be the first vertex of the path $C_k(v_k, u_k)$ that belongs to $V(C_0)$. Let v_0 be the last vertex of $C_k(v_k, u_k)$ belonging to $V(C_0)$. Since G^*f is cubic, we have $\{u_0, v_0\} \cap \{u_1, v_1\} = \varnothing$. There are now six subcases, corresponding to the six possible cyclic orderings of the vertices u_0, v_0, u_1, v_1 on C_0.

(1) Suppose that the vertices occur on C_0 in the cyclic order v_1, u_0, u_1, v_0, v_1. Clearly u_0 is the origin of a C_0C_k-chord. Such a chord cannot contain edges in common with C_1. Since the edge of C_0 with negative end u_1 and the edge of C_0 with positive end v_1 are both in C_1, it follows that there must exist a $\bar{C}_0C_k$-chord with origin o lying on the path $C_0(v_1, u_1)$ and terminus t lying on the path $C_0(u_1, v_1)$. The fact that no two $\bar{C}_0C_k$-chords can interlock implies that $o \in V[C_0(u_0, u_1)]$ and $t \in V[C_0(u_1, v_0)]$.

Since $E(C_1) \cap E(C_k) = \varnothing$, there must be a $\bar{C}_0C_1$-chord with terminus $y \in V[C_0(o, u_1)]$. Choose y so that the length of the path $C_0(o, y)$ is minimal. This $\bar{C}_0C_1$-chord must have edges, and therefore vertices, in common with C_2; let x be the last such vertex. Therefore x is the terminus of a C_1C_2-chord; call the origin of this chord x'. Since $C_2(u_2, v_2)$ is, as we have seen, the unique C_1C_2-chord which is a subpath of $C_2(u_3, v_3)$, it follows that x and x' lie on $C_2(v_3, u_3)$. The vertex x' is the terminus of a $\bar{C}_2C_1$-chord, and this chord must have vertices in common with C_0; let y' be the last such vertex. Thus $C_1(y', y)$ is a $\bar{C}_0C_1$-chord.

Because no two $\bar{C}_0C_1$-chords can interlock, we have $y' \in V[C_0(v_1, y)]$. Suppose $y' \in V[C_0(o, y)]$. Then since y' is the terminus of a C_0C_1-chord, and since $E(C_1) \cap E(C_k) = \varnothing$, there must exist some vertex on the path $C_0(o, y')$ which is the terminus of a $\bar{C}_0C_1$-chord. This result contradicts the choice of y. Therefore $y' \in V[C_0(v_1, o)]$. We distinguish two subcases.

(a) Suppose $y' \in V[C_0(u_0, o)]$. (See Figure 2.)

Then $S' = (C'_0, C'_1, \dots, C'_{n-1})$ becomes an odd ring of directed circuits upon the reorientation of the edges of C_0, where $C'_j = C_j$ for all even j such that $2 < j < k$, $C'_j = C_j(u_j, v_j)C_{j-1}(v_j, u_j)$ for all odd j such that $2 < j < k$, $C'_0 = C_0^{-1}(v_1, u_1)C_1(u_1, v_1)$, $C'_1 = C_0^{-1}(t, o)C_k(o, t)$, $C'_2 = C_1(y', y)C_0^{-1}(y, y')$ and $C'_k = C_k(v_0, u_0)C_0^{-1}(u_0, v_0)$. Furthermore $|S'| = |S|$ and so S' is a maximum odd

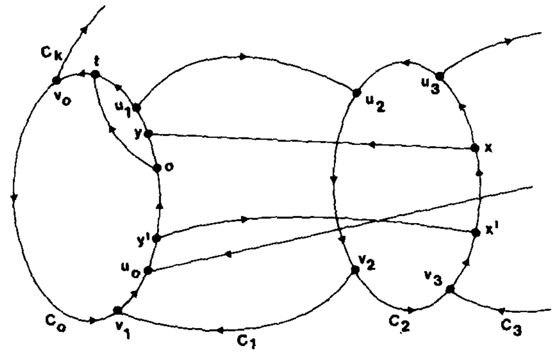

Figure 2

ring. However, there are edges of G^*f that are not in any circuit of S'. An example is the edge of C_2 with positive end u_3. The minimality property of S is contradicted.

