COMPLEMENTED MODULAR LATTICES

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1. Introduction.

1.1. This paper gives a lattice theoretic investigation of "finiteness"[2] and "continuity of the lattice operations" in a complemented modular lattice. Although we usually assume that the lattice is $\aleph$-complete for some infinite $\aleph$,[3] we do not require completeness and continuity, as von Neumann does in his classical memoir on continuous geometry **(3)**; nor do we assume orthocomplementation as Kaplansky does in his remarkable paper **(1)**.

1.2. Our exposition is elementary in the sense that it can be read without reference to the literature. Our brief preliminary § 2 should enable the reader to read this paper independently.

1.3. Von Neumann's theory of independence **(3**, Part I, Chapter II**)** leans heavily on the assumption that the lattice is continuous, or at least upper continuous. We do not assume such continuity and we find it necessary therefore to distinguish several concepts of independence for a family of elements $\{a_\lambda; \lambda \in \Lambda\}$: *independence* shall mean that $a_\lambda \sum(a_\mu; \mu \in F) = 0$ whenever F is a *finite* subset of Λ and $\lambda \notin F$; *residual independence* shall mean that $a_\lambda \sum(a_\mu; \mu \neq \lambda) = 0$ for every λ; and *strong independence* shall mean that $\prod_\lambda \sum(a_\mu; \mu \neq \lambda) = 0$.

Strong independence is sufficiently restrictive that, even without assuming continuity of the lattice operations, many of the continuous geometry arguments remain vaild. For example, if $\{a_\lambda, b_\lambda; \lambda \in \Lambda\}$ is strongly independent and for each λ there is given a perspective mapping of $[0, a_\lambda]$ onto $[0, b_\lambda]$, then these mappings can be imbedded in a single perspective mapping of $[0, \sum a_\lambda]$ onto $[0, \sum b_\lambda]$.

§ 3 is devoted to a discussion of independence.

1.4. Suppose L is complemented, modular, and *countably complete.* Von Neumann's arguments **(3**, Part I, Theorem 4.3**)** show that L is finite, that is,

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[2]In this paper, a lattice is called finite if every independent sequence of pairwise perspective elements has all its elements coinciding with zero. Sequence shall mean *infinite* sequence throughout this paper. We note however that finite families $\{a_1, \ldots, a_m\}$ of pairwise perspective elements were used by von Neumann in his coordinatization theory **(3**, Part II, chapter III**)** and play a key role there.

[3]Throughout this paper $\aleph$ denotes an arbitrary (but fixed) infinite cardinal (that is $\aleph \geqslant \aleph_0$); Ω denotes the least ordinal number whose corresponding cardinal power is $\aleph$.

an independent sequence $\{a_n\}$ of pairwise perspective elements with $a_1 \neq 0$ *cannot* exist, if the lattice is $\aleph_0$-continuous (this means: the lattice is both upper $\aleph_0$-continuous and lower $\aleph_0$-continuous).

If $\aleph_0$-continuity does not hold, then such sequences $\{a_n\}$ can occur. But we find the paradoxical result: *the existence of such sequences actually forces a certain type of continuity to hold.* This situation is described more precisely in the following paragraph.

A *homogeneous* sequence is defined to be a strongly independent sequence $\{a_n\}$ of pairwise perspective elements. We draw attention to two important special cases:

(i) Type (A): all a_n have a common complement, that is, for some element A, $a_n \oplus A = 1$.

(ii) Type (A*): all $a^*_n = \sum(a_m; m \neq n)$ have a common complement.[4]

In § 5 we show: Suppose $\{a_n\}$ is a homogeneous sequence; then the lattice $[0, \sum a_n]$ is upper $\aleph_0$-continuous if and only if $\{a_n\}$ is of type (A), lower $\aleph_0$-continuous if and only if $\{a_n\}$ is of type (A*). Thus, if $\{a_n\}$ is of *both* types (A) and (A*), the above-mentioned result of von Neumann shows that all a_n must be 0.

In § 8 we show that the types (A) and (A*) are *mutually exclusive* in a stronger sense, namely: If $\{a_n\}$ and $\{b_n\}$ are homogeneous sequences of types (A) and (A*) respectively, then $\sum a_n$ and $\sum b_n$ are completely disjoint (this means: a perspective to b with $a \leqslant \sum a_n$ and $b \leqslant \sum b_n$ can occur only when $a = b = 0$). On the other hand, these two types are *exhaustive* in the following sense: every homogeneous sequence $\{a_n\}$ has a unique decomposition $a_n = b_n + c_n$ with $\{b_n\}$ a homogeneous sequence of type (A) and (c_n) a homogeneous sequence of type (A*).

From these facts about homogeneous sequences we can deduce (see § 8): If L is complete then L has a direct sum decomposition $L = L_1 + L_2 + L_3$ where $L_i = (0, a_i)$ with each a_i in the centre of L, and with L_1 upper $\aleph_0$-continuous, L_2 lower $\aleph_0$-continuous and L_3 finite.

1.5. Now suppose L is even $\aleph$-complete for a given infinite $\aleph$. We call L *locally* $\aleph$*-continuous* if for every $a \neq 0$ there exists some $0 \neq a_1 \leqslant a$ with $[0, a_1]$ $\aleph$-continuous. We show (see Corollary 1 to Theorem 7.1): If L is locally $\aleph$-continuous and finite then L must be $\aleph$-continuous.

1.6. In §§ 4, 6 we establish, among other properties of finiteness and continuity, that they are *additive*, that is, if $[0, a]$ and $[0, b]$ enjoys one of these properties then so does $[0, a + b]$.

1.7. Finally, in § 9 we prove theorems somewhat more general than that of Kaplansky **(1)**. Kaplansky proved: (i) every countably complete ortho-

[4]If $(a_1, \ldots, a_m)$ is a finite independent family of pairwise perspective elements, then (i) and (ii) both hold, that is, the a_n have a common complement and the a^*_n have a common complement.

complemented modular lattice is finite and (ii) every complete orthocomplemented modular lattice is necessarily continuous.

Our work gives lattice theoretic proofs for generalizations of both of these results. In particular, (ii) is strengthened to (ii)′ every $\aleph$-complete orthocomplemented modular lattice is $\aleph$-continuous.

More generally, we prove, generalizing (i):

THEOREM 9.1. *A countably complete, complemented modular lattice is finite, if* (*): *for every* $a \neq 0$ *there exists an anti-automorphism* ϕ *of* L *such that* b *perspective to a subelement of* a *occurs for some* $b \neq 0$ *with* $b\,\phi(a) = 0$.

(*) holds, for example, if L possesses an orthocomplementation, or even, in the case that L is complete, if L possesses an anti-automorphism which is an orthocomplementation on the centre of L (see Corollary 1 to Theorem 9.1).

We prove, generalizing (ii)′:

THEOREM 9.5. *An* $\aleph$*-complete complemented modular lattice is* $\aleph$*-continuous if it is finite and possesses an anti-automorphism* ϕ *of period two with the following continuity property*: (**) *for every limit ordinal number* $\Omega_1 \leqslant \Omega$, $x_\beta + \phi(x_\beta) = 1$, $x_\beta\phi(x_\beta) = 0$ *for all* $\beta < \Omega_1$ *and* $x_\beta \leqslant x_\gamma$ *for all* $\beta \leqslant \gamma < \Omega_1$ *together imply* $(\sum x_\beta) + \phi(\sum x_\beta) = 1$, $(\sum x_\beta)\phi(\sum x_\beta) = 0$.

Clearly every orthocomplementation ϕ has the property (**).

1.8. An alternative (but still lattice theoretic) proof of the Kaplansky's finiteness theorem for the *orthocomplemented* case (see (i) in § 1.7 above) is given in an Appendix. This Appendix can be read independently of the rest of this paper and it is somewhat related to Kaplansky's original method.

2. Preliminaries.[5]

2.1. Let L be a set of elements partially ordered by a relation $a \leqslant b$ (written also $b \geqslant a$). By definition, partial ordering means: $a \leqslant b$, $b \leqslant c$ imply $a \leqslant c$, and $a \leqslant b$, $b \leqslant a$ hold if and only if $a = b$ (that is, a and b are the same element).

When a_λ is in L for each $\lambda \in \Lambda$ we call a the *union* of the a_λ and write $a = \sum_{\lambda\epsilon\Lambda} a_\lambda$ (or $\sum a_\lambda$) if a is an element such that: $x \geqslant a_\lambda$ for every λ is equivalent to $x \geqslant a$. We call a the *meet* of the a_λ and write $a = \prod_{\lambda\epsilon\Lambda} a_\lambda$ (or $\prod a_\lambda$) if a is an element such that: $x \leqslant a_\lambda$ for every λ is equivalent to $x \leqslant a$ (each of union and meet is clearly unique if it exists at all).

The zero (unit) in L written as $0(1)$, is defined to be the element (if it exists) such that $0 \leqslant x (x \leqslant 1)$ holds for all x in L.

The dual to any statement or construction concerning elements of L is obtained by replacing $\leqslant$ by $\geqslant$; $\sum, \prod$ by $\prod, \sum$ respectively and 0, 1 by 1, 0, respectively. L' denotes the partially ordered set dual to L. Any theorem implies its dual.

[5]This section is mostly based on the original material of J. von Neumann (see (3) or (2)).