(b) Suppose $y' \in V[C_0(v_1, u_0)]$. (See Figure 3.) Then $C_0(v_1, v_0) \cup C_1(u_1, u_2) \cup C_1(v_2, v_1) \cup C_1(x, y) \cup C_1(y', x') \cup C_2(v_3, v_2) \cup \bigcup_{j=2}^{k/2} C_{2j-1}(u_{2j}, v_{2j-1}) \cup \bigcup_{j=2}^{(k-2)/2} C_{2j}(v_{2j+1}, u_{2j}) \cup C_k(v_0, u_k)$ is a subdivision of H.

(2) Suppose the vertices occur on C_0 in the cyclic order v_1, u_1, v_0, u_0, v_1. Clearly $C_k(v_0, u_0)$ is a $\bar{C}_0C_k$-chord; since no two such chords interlock, it follows that $C_0(v_0, u_0)$ is a $\bar{C}_kC_0$-chord. Therefore $E(C_1) \cap E[C_0(v_0, u_0)] \neq \varnothing$.

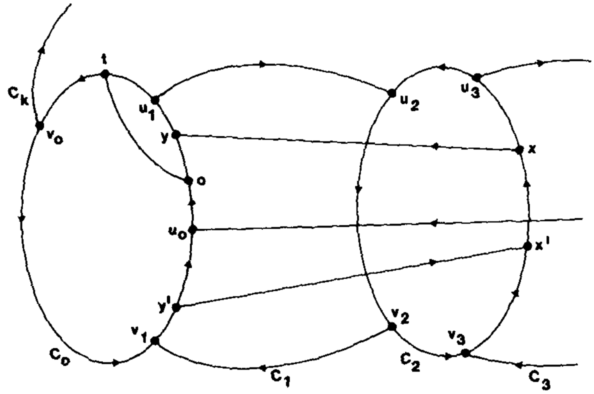

Figure 3

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Now we consider the C_0C_1-chords and the cyclic order in which they occur on C_0 and on C_1. One such chord has terminus u_1; it is succeeded on C_1, and therefore on C_0 also because of the prohibition of interlocking $\bar{C}_0C_1$-chords, by one whose origin is v_1. Hence there is no subpath of $C_0(u_1, v_1)$ which is a C_0C_1-chord. This fact contradicts the conclusion of the preceding paragraph.

(3) Suppose that the vertices occur on C_0 in the cyclic order v_1, u_1, u_0, v_0, v_1. Then a subdivision of H can be produced after a suitable reorientation of edges.

The remaining possibilities reduce to the ones already considered after reorientation of the edges of C_0. The case where $|S| \geqslant 5$ is now complete.

Case II. Suppose $|S| = 3$. Let $S = \{C_0, C_1, C_2\}$. First we consider C_0 and C_1.

Case IIA. Suppose there exist a $\bar{C}_0C_1$-chord with origin o_1 and terminus t_1 and another $\bar{C}_0C_1$-chord with terminus $t \in V[C_0(o_1, t_1)]$.

Since C_1 is a directed circuit, it is then obvious that there must be a $\bar{C}_0C_1$-chord with origin $o_2 \in V[C_0(t_1, o_1)]$ and terminus $t_2 \in V[C_0(o_1, t_1)]$, for otherwise every edge of $E(C_1) \cap E(C_0)$ would be on the path $C_0(t_1, o_1)$ and this result would contradict the existence of t. There may be many choices for this $\bar{C}_0C_1$-chord; we choose the one such that the length of $C_1(o_1, t_2)$ is minimal. Since C_1 is a directed circuit, it is again clear that some subpath of $C_1(t_2, o_1)$ is a $\bar{C}_0C_1$-chord with origin $o_3 \in V[C_0(o_1, t_1)]$ and terminus $t_3 \in V[C_0(t_1, o_1)]$. We choose this path so that the length of $C_1(t_2, o_3)$ is minimal. There now arise several cases and subcases.

(1) Suppose $o_3 \in V[C_0(o_1, t_2)]$.

(a) Suppose $t_3 \in V[C_0(t_1, o_2)]$.

(i) Suppose there exists a $\bar{C}_0C_1$-chord with origin $o_4 \in V[C_0(t_2, t_1)]$ and terminus $t_4 \in V[C_0(o_1, o_3)]$.

Define the circuits

$C_0' = C_0(t_4, o_4)C_1(o_4, t_4)$,

$C_1' = C_0(t_3, o_3)C_1(o_3, t_3)$,

$C_2' = C_0(t_2, o_2)C_1(o_2, t_2)$.

Let $S' = \{C_0', C_1', C_2'\}$. Then S' is a maximum odd ring of directed circuits. However, no edge of the path $C_1(o_1, t_1)$ belongs to any circuit of S'. This result contradicts the minimality property of S.

(ii) Suppose there is no $\bar{C}_0C_1$-chord with origin lying on $C_0(t_2, t_1)$ and terminus lying on $C_0(o_1, o_3)$. Since C_1 is a directed circuit, there must be a $\bar{C}_0C_1$-chord with origin $u_1 \in V[C_0(t_2, t_1)]$. Let v_1 be the terminus of this path. Since $u_1 \in V[C_0(t_2, t_1)]$, $u_1 \neq o_3$ and therefore if we choose u_1 so that $u_1 \in V[C_1(t_2, o_3)]$, then, by the choice of o_3, v_1 cannot lie on the path $C_0(t_1, o_1)$. By the assumption defining case (ii), v_1 cannot lie on the path $C_0(o_1, o_3)$. If we choose u_1 to be the last vertex of the path $C_1(t_2, o_3)$ that lies on the path $C_0(t_2, t_1)$, then v_1 cannot lie on the path $C_0(t_2, t_1)$ either. Hence $v_1 \in V[C_0(o_3, t_2)]$.

Again since C_1 is a directed circuit, there must be a subpath of $C_1(v_1, o_3)$ which is a $\bar{C}_0C_1$-chord with origin $u_2 \in V[C_0(v_1, t_2)]$. Again we can choose u_2 to be the last vertex of $C_1(t_2, o_3)$ lying on $C_0(v_1, t_2)$. If the terminus of this $\bar{C}_0C_1$-chord is v_2, then as before $v_2 \in V[C_0(o_1, v_1)]$. Suppose $v_2 \in V[C_0(o_3, v_1)]$. Then, since C_1 is a directed circuit, there must be a subpath of $C_1(v_2, o_3)$ which is a $\bar{C}_0C_1$-chord with origin $u_3 \in V[C_0(v_2, v_1)]$. Let the terminus of this path be v_3. We choose u_3 again so that it is the last vertex of $C_1(t_2, o_3)$ lying on the path $C_0(v_2, v_1)$. Then $v_3 \in V[C_0(o_1, v_2)]$. If $v_3 \in V[C_0(o_3, v_2)]$, then we repeat the argument. By the finiteness of the graph, there must exist an integer $n \geqslant 2$ such that $v_n \in V[C_0(o_1, o_3)]$.

Define the following circuits:

$$C'_j = C_0(v_j, u_j)C_1(u_j, v_j) \text{ for all } j \in \{1, 2, \dots, n\},$$
$$C'_{n+1} = C_0(t_3, o_3)C_1(o_3, t_3),$$
$$C'_{n+2} = C_0(t_2, o_2)C_1(o_2, t_2).$$

Let $S' = \{C'_1, C'_2, \dots, C'_{n+2}\}$. Then $|S'| = n + 2 \geqslant 4$. Now S' is clearly an odd ring of directed circuits, and so the maximality of S is contradicted.

(b) Suppose $t_3 \in V[C_0(o_2, o_1)]$. (This is the only other possibility since by assumption $t_3 \in V[C_0(t_1, o_1)]$.) In this case, we define vertices u_i and v_i as in Case (a)(ii) for all $i \in \{1, 2, \dots, n\}$. Again $v_n \in V[C_0(o_1, o_3)]$ but in this case $n \geqslant 1$. Now define the following circuits:

$$C'_j = C_0(v_j, u_j)C_1(u_j, v_j) \text{ for all } j \in \{1, 2, \dots, n\},$$
$$C'_{n+1} = C_0(t_3, o_3)C_1(o_3, t_3),$$
$$C'_{n+2} = C_0(t_1, o_1)C_1(o_1, t_1),$$
$$C'_{n+3} = C_0(t_2, o_2)C_1(o_2, t_2).$$

Let $S' = \{C'_1, C'_2, \dots, C'_{n+3}\}$. Then $|S'| = n + 3 \geqslant 4$. Again the maximality of S is contradicted.