L is called complete if $\sum a_\lambda, \prod a_\lambda$ exist for all families $\{a_\lambda; \lambda \in \Lambda\}$; *$\aleph$-complete* if these elements exist whenever $\bar{\Lambda} \leqslant \aleph$;[6] a *lattice* if it is 2-complete (hence n-complete for every $n = 2, 3, \ldots$).

A lattice is called modular if $a(b + c) = b + ac$ whenever $a \geqslant b$ (equivalently, if: $a(b + c) = a(b(a + c) + c)$ for all a, b, c).

When $a \leqslant b$ we write $L(a, b)$ or $[a, b]$ to denote the sub-partially-ordered set of all x with $a \leqslant x \leqslant b$; clearly, it has a, b as zero and unit respectively.

2.2. *Let L be a lattice with zero element.* Elements a, b are called *disjoint* if $ab = 0$ ($\oplus$ shall mean $+$ but shall imply that the summands are disjoint).

If $a \leqslant c$, $[c - a]$ will denote any element A, to be called a *complement of a in c* (sometimes called a *relative complement* of a in c), for which $a \oplus A = c$. If L has a unit, $[1 - a]$ (if it exists) is called a *complement* of a.

L is called *complemented* if 0, 1 exist in L and every a has at least one complement. L is called *orthocomplemented* if 0, 1 exist in L and L possesses an anti-automorphism ϕ of period 2 with $\phi(a) \oplus a = 1$ for all a.

If L is complemented and modular, a relative complement $[c - a]$ exists always ($c[1 - a]$ will do); then, whenever $ab = 0$ there exists a complement A of a with $A \geqslant b$ (indeed, $b + [1 - (a + b)]$ will do for A).

If L is modular and A is a complement of a then $[0, a]$ and $[A, 1]$ are lattice isomorphic under the mutually inverse mappings:

$$a_1 \to a_1 + A \text{ if } a_1 \leqslant a; \quad A_1 \to aA_1 \text{ if } A_1 \geqslant A. \tag{2.1}$$

2.3. *Let L be a modular lattice with zero element.* The elements a and b are called *perspective* with axis x (we write $a \sim b$), if $a \oplus x = b \oplus x$; we may replace x by $x(a + b)$ to obtain $a \oplus x = b \oplus x = a + b$.

If a, b are perspective with axis x then $[0, a]$ and $[0, b]$ are lattice isomorphic under the mutually inverse perspective mappings:

$$a_1 \to (a_1 + x)b \text{ if } a_1 \leqslant a; \quad b_1 \to (b_1 + x)a \text{ if } b_1 \leqslant b$$

(clearly, $a_1 \sim b_1$ with the same axis x). We note:

$$a \sim c,\ c \sim b,\ (a + c)b = 0 \text{ imply } a \sim b; \tag{2.2}$$

for if $a \oplus x = x \oplus c = a + c$ and $c \oplus y = y \oplus b = c + b$ then

$$a \oplus (x + y)(a + b) = b \oplus (x + y)(a + b).$$

Elements a, b are called *projective* (we write $a \approx b$) if $a = a_1$ and $b = a_m$ for some finite family $a_1, \ldots, a_m$ with $a_i \sim a_{i+1}$ for $i < m$.

We shall say that an element a can be *doubled in L* if

$$a \sim u \text{ holds for some } u \text{ in } L \text{ with } ua = 0. \tag{2.3}$$

If a modular lattice L with zero has a unit, we shall say the lattice L *can*

[6]$\bar{\Lambda}$ denotes the cardinal power of Λ. For every family of lattice elements $\{a_\lambda; \lambda \in \Lambda\}$ which we consider, we shall suppose that $\bar{\Lambda} \leqslant \aleph$. Von Neumann defines L to be an $\aleph$-lattice if it is $\aleph_1$-complete for every $\aleph_1 < \aleph$ (see (**3**, Part III, Definition A. 1)).

be doubled if there exists a modular lattice L_1 with zero such that for some u in L_1, L is lattice isomorphic to $[0, u]$ and u can be doubled in L_1. Clearly if a can be doubled in L, the lattice $[0, a]$ can be doubled.

We write (as in von Neumann (**3**, Part II, Definition 3.4)), $(a, b, x)C$ to mean: $a \oplus x = b \oplus x = a \oplus b$.

We call a and b *completely disjoint* and write $(a, b)P$ to mean:

(2.4) $\quad a_1 \sim b_1,\ a_1 \leqslant a,\ b_1 \leqslant b$ together imply $a_1 = 0$.

Clearly $(a, b)P$ implies $ab = 0$.

We say:

(2.5) $\quad a$ is in the *centre* of L if $(a, b)P$ holds whenever $ab = 0$.

2.4. *Let L be a complemented modular lattice.* Then $(a, b)P$ holds if and only if:

(2.6) $\quad$ every complement of b contains a.

(Suppose (2.6) fails: if B is a complement of b with $B \geqslant a$ false, then $a_1 = [a - aB] \neq 0$ and $a_1 \sim (B + a_1)b$ (with axis B) so $(a, b)P$ does not hold. Suppose, on the other hand, (2.6) does hold: then if $a_1 \leqslant a$, $b_1 \leqslant b$, and $a_1 \sim b_1$ with axis x, we have in succession: $a_1 b \leqslant ab = 0$; $b(a_1 + b_1)x = 0$; there exists a complement B of b with $B \geqslant (a_1 + b_1)x$; $B \geqslant (a_1 + b_1)x + a_1$; $B \geqslant b_1$; $b_1 = 0$; $a_1 = 0$; hence $(a, b)P$ holds.)

(2.6) is also equivalent to: every complement of a contains b (consequently, a is in the centre of L if and only if it has a unique complement, necessarily also in the centre of L, and a is in the centre of L if and only if it is in the centre of L').

Hence in a complemented modular lattice, if $(a, b_\lambda)P$ holds for every λ, and $\sum b_\lambda$ exists, then every complement of a contains $\sum b_\lambda$ along with all b_λ so $(a, \sum b_\lambda)P$ holds; therefore, if $\sum a_\lambda$ and $\sum b_\mu$ both exist and $(\sum a_\lambda, \sum b_\mu)P$ is false, we must have $(a_\lambda, b_\mu)P$ false for some particular λ, μ.

Consequently, although this fact is not needed in the present paper, if b_λ are all in the centre of L then $\sum b_\lambda$, if it exists, is also in the centre of L and, by duality, $\prod b_\lambda$, if it exists, is also in the centre of L.

If, in a complemented modular lattice, $(a, b)P$ is false and $b \sim c$ then $(a, c)P$ is also false; this follows from:

(2.7) $\quad a \sim b,\ b \sim c,\ a \neq 0$ together imply $a_1 \sim c_1$ for some $a_1 \leqslant a$, $c_1 \leqslant c$ with $a_1 \neq 0$.

Clearly, we need prove (2.7) only for the case $ac = ba = bc = 0$. Because of (2.2), we may also suppose $a \leqslant b + c$, $c \leqslant a + b$.[7] Now it follows that $a \oplus b = c \oplus b$ so $a \sim c$ (axis b).

Hence, in a complemented modular lattice, $(a, b)P$ holds if and only if

[7]For example, if $a \leqslant b + c$ is false then $a' = [a - a(b + c)] \neq 0$, and $a' \sim b'$, $b' \sim c'$ for some $b' \leqslant b$ and $c' \leqslant c$. Since $a'(b' + c') \leqslant a'(b + c) = 0$, it follows from (2.2) that $a' \sim c'$.

$ac = 0$ whenever $c \sim b$. Indeed, $(a, b)P$ and $b \sim c$ imply $ac = 0$ by (2.7). On the other hand, if $ac = 0$ for all $c \sim b$ then $(a, b)P$ does hold; for then $ab = 0$, and if $a_1 \leqslant a$, $b_1 \leqslant b$ with $a_1 \sim b_1$ we have $b \sim (a_1 + [b - b_1])$ (use (2.2)); since $a(a_1 + [b - b_1]) = a_1$, then we must have $a_1 = 0$, proving $(a, b)P$ does hold.

If L is complete then for each element a there exists a central element $z \geqslant a$ (namely, $z = \sum a'$ for all a' perspective to subelements of a) such that:

(2.8) $(a, b)P$ holds if and only if $zb = 0$. This z is the least element in the centre with property $z \geqslant a$.

2.5. *Let L be an $\aleph$-complete lattice.* A family $\{a_\lambda; \lambda < \Omega_1\}$ with $\Omega_1 \leqslant \Omega$, either increasing, that is, $\lambda \leqslant \mu$ implies $a_\lambda \leqslant a_\mu$ (written $a_\lambda \uparrow a$ to denote also $a = \sum a_\lambda$) or decreasing, that is, $\lambda \leqslant \mu$ implies $a_\lambda \geqslant a_\mu$ (written $a_\lambda \downarrow a$ to denote also $a = \prod a_\lambda$) is said to *converge continuously* if for every x, $\sum(xa_\lambda) = xa$ or $\prod(x + a_\lambda) = x + a$, respectively; L is said to be *upper $\aleph$-continuous* or *lower $\aleph$-continuous* if every such increasing or decreasing family respectively, converges continuously (upper $\aleph$-continuity of L is clearly equivalent to lower $\aleph$-continuity of L'.[8]

If $a_\lambda \uparrow$ continuously, then for every c, $ca_\lambda \uparrow$ continuously; indeed, for every x, $x(\sum ca_\lambda) = cx(\sum a_\lambda) = \sum(cx)a_\lambda = \sum x(ca_\lambda)$.