(2) Suppose $o_3 \in V[C_0(t_2, t_1)]$. There are two subcases.

(a) Suppose $t_3 \in V[C_0(o_2, o_1)]$. Then $C_0 \cup C_1(o_1, t_1) \cup C_1(o_2, t_2) \cup C_1(o_3, t_3)$ is a subdivision of H.

(b) The only other possibility is the case where $t_3 \in V[C_0(t_1, o_2)]$.

Define $u_1 = o_2$, $v_1 = t_2$, $u_2 = o_1$, $v_2 = t_1$, $u_3 = o_3$, $v_3 = t_3$. Since we have disposed of all the other possibilities we are free to assume that all $\bar{C}_0C_1$-chords considered in Case A satisfy Case (2)(b).

Since C_1 is a circuit, there must clearly be a $\bar{C}_0C_1$-chord with origin $u_4 \in V[C_0(u_3, v_3)]$ and terminus $v_4 \in V[C_0(v_3, u_3)]$. We choose u_4 so that the length of the path $C_1(u_2, u_4)$ is minimal. Furthermore $u_4 \in V[C_0(v_2, v_3)]$, because if

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$u_4 \in V[C_0(u_3, v_2)]$, then the $\bar{C}_0C_1$-chords $C_1(u_3, v_3)$, $C_1(u_2, v_2)$ and $C_1(u_4, v_4)$ together would satisfy Case (1).

Suppose $v_4 \in V[C_0(v_3, u_1)]$. Then since C_1 is a circuit, there must be a $\bar{C}_0C_1$-chord with origin $u_5 \in V[C_0(u_4, v_4)]$ and terminus lying on the path $C_0(v_4, u_4)$. Again, choose u_5 in such a way that the length of the path $C_1(u_3, u_5)$ is minimal. We have $u_5 \in V[C_0(v_3, v_4)]$ because if $u_5 \in V[C_0(u_4, v_3)]$ then the $\bar{C}_0C_1$-chords $C_1(u_4, v_4)$, $C_1(u_3, v_3)$ and $C_1(u_5, v_5)$ together would satisfy Case (1).

If $v_5 \in V[C_0(v_4, u_1)]$, we repeat the argument. By the finiteness of the graph, there is an integer n such that $v_n \notin V[C_0(v_{n-1}, u_1)]$. Therefore there must exist an integer m such that $v_n \in V[C_0(u_m, u_{m+1})]$. Since by definition $v_n \notin V[C_0(u_{n-1}, u_n)]$, we must have $m \leqslant n - 2$.

(i) Suppose $m = n - 2$. Then $C_0 \cup C_1(u_{n-2}, v_{n-2}) \cup C_1(u_{n-1}, v_{n-1}) \cup C_1(u_n, v_n)$ is a subdivision of H.

(ii) Suppose $m < n - 2$. Then the set

$$S' = \{C_0^{-1}(v_m, u_m)C_1(u_m, v_m), C_0^{-1}(v_{m+1}, u_{m+1})C_1(u_{m+1}, v_{m+1}), \ldots, C_0^{-1}(v_n, u_n)C_1(u_n, v_n)\}$$

becomes an odd ring of directed circuits upon reorientation of every edge of C_0. Since $m \leqslant n - 3$, we have $|S'| = n - m + 1 \geqslant 4$, so that the maximality of $|S|$ is contradicted.

Case IIB. We now assume that for every $\bar{C}_0C_1$-chord with origin o_1 and terminus t_1 there does not exist a $\bar{C}_0C_1$-chord whose terminus lies on the path $C_0(o_1, t_1)$. It obviously follows that there is no $\bar{C}_0C_1$-chord with origin in $V[C_0(o_1, t_1)]$. Thus no two $\bar{C}_0C_1$-chords interlock. We can clearly assume also that no two $\bar{C}_1C_0$-, $\bar{C}_2C_1$-, $\bar{C}_1C_2$-, $\bar{C}_0C_2$- or $\bar{C}_2C_0$-chords interlock either, for otherwise we simply apply Case A to the appropriate pair of circuits.