L is called $\aleph$-continuous if it is both upper and lower $\aleph$-continuous.

2.6. *Let L be a complemented, modular, and $\aleph$-complete lattice.* If $\{a_\lambda\}$ is increasing or decreasing, then $\{a_\lambda\}$ does converge continuously if: $xa_\lambda = 0$ for every λ implies $x\sum a_\lambda = 0$ or if $x + a_\lambda = 1$ for every λ impiles $x + \prod a_\lambda = 1$, respectively. Also, L is upper $\aleph$-continuous if $a_\lambda \uparrow 1$ implies a_λ converges continuously, lower $\aleph$-continuous if $a_\lambda \downarrow 0$ implies a_λ converges continuously.

2.7. *Let L be an $\aleph$-complete lattice with zero.* L is called *locally $\aleph$-continuous* (*upper $\aleph$-continuous, lower $\aleph$-continuous*) if $a \neq 0$ implies $[0, a_1]$ is $\aleph$-continuous (upper $\aleph$-continuous, lower $\aleph$-continuous) for some $0 \neq a_1 \leqslant a$.

If L is also complemented and modular, then L is locally $\aleph$-continuous (upper $\aleph$-continuous, lower $\aleph$-continuous) if and only if the dual L' is locally $\aleph$-continuous (lower $\aleph$-continuous, upper $\aleph$-continuous); for if $A \neq 1$, let a be a complement of A. Then $a \neq 0$, and $[0, a_1]$ is $\aleph$-continuous (upper $\aleph$-continuous, lower $\aleph$-continuous) for some $0 \neq a_1 \leqslant a$. Let $A_1 = A + [a - a_1]$. Then $A \leqslant A_1 \neq 1$ and $[A_1, 1]$ is $\aleph$-continuous (upper $\aleph$-continuous, lower $\aleph$-continuous) by (2.1). This shows that L' is locally $\aleph$-continuous (lower $\aleph$-continuous, upper $\aleph$-continuous) since $\leqslant$ in L means $\geqslant$ in L'.

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[8]Transfinite induction shows that L is upper $\aleph$-continuous if and only if, for arbitrary $\{a_\lambda; \lambda \in \Lambda\}$ and for every x, $\Sigma_F(x\Sigma(a_\lambda; \lambda \in F))$ exists and equals $x\Sigma\, a_\lambda$ (F varies over all finite subsets of Λ), lower $\aleph$ -continuous if and only if, for arbitrary $\{a_\lambda; \lambda \in \Lambda\}$ and for every x, $\Pi_F(x + \Pi(a_\lambda; \lambda \in F))$ exists and equals $x + \Pi a_\lambda$.

An equivalent definition of continuity in terms of *directed* families $\{a_\lambda\}$ was given by U. Sasaki who used a lemma of T. Iwamura (see **(4)** or **(2**, Appendix II**)**).

If $[0, a_\lambda]$ is $\aleph$-continuous (upper $\aleph$-continuous, lower $\aleph$-continuous) for every λ then $[0, \sum a_\lambda]$ must be locally $\aleph$-continuous (upper $\aleph$-continuous, lower $\aleph$-continuous); for if $x \neq 0$ and $x \leqslant \sum a_\lambda$ then $(x, a_\lambda)P$ is false for some λ, so $[0, x_1]$ is lattice isomorphic to $[0, a_\lambda']$ for some $a_\lambda' \leqslant a_\lambda$ and some $x_1 \neq 0$ with $x_1 \leqslant x$. But $[0, a_\lambda']$ is $\aleph$-continuous (upper $\aleph$-continuous, lower $\aleph$-continuous) along with $[0, a_\lambda]$, so $[0, x_1]$ has the same property. This proves that $[0, \sum a_\lambda]$ is locally $\aleph$-continuous (upper $\aleph$-continuous, lower $\aleph$-continuous).

3. Independence theory. In this section we assume L is an $\aleph$-complete modular lattice with zero. Since we do not make any continuity assumptions we need to refine the von Neumann independence theory.[9] in so far as it applies to infinite families of elements. In particular, in Theorem 3.1 below, we use a complementation argument to replace the usual "continuity" argument.

If $\{a_\lambda; \lambda \in \Lambda\}$ is a set of elements in L we use the following notation:

$$a_\lambda^* \text{ denotes } \textstyle\sum(a_\mu; \mu \neq \lambda),$$

$$a_\Gamma \text{ denotes } \textstyle\sum(a_\lambda; \lambda \in \Gamma) \quad \text{if} \quad \Gamma \subset \Lambda,$$

$$a_\Gamma^* \text{ denotes } \textstyle\sum(a_\lambda; \lambda \notin \Gamma) \quad \text{if} \quad \Gamma \subset \Lambda \text{ (in particular, } a_\Lambda^* = 0).$$

Definition 3.1. A family $\{a_\lambda; \lambda \in \Lambda\}$ is called *independent* if $a_\lambda a_F = 0$ whenever F is a finite subset of Λ and $\lambda \notin F$; *residually independent* if $a_\lambda a_\lambda^* = 0$ for every λ.

Definition 3.2. If $\{a_\lambda\}$ is residually independent the residual element of $\{a_\lambda\}$ is defined to be $\prod a_\lambda^*$; an element x is called a *residual element in* L (more precisely, *an* $\aleph$*-residual element in* L) if x is the residual element of some residually independent family $\{a_\lambda; \lambda \in \Lambda\}$ with $\bar{\Lambda} = \aleph$.

If $\{a_\lambda\}$ is residually independent with residual element 0 then $\{a_\lambda\}$ is called *strongly independent.*

Because of the modular law, the following statements follow easily:

Independence of $\{a_\lambda\}$ is equivalent to: $a_F a_G = 0$ whenever F, G are finite, disjoint subsets of Λ, and also to:

$$\textstyle\prod_\tau \sum(a_\lambda; \lambda \in F_\tau) = \sum(a_\lambda; \lambda \in \bigcap F_\tau)$$

for every *finite* collection of finite subsets F_τ of Λ.

Residual independence implies independence and is equivalent to: $a_F a_F^* = 0$ for every finite subset F of Λ.

Strong independence implies residual independence and is equivalent to the single condition $\Pi a^* = 0$. (It will follow from Theorem 3.1 below that strong independence of $\{a_\lambda\}$ is equivalent to: for every collection of subsets I_γ of Λ, $\prod_\gamma \sum(a_\lambda; \lambda \in I_\gamma)$ exists and equals $\sum(a_\lambda; \lambda \in \bigcap I_\gamma)$.)

[9]The von Neumann theory of independence can be found in (**3**, Part I, chapter II).

If $a_i(a_1 + \ldots + a_{i-1}) = 0$ for $i \geqslant 2$, then the finite or infinite family $\{a_n; n \geqslant 1\}$ is independent. Hence:

if $\{b_1, \ldots, b_m, a_1, \ldots, a_r\}$ *is independent and* $c_1 + \ldots + c_p \leqslant b_1 + \ldots + b_m$ *and* $\{c_1, \ldots, c_p\}$ *is independent, then* $\{c_1, \ldots, c_p, a_1, \ldots, a_r\}$ *is independent*; a generalization of this fact is proved in the Corollary to Theorem 3.1.

If $b_\lambda \leqslant a_\lambda$ for each λ and $\{a_\lambda\}$ is independent (residually independent, strongly independent), then $\{b_\lambda\}$ is independent (residually independent, strongly independent).

If $\{b_\lambda, c_\lambda; \lambda \in \Lambda\}$ is independent (residually independent, strongly independent) then $\{b_\lambda + c_\lambda\}$ is independent (residually independent, strongly independent).

$\{a_n; n = 1, 2, \ldots\}$ is residually independent if and only if $a_n\sum(a_m; m > n) = 0$ for every $n = 1, 2, \ldots$, strongly independent if and only if residually independent with $\prod_n(\sum(a_m; m \geqslant n)) = 0$. If $\{a_n\}$ is strongly independent, then

$$c_n \leqslant \sum_{i=n}^{\infty} a_i,\ c_n\left(\sum_{i=n+1}^{\infty} a_i\right) = 0 \tag{3.1}$$

for every n imply $\{c_n\}$ is strongly independent.

If $\{a_\lambda\}$,$\{b_\lambda\}$ are both independent (residually independent) and $(\sum a_\lambda)(\sum b_\lambda) = 0$ then $\{a_\lambda + b_\lambda; \lambda \in \Lambda\}$ and $\{a_\lambda, b_\lambda; \lambda \in \Lambda\}$ are both independent (residually independent) (the Corollary to Theorem 3.1 below shows that if $\{a_\lambda\}$,$\{b_\lambda\}$ are both strongly independent with $(\sum a_\lambda)(\sum b_\lambda) = 0$ then $\{a_\lambda + b_\lambda; \lambda \in \Lambda\}$ and $\{a_\lambda, b_\lambda; \lambda \in \Lambda\}$ are both strongly independent).

If L is upper $\aleph$-continuous then independence implies strong independence for families $\{a_\lambda; \lambda \in \Lambda\}$ with $\bar{\Lambda} \leqslant \aleph$ (this was shown first by von Neumann (**3**, Part I, Chapter II)).