We recall that every $\bar{C}_0C_1$-chord contains edges in common with C_2. Of course, corresponding statements hold for $\bar{C}_1C_0$-, $\bar{C}_0C_2$-, $\bar{C}_2C_0$-, $\bar{C}_2C_1$- and $\bar{C}_1C_2$-chords.

Suppose there are distinct $\bar{C}_0C_1$-chords P_1 and P_2. Let P_1 have origin u_1 and terminus v_1. Since P_1, P_2 are both $\bar{C}_0C_1$-chords, each of them contains edges in common with C_2. Hence some subpath of P_1 is a C_1C_2-chord; let this chord have origin y_1 and terminus z_1. Similarly, since $C_0(u_1, v_1)$ is a $\bar{C}_1C_0$-chord, it contains edges in common with C_2. Hence some subpath of $C_0(u_1, v_1)$ is a C_0C_2-chord; let this subpath have origin w_1 and terminus x_1.

Define the following circuits:

$$C_0' = C_0(v_1, u_1)C_1(u_1, v_1), \qquad C_1' = C_1(v_1, u_1)C_0(u_1, v_1).$$

Let $S' = \{C_0', C_1', C_2\}$. S' is clearly a maximum odd ring of directed circuits.

We now consider the set of $C_0'C_2$- and $C_1'C_2$-chords, and the cyclic order in which they occur on the directed circuit C_2. $C_0(w_1, x_1)$ is an example of a

$C_1'C_2$-chord. Since P_2 is a $\bar{C}_0'C_1'$-chord, some subpath of P_2 must be another $C_1'C_2$-chord distinct from $C_0(w_1, x_1)$. Therefore the $C_1'C_2$-chord $C_0(w_1, x_1)$ cannot be both preceded and followed on C_2 by the $C_0'C_2$-chord $C_1(y_1, z_1)$ without any intervening $C_0'C_2$- or $C_1'C_2$-chords. For the sake of concreteness, suppose that $C_0(w_1, x_1)$ is followed on C_2 by P_3 without any intervening $C_0'C_2$- or $C_1'C_2$-chords, where $P_3 \neq C_0'(y_1, z_1)$ and P_3 is either a $C_0'C_2$- or a $C_1'C_2$-chord. Suppose the former. Then P_3 cannot be a subpath of $C_1(u_1, v_1)$ because $C_1(y_1, z_1)$ is the unique C_1C_2-chord which is a subpath of $C_1(u_1, v_1)$. Thus if P_3 is a $C_0'C_2$-chord, it is clearly a C_0C_2-chord. If P_3 has origin o, then if follows from the definition of P_3 that $C_2(x_1, o)$ is a $\bar{C}_0C_2$-chord that contains no edges in common with C_1. This is a contradiction; hence P_3 must be a $C_1'C_2$-chord. But now the path $C_2(x_1, o)$ is a $\bar{C}_1C_2$-chord containing no edges in common with C_0', contradicting the fact that S' is a maximum odd ring. Thus we have a contradiction in either case. The argument is similar if $C_0(w_1, x_1)$ is not preceded on C_2 by $C_1(y_1, z_1)$ without any intervening $C_0'C_2$- or $C_1'C_2$-chords.

We conclude that there is exactly one $\bar{C}_0C_1$-chord, P, say. Let P have origin u and terminus v. Then C_2 must contain a C_1C_2-chord which is a subpath of P. Let this chord have origin y and terminus z. Similarly C_2 must contain a C_0C_2-chord which is a subpath of $C_0(u, v)$; let this chord have origin w and terminus x. Then $C_2(x, y)$ and $C_2(z, w)$ have no internal vertices in common with C_1 or C_2. Therefore the graph $C_0 \cup C_1 \cup C_2$ is a subdivision of H. The theorem is proved.

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Department of Mathematics and Statistics
Massey University
Palmerston North
New Zealand