THEOREM 3.1. *Suppose* $\{a_\lambda\}$ *is strongly independent and for an arbitrary set of* μ, $a_{\lambda,\mu} \leqslant a_\lambda$ *for all* λ, μ. *Then* $\prod_\mu a_{\lambda,\mu}$ *exists for each* λ *if* $\prod_\mu(\sum_\lambda a_{\lambda,\mu})$ *exists. On the other hand,*

$$\prod_\mu(\textstyle\sum_\lambda a_{\lambda,\mu}) = \sum_\lambda(\prod_\mu a_{\lambda,\mu})$$

(*that is, both sides exist and are equal*) *provided that for each* λ, *the element* $\prod_\mu a_{\lambda,\mu}$ *exists and has a complement in* a_λ (*in particular, if* $\prod_\mu a_{\lambda,\mu} = 0$ *for every* λ).

Proof. 1. If $\prod_\mu a_{\lambda,\mu}$ exists let it be denoted as b_λ.

Clearly, if $\prod_\mu(\sum_\lambda a_{\lambda,\mu})$ exists, then for each ν,

$$a_\nu\prod_\mu(\textstyle\sum_\lambda a_{\lambda,\mu}) = \prod_\mu(a_{\nu,\mu} + 0)$$

since $\{a_\lambda\}$ is strongly independent, so b_ν exists for each ν.

2. Clearly $\sum_\lambda a_{\lambda,\mu} \geqslant \sum_\lambda b_\lambda$ for every μ. So we need only show: if for some x, $\sum_\lambda a_{\lambda,\mu} \geqslant x$ for all μ, then $x \leqslant \sum_\lambda b_\lambda$.

But if for all μ, $\sum_\lambda a_{\lambda,\mu} \geqslant x$ then for all λ, μ,

$$x \leqslant a_{\lambda,\mu} + a_\lambda^*,$$
$$(x + a_\lambda^*)a_\lambda \leqslant (a_{\lambda,\mu} + a_\lambda^*)a_\lambda = a_{\lambda,\mu};$$

then for every λ,

$$(x + a_\lambda^*)a_\lambda \leqslant \prod_\mu a_{\lambda,\mu} = b_\lambda,$$
$$x \leqslant x + a_\lambda^* \leqslant b_\lambda + a_\lambda^*;$$

finally,

$$x \leqslant \prod_\lambda (b_\lambda + a_\lambda^*).$$

Thus the theorem will be completely proved if we establish

$$\prod_\lambda (b_\lambda + a_\lambda^*) = \sum_\lambda b_\lambda.$$

3. Now suppose $c_\lambda \oplus b_\lambda = a_\lambda$ for each λ. Then

$$\sum b_\lambda + \sum c_\lambda \geqslant \prod (b_\lambda + a_\lambda^*) \geqslant \sum b_\lambda,$$

and the modular law now shows that we need only prove

$$(\prod (b_\lambda + a_\lambda^*))(\sum c_\lambda) = 0.$$

And this does hold because $(b_\lambda + a^*_\lambda)(\sum c_\lambda) \leqslant a^*_\lambda$ for each λ, and $\prod a^*_\lambda = 0$.

COROLLARY. (*Strong independence under substitution*). *Suppose L is an $\aleph$-complete modular lattice with zero, $\{a_\lambda; \lambda \in \Lambda\}$ is strongly independent and $\Gamma, \Delta, \ldots$ are mutually disjoint subsets of Λ. If $\{c_\mu\}, \{d_\nu\}, \ldots$ are each strongly independent with sets of indices $\mu, \nu, \ldots$ each of cardinal power $\leqslant \aleph$ and $a_\Gamma \geqslant \sum c_\mu$, $a_\Delta \geqslant \sum d_\nu, \ldots$, then the set of all elements* {all c_μ, all $d_\nu, \ldots$} *is strongly independent.*

Proof. Since $a^*_\Gamma \leqslant \prod (a^*_\lambda; \lambda \in \Gamma)$ the meet $a^*_\Gamma a^*_\Delta \ldots \leqslant \prod (a^*_\lambda; \lambda \in \Lambda) = 0$. So $\{a_\Gamma, a_\Delta, \ldots\}$ is strongly independent.

To prove {all c_μ, all $d_\nu, \ldots$} strongly independent we form the union $\sum^*$ of *all c_μ, all d_ν omitting one of these elements* and we need only prove that all such $\sum^*$ have 0 as meet.

But if c_μ is omitted, $\sum^* = c^*_\mu + a^*_\Gamma$. Then Theorem 3.1, applied to the family $\{a_\Gamma, a_\Delta, \ldots\}$ shows that

$$\prod \sum^* = \prod_\mu c_\mu^* + \prod_\nu d_\nu^* + \ldots = 0 + 0 + \ldots = 0.$$

Remark. In the case that L is *complemented* Theorem 3.1 *is equivalent to the statement:* if $\{a_\lambda\}$ is strongly independent, then the set L_0 of all $\sum x_\lambda$ with $x_\lambda \leqslant a_\lambda$, is a sublattice of L, isomorphic by the correspondence $\sum x_\lambda \leftrightarrow \{x_\lambda\}$ to the direct product of the lattices $[0, a_\lambda]$; L_0 has the property that if *any* family of elements in L_0 has a union or meet in L, then this union or meet is in L_0.

At this point we introduce an important generalization of the conjointness relationship "$(a, b, c)C$" of von Neumann (see § 2).

Definition 3.3. A family of ordered triplets $\{(x_\lambda, c_\lambda, b_\lambda), \lambda \in \Lambda\}$ is called a *C-system* (more precisely, an $\aleph$ *C-system* if $\bar{\Lambda} = \aleph$, and sometimes a *C-sequence* if Λ is countable) if:

(i) $(x_\lambda, c_\lambda, b_\lambda)C$ for every λ, as defined in § 2;

(ii) $\{\sum x_\mu, b_\lambda; \lambda \in \Lambda\}$ is strongly independent.

We shall write: $\{(x_\lambda, c_\lambda, b_\lambda); \lambda \in \Lambda\}C$ to denote that $\{(x_\lambda, c_\lambda, b_\lambda); \lambda \in \Lambda\}$ is a C-system.

Clearly, if x denotes $\sum x_\lambda$ and $\{(x_\lambda, c_\lambda, b_\lambda)\}C$, then:

(iii) $x_\lambda = x(c_\lambda + b_\lambda)$ for every λ,

(iv) $x \oplus c_\lambda = x \oplus b_\lambda$ for every λ,

(v) $x \leqslant \sum c_\lambda + \sum b_\lambda$.

Conversely, if some given x, $\{b_\lambda\}$, $\{c_\lambda\}$ satisfy (iv), (v), and

(ii)′ $\{x, b_\lambda; \lambda \in \Lambda\}$ is strongly independent then, with x_λ defined by (iii), it is so that $\{(x_\lambda, c_\lambda, b_\lambda)\}C$ holds.

Thus in a C-system the x_λ are uniquely determined by the elements $\{c_\lambda, b_\lambda; \lambda \in \Lambda\}$ and the union $\sum x_\lambda$. We shall sometimes write $\{(x|c_\lambda, b_\lambda)\}C$ with $x = \sum x_\lambda$ in place of $\{(x_\lambda, c_\lambda, b_\lambda)\}C$.

LEMMA 3.1. *If $\{(x|c_\lambda, b_\lambda)\}C$ holds, then $\{c_\lambda\}$ is residually independent and has residual element $x\sum c_\lambda$.*

Proof.

$$c_\lambda^* \leqslant b_\lambda^* + x.$$

Hence, (by (iv), Definition 3.3),

$$c_\lambda c_\lambda^* \leqslant c_\lambda(b_\lambda + x)(b_\lambda^* + x) = c_\lambda x$$

and (by (iii), Definition 3.3),

$$c_\lambda x = c_\lambda x_\lambda = 0.$$

Thus for each λ, $c_\lambda c^*_\lambda = 0$ and hence $\{c_\lambda\}$ is residually independent. Next,

$$x\sum c_\lambda = x(c_\mu^* + c_\mu)(c_\mu^* + x) = x(c_\mu^* + c_\mu(c_\mu^* + x)) = x(c_\mu^* + c_\mu(b_\mu^* + x)).$$

But $\{b_\mu, b^*_\mu, x\}$ is independent, so, by the Corollary to Theorem 3.1, $\{c_\mu, b^*_\mu, x\}$ is independent; thus $c_\mu(b^*_\mu + x) = 0$ and

$$x\sum c_\lambda = x\, c_\mu^* \qquad \text{for each } \mu.$$

Thus $x\sum c_\lambda = x\prod_\mu c^*_\mu \leqslant$ (residual element of $\{c_\lambda\}$). On the other hand,

$$(\text{residual element of } \{c_\lambda\}) = \prod c_\lambda^* \leqslant \prod(b_\lambda^* + x) = x$$

since $\{x, b_\lambda; \lambda \in \Lambda\}$ is strongly independent. Thus

$$(\text{residual element of } \{c_\lambda\}) \leqslant (\sum c_\lambda)x$$

and so equality holds.

Definition 3.4. A C-system $\{(x_\lambda, c_\lambda, b_\lambda)\}$ is called a *residual C-system*, if

$$\sum x_\lambda = (\text{residual element of } \{c_\lambda\})$$

equivalently (by Lemma 3.1), if $\sum x_\lambda \leqslant \sum c_\lambda$.

Remark. It is easy to see that a residual C-system with Λ *finite* must have all $x_\lambda, c_\lambda, b_\lambda$ identically 0. But a non-trivial residual C-system can be constructed whenever there exists an increasing sequence $\{a_n\}$ which does not converge continuously (this will follow immediately from Theorem 3.6 and the Corollary to Theorem 3.2 below).

THEOREM 3.2. *If $\{a_\lambda\}$ is residually independent and $a_\lambda \geqslant b_\lambda \oplus c_\lambda$ for every λ, then the residual elements a, b, c* (of $\{a_\lambda\}, \{b_\lambda\}, \{c_\lambda\}$ *respectively*) *satisfy:*

(i) $(\sum b_\lambda)(\sum c_\lambda) = bc$;

(ii) $a \geqslant b + c$ *with equality if $a_\lambda = b_\lambda \oplus c_\lambda$ for every* λ.

Proof. Each of $\{b_\lambda\}, \{c_\lambda\}$ is residually independent along with $\{a_\lambda\}$. Now:

(i) For each fixed μ, $\{a^*_\mu, b_\mu, c_\mu\}$ is independent. Hence

$$(\textstyle\sum b_\lambda)(\sum c_\lambda) = b^*_\mu c^*_\mu;\ (\sum b_\lambda)(\sum c_\lambda) = (\prod_\mu b^*_\mu)(\prod_\mu c^*_\mu) = bc.$$

(ii) $a \geqslant b + c$ is clear. But if $a_\lambda = b_\lambda \oplus c_\lambda$ for every λ then also $a \leqslant b + c$ for:

$$(a + \textstyle\sum b_\lambda)(\sum c_\lambda) \leqslant (b_\mu + a^*_\mu)(\sum c_\lambda) = c^*_\mu.$$

Hence

$$c \geqslant (a + \textstyle\sum b_\lambda)(\sum c_\lambda),$$

$$(\textstyle\sum b_\lambda) + c \geqslant (a + \sum b_\lambda)(\sum c_\lambda + \sum b_\lambda) \geqslant a.$$

Similarly $(\sum c_\lambda) + b \geqslant a$. Thus

$$a \leqslant ((\textstyle\sum b_\lambda) + c)((\sum c_\lambda) + b) = b + c + (\sum b_\lambda)(\sum c_\lambda) = b + c.$$

COROLLARY. *If L is complemented and $\{a_\lambda\}$ is residually independent with residual element x, there exists a residual C-system* $\{(x|c_\lambda, b_\lambda)\}$ with $c_\lambda \leqslant a_\lambda$.

Proof. 1. Choose X to be a complement of x and define:

$$c_\lambda = [a_\lambda - a_\lambda X],$$

$$b_\lambda = (x + c_\lambda)X.$$

2. Then $a_\lambda = a_\lambda X \oplus c_\lambda$ and each of $\{a_\lambda X\}, \{c_\lambda\}$ is residually independent since $\{a_\lambda\}$ is residually independent by hypothesis.

But

$$(\text{residual element of } \{a_\lambda X\}) \leqslant (\text{residual element of } \{a_\lambda\}) = x;$$

also $\leqslant X$, so $\leqslant xX = 0$. Now Theorem 3.2 shows that $\{c_\lambda\}$ is residually independent with x as residual element.

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3. $\{x, b_\lambda; \lambda \in \Lambda\}$ is strongly independent because

$$b_\lambda^* \leqslant x + c_\lambda^* = c_\lambda^*,$$
$$\prod b_\lambda^* \leqslant \prod c_\lambda^* = x.$$

But $\prod b^*_\lambda \leqslant X$, hence $\prod b^*_\lambda \leqslant xX = 0$. Therefore $\{b_\lambda\}$ is strongly independent.

Since $x\sum b_\lambda \leqslant xX = 0$, the Corollary to Theorem 3.1 shows that $\{x, b_\lambda; \lambda \in \Lambda\}$ is strongly independent.

4. $c_\lambda x \leqslant a_\lambda x = 0$, $b_\lambda x = 0$ and

$$b_\lambda \oplus x = (x \oplus c_\lambda)(X + x) = x \oplus c_\lambda.$$

Finally, $\sum b_\lambda + \sum c_\lambda \geqslant x$ since, as we have already shown, $\{c_\lambda\}$ is residually independent with x as residual element. Thus (iv), (v), and (ii)$'$ of Definition 3.3. hold and it follows that $\{(x|c_\lambda, b_\lambda)\}$ is a residual C-system, as required.

THEOREM 3.3. *Suppose $\{a_\lambda\}$ is residually independent with residual element t. If $t = x \oplus y$ and Y is an element with $Y \geqslant x$ and $Y \oplus y \geqslant \sum a_\lambda$, then*

$$\{Y(a_\lambda + y)\}$$

is residually independent with residual element x.

Remark. If L is complemented, Y could be chosen to be $x + [(\sum a_\lambda) - t]$.

Proof. Put $b_\lambda = Y(a_\lambda + y)$. Then

$$b_\lambda + y = a_\lambda + y,\ b_\lambda^* + y = a_\lambda^* + y = a_\lambda^*,$$
$$b_\lambda b_\lambda^* \leqslant Ya_\lambda^*(a_\lambda + y) = Y(a_\lambda^* a_\lambda + y) = Yy = 0.$$

Thus $\{b_\lambda\}$ is residually independent.

Now $b^*_\lambda \leqslant Y$, and $Yy = 0$, so $b^*_\lambda = Y(b^*_\lambda + y) = Ya^*_\lambda$. Hence

$$(\text{the residual element of } \{b_\lambda\}) = \prod(Ya^*_\lambda) = Yt = x + Yy = x.$$

COROLLARY 1. *If L is complemented then every subelement of an $\aleph$-residual element is also an $\aleph$-residual element.*

COROLLARY 2. *If L is complemented and $\{a_\lambda\}$ is residually independent there exists a strongly independent family $\{b_\lambda\}$ with $\sum b_\lambda \leqslant \sum a_\lambda$ and b_λ perspective to a_λ for every λ, with a common axis of perspectivity.*

Proof. Let t be the residual element of $\{a_\lambda\}$ and choose $Y = [(\sum a_\lambda) - t]$, that is, let $x = 0$, $y = t$ in Theorem 3.3. Then $b_\lambda = Y(a_\lambda + y)$ satisfies our requirements and for every λ, b_λ is perspective to a_λ with axis Y.

THEOREM 3.4. *Additivity of perspectivity. Suppose $\{a_\lambda + b_\lambda; \lambda \in \Lambda\}$ is residually independent and $a_\lambda \sim b_\lambda$ for every λ. If $\{a_\lambda\}$ and $\{b_\lambda\}$ are both strongly independent (in particular, if $\{a_\lambda + b_\lambda\}$ is strongly independent),*[10] *then there*

[10]Strong independence of $\{a_\lambda + b_\lambda\}$ obviously implies that of each of $\{a_\lambda\}$, $\{b_\lambda\}$. The interested reader can verify, using Theorem 3.1 and Theorem 3.2, that if L is complemented then residual independence of $\{a_\lambda + b_\lambda\}$ together with strong independence of each of $\{a_\lambda\}$, $\{b_\lambda\}$, actually forces $\{a_\lambda + b_\lambda\}$ to be strongly independent.

exists a perspective mapping of $[0, \sum b_\lambda]$ *onto* $[0, \sum a_\lambda]$ *which maps* b_λ *on* a_λ *for each* λ. *If* $\{b_\lambda\}$ *is strongly independent and* L *is complemented then* $\sum b_\lambda$ *is perspective to a subelement of* $\sum a_\lambda$.

Proof. Suppose $a_\lambda \oplus x_\lambda = b_\lambda \oplus x_\lambda$ and $\{a_\lambda\}$, $\{b_\lambda\}$ are both residually independent. Then for every fixed μ, $(a_\mu + b_\mu)(a^*{}_\mu + b^*{}_\mu) = 0$. Hence

$$(\textstyle\sum x_\lambda)(\sum b_\lambda) = x_\mu b_\mu + x^*_\mu b^*_\mu = 0 + x^*_\mu b^*_\mu \leqslant b^*_\mu;$$

$$(\textstyle\sum x_\lambda)(\sum b_\lambda) \leqslant (\text{residual element of } \{b_\lambda\}).$$

Similarly,

$$(\textstyle\sum x_\lambda)(\sum a_\lambda) \leqslant (\text{residual element of } \{a_\lambda\}).$$

Thus if $\{b_\lambda\}$ is strongly independent, $\sum b_\lambda$ is perspective to $[\sum a_\lambda - (\sum x_\lambda)(\sum a_\lambda)]$ with axis $\sum x_\lambda$. If $\{a_\lambda\}$ is also strongly independent then $(\sum x_\lambda)(\sum a_\lambda) = 0$; in this case $(\sum b_\lambda) \sim (\sum a_\lambda)$ with axis $\sum x_\lambda$, and the corresponding perspective mapping maps b_λ on a_λ for each λ.

Remark. If $(\sum b_\lambda)(\sum a_\lambda) = 0$ and $\{b_\lambda\}$ is strongly independent, then residual independence of $\{a_\lambda + b_\lambda\}$ is equivalent to residual independence of $\{a_\lambda\}$, by application of Theorem 3.1.

COROLLARY. *Suppose* $\{a_\lambda\}$, $\{b_\lambda\}$ *are both strongly independent families.*

(i) *If* $a_\lambda \sim b_\lambda$ *for each* λ *and* $(\sum a_\lambda)(\sum b_\lambda) = 0$ *then there is a perspective mapping of* $[0, \sum a_\lambda]$ *onto* $[0, \sum b_\lambda]$ *which maps* a_λ *on* b_λ *for each* λ.

(ii) *If* $a_\lambda \approx b_\lambda$ *and* L *can be doubled then there is a lattice isomorphism of* $[0, \sum a_\lambda]$ *onto* $[0, \sum b_\lambda]$ *which maps* a_λ *on* b_λ *for each* λ; *if also* $\sum b_\lambda \leqslant \sum a_\lambda$ *and* L *is complemented then* $[\sum a_\lambda - \sum b_\lambda]$ *is a member of an independent sequence of mutually perspective elements.*

Proof of (i): Theorem 3.4 shows this since $\{a_\lambda + b_\lambda\}$ is strongly independent by the Corollary to Theorem 3.1, under the present hypotheses.

Proof of (ii): We may suppose that $L = [0, c]$ with c an element in a modular lattice L_1 such that $[0, c]$ can be mapped by a perspective mapping ϕ onto $[0, u]$ for some u in L_1 with $cu = 0$. Then $a_\lambda \sim \phi(b_\lambda)$ for each λ (by repeated applications of (2.2)).

Since $(\sum a_\lambda)(\sum \phi(b_\lambda)) = 0$, there exists, by (i) above, a perspective mapping ψ of $[0, \sum a_\lambda]$ onto $[0, \sum \phi(b_\lambda)]$ which maps each a_λ on $\phi(b_\lambda)$.

Now $\phi^{-1}\psi$ is a lattice isomorphism of $[0, \sum a_\lambda]$ onto $[0, \sum b_\lambda]$ as required.

If finally $\sum b_\lambda \leqslant \sum a_\lambda$ and L is complemented, let $x_1 = [(\sum a_\lambda) - (\sum b_\lambda)]$; and for $n \geqslant 1$ define x_{n+1} by induction:

$$x_{n+1} = \phi^{-1}\psi(x_n).$$

Then $\{x_n\}$ is an independent sequence since $x_n(\sum(x_m; m > n)) = 0$ (this follows from repeated applications of $\phi^{-1}\psi$ to the identity $x_1(\sum(x_m; m > 1)) \leqslant x_1(\sum b_\lambda) = 0$).

Since $x_n \sim \phi(x_n)$ and $\phi(x_n) \sim x_{n+1}$ and $\{x_n, \phi(x_n), x_{n+1}\}$ is independent for each n, therefore $x_n \sim x_{n+1}$. Then by (2.2), $x_n \sim x_m$ for all n, m. This proves (ii).

THEOREM 3.5. *Extension of perspective mapping.*

Suppose $\{(x_\lambda, c_\lambda, b_\lambda)\}$ *and* $\{(x_\lambda', c_\lambda', b_\lambda')\}$ *are both C-systems and* $\sum x_\lambda \leqslant x$, $\sum x_\lambda' \leqslant x'$ *and* $\{x + x', \sum b_\lambda, \sum b_\lambda'\}$ *is independent. Then any perspective mapping of* $[0, x]$ *onto* $[0, x']$ *which maps* x_λ *on* x_λ' *for every* λ, *can be extended to a perspective mapping of* $[0, x + \sum b_\lambda]$ *onto* $[0, x' + \sum b_\lambda']$ *which maps* b_λ *on* b_λ' *and* c_λ *on* c_λ' *for every* λ.

Proof. 1 The given perspective mapping of $[0, x]$ onto $[0, x']$ is determined by some axis of perspectivity a with:

$$x \oplus a = x' \oplus a = x + x'.$$

2. We shall choose y_λ below so that

$$y_\lambda \oplus b_\lambda = y_\lambda \oplus b_\lambda' = b_\lambda \oplus b_\lambda'; \tag{3.2}$$

it will then follow immediately, as in the proof of Theorem 3.4, that the axis $a + \sum y_\lambda$ gives a perspective mapping of $[0, x + \sum b_\lambda]$ onto $[0, x' + \sum b_\lambda']$ which fulfills all our requirements except possibly for the requirement:

(3.3) c_λ should be mapped onto c_λ' for each λ.

3. Our choice of y_λ is:

$$y_\lambda = (a + c_\lambda + c_\lambda')(b_\lambda + b_\lambda')$$

and we verify that (3.2) holds, as follows:

(i) $$y_\lambda + b_\lambda = (a + b_\lambda + c_\lambda + c_\lambda')(b_\lambda + b_\lambda')$$

and

$$a + b_\lambda + c_\lambda + c_\lambda' = a + x_\lambda + b_\lambda + c_\lambda' = a + x_\lambda' + b_\lambda + c_\lambda' = a + x_\lambda' + b_\lambda + b_\lambda'$$

so $y_\lambda + b_\lambda = b_\lambda + b_\lambda'$.

Similarly $y_\lambda + b_\lambda' = b_\lambda + b_\lambda'$ so $y_\lambda + b_\lambda = y_\lambda + b_\lambda' = b_\lambda + b_\lambda'$.

(ii) The hypotheses imply that $\{b_\lambda, b_\lambda', x + x'\}$ is an independent family for each λ. Now successive applications of the Corollary to Theorem 3.1 show in turn that each of the following is independent:

$$\{b_\lambda, b_\lambda', x_\lambda', a\}, \qquad \{b_\lambda, c_\lambda', x_\lambda', a\},$$
$$\{b_\lambda, c_\lambda', x_\lambda, a\}, \qquad \{b_\lambda, c_\lambda', c_\lambda, a\}.$$

Therefore

$$b_\lambda y_\lambda = (a + c_\lambda + c_\lambda')b_\lambda = 0.$$

Similarly

$$b_\lambda y_\lambda = 0.$$

(i) and (ii) prove that (3.2) holds.

4. Finally, we verify that $a + \sum y_\lambda$ does satisfy (3.3), as follows:

$$\begin{aligned}(\text{the map of } c_\lambda) &= (a + \textstyle\sum y_\mu + c_\lambda)(x' + \sum b'_\mu) \\ &\geqslant (a + y_\lambda + c_\lambda)c'_\lambda = (a + c_\lambda + c'_\lambda)(b_\lambda + b'_\lambda + a + c_\lambda)c'_\lambda \\ &\geqslant (b'_\lambda + x_\lambda + a)c'_\lambda \geqslant (b'_\lambda + x'_\lambda)c'_\lambda = c'_\lambda,\end{aligned}$$

that is, (map of c_λ) $\geqslant c_\lambda'$. Similarly: (map of c_λ') $\geqslant c_\lambda$. Since the mappings are inverse perspective mappings, equality must then hold in the preceding two relations and the theorem is completely proved.

THEOREM 3.6. *Suppose L is complemented and $\aleph'$-continuous for every $\aleph' < \aleph$. Suppose also that $\{c_\beta; \beta < \Omega\}$ is an increasing family with $xc_\beta = 0$ for every $\beta < \Omega$ for some fixed x with $x \leqslant \sum c_\beta$. Then there exists a residually independent family $\{a_\beta; \beta < \Omega\}$ such that:*

$$\sum_{\gamma \leqslant \beta} a_\gamma = c_\beta \qquad \text{for ever } \beta < \Omega, \tag{3.4}$$

(3.5) *the residual element of $\{a_\beta\}$ is $\geqslant x$.*

Proof. By transfinite induction we shall define for each $\beta < \Omega$ a complement C_β of c_β such that $C_\beta \geqslant x$ and $C_\gamma \geqslant C_\beta$ for all $\gamma \leqslant \beta$.

We choose C_1 to be any complement of c_1 with $C_1 \geqslant x$. Then for $\beta > 1$, by transfinite induction, we choose C_β to be a relative complement $[\prod_{\delta<\beta} C_\delta - c_\beta(\prod_{\delta<\beta} C_\delta)]$ with $C_\beta \geqslant x$. This is possible since, by the inductive assumption, $\prod_{\delta<\beta} C_\delta \geqslant x$ and $xc_\beta(\prod_{\delta<\beta} C_\delta) \leqslant xc_\beta = 0$; this choice of C_β does give a complement of c_β because

$$\Big(\prod_{\delta<\beta} C_\delta\Big) + c_\beta = \prod_{\delta<\beta} (C_\delta + c_\beta) = \prod_{\delta<\beta} (1) = 1$$

due to the assumption that L is lower $\aleph'$-continuous for $\aleph' < \aleph$.

Now choose $a_1 = c_1$, and for $1 < \beta < \Omega$, choose $a_\beta = c_\beta(\prod_{\delta<\beta} C_\delta)$.

Then (3.4) holds; for by transfinite induction on β, it follows that for every β:

$$\Big(\sum_{\gamma<\beta} c_\gamma + \prod_{\delta<\beta} C_\delta\Big) = \prod_{\delta<\beta} \Big(\sum_{\gamma<\beta} c_\gamma + C_\delta\Big) = \prod_{\delta<\beta} (1) = 1;$$

$$c_\beta = c_\beta\Big(\sum_{\gamma<\beta} c_\gamma + \prod_{\delta<\beta} C_\delta\Big) = \sum_{\gamma<\beta} c_\gamma + c_\beta\Big(\prod_{\delta<\beta} C_\delta\Big) = \sum_{\gamma<\beta} a_\gamma + a_\beta = \sum_{\gamma\leqslant\beta} a_\beta.$$

Next, $\{a_\beta; \beta < \Omega\}$ is residually independent; for

$$\begin{aligned}a_\beta(\textstyle\sum(a_\gamma; \gamma \neq \beta)) &= a_\beta c_\beta(\textstyle\sum(a_\gamma; \gamma \neq \beta)) \\ &= a_\beta(\textstyle\sum(a_\gamma; \gamma < \beta) + c_\beta(\sum(a_\gamma; \gamma > \beta))C_\beta) \\ &= a_\beta(\textstyle\sum(a_\gamma; \gamma < \beta)) = 0\end{aligned}$$

since L is upper $\aleph'$-continuous for $\aleph' < \aleph$.

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Finally, for each $\gamma < \Omega$,

$$x = x(\sum c_\beta) \leqslant C_\gamma(\sum a_\beta) = \sum (a_\beta; \beta > \gamma)$$

so (3.5) holds.

COROLLARY 1. *Suppose L is a complemented, $\aleph_0$-complete modular lattice. Then L is upper $\aleph_0$-continuous if and only if every residually independent sequence is strongly independent. More generally, if L is a complemented $\aleph$-complete modular lattice, and L is $\aleph'$-continuous for all $\aleph' < \aleph$, then L is upper $\aleph$-continuous if and only if every $\aleph$-residual element is* 0.[11]

COROLLARY 2. *Suppose L is complemented and $\aleph'$-continuous for every $\aleph' < \aleph$. If $\{a_\lambda\}$ is independent then there exists a strongly independent family $\{b_\lambda\}$ such that $a_\lambda \approx b_\lambda$ for each λ.*

Proof. We may suppose the a_λ are well-ordered and indexed as $\{a_\beta; \beta < \Omega\}$. Let $c_\beta = \sum (a_\gamma; \gamma \leqslant \beta)$. Then since $\{a_\beta\}$ is independent and L is upper $\aleph'$-continuous for every $\aleph' < \aleph$ it follows that for every $\beta < \Omega$,

$$a_\beta \sum (c_\gamma; \gamma < \beta) = 0, \qquad a_\beta + \sum (c_\gamma; \gamma < \beta) = c_\beta.$$

Now apply Theorem 3.6 (with $x = 0$) to the increasing family $\{c_\beta\}$; it follows that there exists a residually independent family $\{a_\beta'; \beta < \Omega\}$ with

$$\sum (a_\gamma'; \gamma \leqslant \beta) = c_\beta \qquad \text{for every } \beta < \Omega.$$

Clearly for every β, $a_\beta \sim a_\beta'$ with axis $\sum (c_\gamma; \gamma < \beta)$.

Now let the residual element of $\{a_\beta'\}$ be denoted as y and let Y be a complement of y. Then by Theorem 3.3 the elements $b_\beta = Y(a_\beta' + y)$ form a strongly independent family.

Since $b_\beta \sim a_\beta'$ (with axis y) for each β, and $a_\beta' \sim a_\beta$ (as shown above), therefore $b_\beta \approx a_\beta$ and so the b_β satisfy our requirements.

4. Additivity of continuity. In this section and in §§ 5, 6, 7, we assume that L is a complemented, $\aleph$-complete modular lattice.

THEOREM 4.1. *Suppose $a_\lambda \uparrow a$, $b_\lambda \uparrow b$, and both a_λ, b_λ converge continuously. If $a_\lambda b_\lambda = 0$ for every λ (equivalently, if $ab = 0$), then $a_\lambda + b_\lambda$ also converges continuously.*

Proof. We may suppose $x(a_\lambda + b_\lambda) = 0$ for every λ (which implies $x(a_\lambda + b_\mu) = 0$, $(x + a_\lambda)b_\mu = 0$ for all λ, μ) and need only prove that $x(a + b) = 0$. But the continuous convergence of b_μ yields for every λ, $(x + a_\lambda)b = 0$, and so $(x + b)a_\lambda = 0$; continuous convergence of a_λ yields $(x + b)a = 0$, hence

$$x(a + b) = x(a(x + b) + b) = xb = \sum (xb_\lambda) = 0 \text{ as required.}$$

THEOREM 4.2. *If $[a, 1]$ is upper $\aleph$-continuous and $c_\lambda \uparrow 1$ in L then* $\sum (ac_\lambda) = a$.

Proof. 1. First consider the case that for every λ, $ac_\lambda = 0$. We shall show

[11] In the case $\aleph = \aleph_0$ *no* continuity assumption is implied.

that in this case $(a, c_\mu)P$ holds for each μ. Then since $\sum c_\mu = 1$, this implies $(a, 1)P$, hence $a = 0$, as required.

To show $(a, c_\mu)P$ holds we let C_μ be an arbitrary complement of c_μ and we need only prove that $C_\mu \geqslant a$ (see 2.6)). But if $\lambda \geqslant \mu$,

$$c_\lambda = c_\lambda(c_\mu \oplus C_\mu) = c_\mu + c_\lambda C_\mu.$$

Hence

$$\begin{aligned} c_\mu \oplus C_\mu = 1 &= \textstyle\sum_\lambda c_\lambda = \sum_\lambda(c_\mu + c_\lambda C_\mu) \\ &= c_\mu \oplus \textstyle\sum_\lambda(c_\lambda C_\mu) \end{aligned}$$

by the definition of lattice union. Since $C_\mu \geqslant \sum_\lambda(c_\lambda C_\mu)$, the modular law implies that $C_\mu = \sum_\lambda(c_\lambda C_\mu)$. Hence $a + C_\mu = a + \sum_\lambda(c_\lambda C_\mu) = \sum_\lambda(a + c_\lambda C_\mu)$, by the definition of lattice union. Then

$$\begin{aligned} (a + C_\mu)(a + c_\mu) &= (\textstyle\sum_\lambda(a + c_\lambda C_\mu))(a + c_\mu) \\ &= \textstyle\sum_\lambda((a + c_\lambda C_\mu)(a + c_\mu)) \end{aligned}$$

since $[a, 1]$ is upper $\aleph$-continuous,

$$\begin{aligned} &= \textstyle\sum_\lambda(a + c_\mu(a + c_\lambda C_\mu)) \\ &= a \end{aligned}$$

because

$$c_\mu(a + c_\lambda C_\mu) = c_\mu(a(c_\mu + c_\lambda C_\mu) + c_\lambda C_\mu) \leqslant c_\mu(a c_\lambda + C_\mu) = c_\mu(0 + C_\mu) = 0.$$

Thus, in turn,

$$\begin{aligned} (a + C_\mu)c_\mu &= (a + C_\mu)(a + c_\mu)c_\mu = a c_\mu = 0; \\ (a + C_\mu)c_\mu + C_\mu &= C_\mu; \\ a + C_\mu &= C_\mu; \\ a &\leqslant C_\mu \end{aligned}$$

as required.

2. In the general case, let $a_0 = \sum_\lambda(a c_\lambda)$. Then $a_0 \leqslant a$ and $(a_0 + c_\lambda)a = a_0$ for every λ. Since $(a_0 + c_\lambda) \uparrow 1$ in the lattice $[a_0, 1]$, we can apply the argument of the preceding paragraph with $[a_0, 1]$ in place of L. We obtain: $a = a_0$, that is, $\sum(a c_\lambda) = a$, as required.

THEOREM 4.3. *Additivity of upper $\aleph$-continuity. If both* $[0, a]$, $[0, b]$ *are upper $\aleph$-continuous then* $[0, a + b]$ *is also upper $\aleph$-continuous.*

Proof. We may clearly suppose $a \oplus b = 1$, $c_\lambda \uparrow 1$ and need only prove $(x c_\lambda) \uparrow x$ for every x. But $a c_\lambda$, $b c_\lambda$ both converge continuously; hence, by Theorem 4.1, $a c_\lambda + b c_\lambda$ converges continuously.

By (2.1), $[a, 1]$ is lattice isomorphic to $[0, b]$ and hence is upper $\aleph$-continuous. Then, by Theorem 4.2, $\sum(a c_\lambda) = a$. Similarly $\sum(b c_\lambda) = b$. So $\sum(a c_\lambda + b c_\lambda) = a + b = 1$.

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But we have shown that $ac_\lambda + bc_\lambda$ converges continuously; so for every x, $x \geqslant \sum(xc_\lambda) \geqslant \sum x(ac_\lambda + bc_\lambda) = x$. This shows that $(xc_\lambda) \uparrow x$ and proves Theorem 4.3.

THEOREM 4.4. (Generalization of Theorem 4.3.) *Suppose L is upper $\aleph'$-continuous for some $\aleph' < \aleph$ and $[0, a_\mu]$ is upper $\aleph$-continuous for each $\mu \in \Gamma$ with $\bar{\Gamma} \leqslant \aleph'$. Then $[0, \sum a_\mu]$ is upper $\aleph$-continuous.*

Proof. 1. We may suppose that $\aleph'$ is infinite since Theorem 4.3 shows that Theorem 4.4 holds for finite $\aleph'$.

2. We shall prove Theorem 4.4 by transfinite induction on $\aleph'$; we may therefore suppose that Theorem 4.4 holds for all cardinals less than the given infinite $\aleph'$.

3. We may now suppose that the indices μ are arranged as the set of ordinal numbers $\beta < \Omega_1$, where Ω_1 is the least ordinal number of corresponding cardinal power $\aleph'$.

4. Since $[0, \sum(a_\beta; \beta < \gamma)]$ is upper $\aleph$-continuous for every $\gamma < \Omega_1$ (by the inductive assumption), we may assume that $\{a_\beta\}$ is increasing, say $a_\beta \uparrow a$.

5. Thus we may suppose:

(i) For each $\beta < \Omega_1$, $[0, a_\beta]$ is upper $\aleph$-continuous and $a_\beta \uparrow a$ with continuous convergence (since L is assumed to be upper $\aleph'$-continuous).

And we need only prove that $[0, a]$ is upper $\aleph$-continuous.

It is sufficient to prove:

(ii) $c_\gamma \uparrow a$, $xc_\gamma = 0$ for all $\gamma < \Omega_2$ for some $\Omega_2 \leqslant \Omega$ together imply $xa = 0$.

6. For each β, $(c_\gamma a_\beta) \uparrow \bar{a}_\beta$ where $\bar{a}_\beta = \sum_\gamma(c_\gamma a_\beta) \leqslant a_\beta$.

Clearly $\{\bar{a}_\beta; \beta < \Omega_1\}$ is an increasing family, along with $\{a_\beta\}$, and converges continuously since L is assumed to be upper $\aleph'$-continuous. Hence, for every γ,

$$c_\gamma(\textstyle\sum_\beta \bar{a}_\beta) = \sum_\beta(c_\gamma \bar{a}_\beta).$$

Now

$$\begin{aligned}\textstyle\sum \bar{a}_\beta &= \textstyle\sum_{\gamma,\beta}(c_\gamma a_\beta) = \sum_\gamma(\sum_\beta c_\gamma a_\beta)\\ &= \textstyle\sum_\gamma(c_\gamma a) \quad \text{since } a_\beta \uparrow a, \text{ continuous convergence},\\ &= \textstyle\sum_\gamma c_\gamma \quad \text{since } c_\gamma \leqslant a \text{ for every } \gamma,\\ &= a \quad \text{since } c_\gamma \uparrow a, \text{ by hypothesis.}\end{aligned}$$

Thus $\bar{a}_\beta \uparrow a$ and the convergence is continuous.

Next, for each β,

$$\begin{aligned}x\bar{a}_\beta &= x\textstyle\sum_\gamma(c_\gamma a_\beta)\\ &= \textstyle\sum_\gamma(xc_\gamma a_\beta) \quad \text{since } [0, a_\beta] \text{ is upper } \aleph\text{-continuous},\\ &= \textstyle\sum_\gamma(0) \quad \text{since } xc_\gamma = 0 \text{ for very } \gamma,\\ &= 0.\end{aligned}$$

This proves the theorem.

5. Homogeneous sequences. We assume, as in § 4, that L is a complemented $\aleph$-complete, modular lattice but most of this section involves only the complemented countably complete modular lattices.

Definition 5.1. A sequence $\{a_n\}$ is called *homogeneous* if $\{a_n\}$ is strongly independent and the a_n are pairwise perspective.

Definition 5.2. If $\{a_n\}$ is a sequence in L then for any complement A of $\sum a_n$, the sequence $\{a^*_n + A\}$ is called a *dual sequence* of $\{a_n\}$.

Remark 1. Each dual sequence of $\{a_n\}$ is strongly independent in L'; if $\{a_n\}$ is strongly independent in L then each of its dual sequences, considered in L', has the original $\{a_n\}$ as a dual sequence.

Remark 2. If $\{a_n\}$ is homogeneous in L then each of its dual sequences is homogeneous in L'.

Definition 5.3. A homogeneous sequence $\{a_n\}$ is said to be of type (A) if all the a_n possess a common complement (equivalently, a common relative complement in $\sum a_n$), that is, there exists an element A such that $a_n \oplus A = 1$ for all n.

A homogeneous sequence $\{a_n\}$ is said to be of type (A*) if all the a^*_n have a common complement (equivalently, a common relative complement in $\sum a_n$).

Remark 1. Clearly if $\{a_n\}$ is strongly independent, then $\{a_n\}$ is homogeneous and of type (A), or (A*), if and only if one (hence all) of its dual sequences is homogeneous and of type (A*), or (A) respectively, in L'.

Hence, if every homogeneous sequence in L is of type (A), or if every homogeneous sequence in L is of type (A*), then every homogeneous sequence in L' is of type (A*) or (A), respectively.

Remark 2. If $\{a_n\}$ is a homogeneous sequence and $x_1 \leqslant a_1$, then any set of perspective mappings of $[0, a_1]$ onto $[0, a_n]$ when applied to x_1 will yield a homogeneous sequence $\{x_n\}$ (Theorem 5.1 below and its Corollary 1 will imply that if $\{a_n\}$ is of type (A), or (A*), then $\{x_n\}$ has the same property).

Remark 3. If $\{a_n\}$ is a homogeneous sequence then every infinite subsequence is also homogeneous; and if $\{a_n\}$ is of type (A), or (A*), then every infinite subsequence is of the same type.

LEMMA 5.1. *If* $\{a_0, a_1, \ldots, a_n, \ldots\}$ *is a homogeneous sequence of type* (A) *there exists at least one C-sequence* $\{(a_0|c_n, a_n)\}$ *such that* $a_0\sum c_n = 0$.

Remark. From Theorem 5.1 below it will follow that under the hypothesis of Lemma 5.1 *every* C-sequence $\{(a_0|c_n, a_n)\}$ has the property $a_0\sum c_n = 0$.

Proof. Let A be a common complement of the a_n. Choose $c_n = A(a_0 + a_n)$ for $n \geqslant 1$. Then

$$a_0c_n = a_nc_n = 0, \qquad a_0 \oplus c_n = a_0 \oplus a_n = c_n \oplus a_n.$$

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The lemma follows since $\{a_0, a_1, \ldots\}$ is strongly independent (by hypothesis) and $a_0\sum c_n \leqslant a_0A = 0$.

LEMMA 5.2. *Suppose* $\{a_0, a_1, \ldots\}$ *is a homogeneous sequence of type* (A) *and* $A\sum a_n = 0$. *If* $x \leqslant A$, $(x,\sum a_n)P$ *together imply* $x = 0$ (*in particular, if* A *is perspective to a subelement of* $\sum a_n$), *then* $[0, A]$ *is upper* $\aleph_0$*-continuous.*

Proof 1. By Corollary 1 to Theorem 3.6 we need only prove that every residually independent sequence in $[0, A]$ has residual element zero. We may therefore suppose that x_1 ($\neq 0$) is the residual element of some residually independent sequence in $[0, A]$ and we need only derive a contradiction.

2. The hypotheses imply that $(x_1,\sum a_n)P$ is false; therefore $(x_1, a_n)P$ is false for some n, hence $(x_1, a_0)P$ is false since $a_n \sim a_0$. Thus there exists $x \neq 0$ with $x \leqslant x_1$ and x perspective to a subelement of a_0. Theorem 3.3 shows that x is the residual element of some residually independent sequence in $[0, A]$.

By Remark 2 following Definition 5.3 we may suppose (by replacement of a_n by suitable subelements) that x is perspective to a_0, say by a perspective mapping ϕ.

3. By the Corollary to Theorem 3.2 there exists a residual C-sequence $\{(x|c_n, b_n)\}$ with $\sum c_n + \sum b_n \leqslant A$; then x is the residual element of $\{c_n\}$ and $x = \sum x_n$ for suitable x_n such that $\{(x_n, c_n, b_n)\}C$ holds.

4. By Lemma 5.1 there exists a C-sequence $\{(a_0|d_n, a_n)\}$ with $a_0\sum d_n = 0$. We shall derive a contradiction in the following way: we shall construct a C-sequence $\{(x_n', c_n', b_n')\}$ with:

(i) $$x_n' = \phi(x_n)$$

(ii) $$c_n' \leqslant d_n,\ b_n' \leqslant a_n.$$

(i) will imply that $\sum x_n' = \phi(\sum x_n) = \phi(x) = a_0$ and (ii) will imply that $(\sum c_n')\ (\sum x_n') = 0$. Then the "extension of perspective mapping" Theorem 3.5 will apply and give an extension of ϕ (which we write again as ϕ) such that $\phi(c_n) = c_n'$ for all n. This will yield:

$$\phi(x) = \phi(x\sum c_n) = \phi(x)\sum \phi(c_n)$$
$$= a_0 \sum c_n' \leqslant a_0 \sum d_n = 0$$

and imply that $x = 0$, the desired contradiction.

5. The reader can verify easily that the elements $x_n' = \phi(x_n)$, $c_n' = (x_n' + a_n)d_n$, $b_n' = (a_0 + c_n')a_n$ form a C-sequence satisfying (i), (ii) above. Thus Lemma 5.2 is proved.

LEMMA 5.3. *Suppose* $\{x, a_n; n \geqslant 1\}$ *and* $\{x, b_n; n \geqslant 1\}$ *are homogeneous sequences with* $\{x, \sum a_n, \sum b_n\}$ *independent. Then* $S = \{x, a_1, b_1, a_2, b_2, \ldots\}$ *is a homogeneous sequence. Moreover if both* $\{x, a_n; n \geqslant 1\}$ *and* $\{x, b_n; n \geqslant 1\}$ *are of type* (A) *then* S *is also of type* (A).

Proof. S is strongly independent, by application of the Corollary to Theorem 3.1. Moreover, all of x, $a_n (n \geqslant 1)$, b_n $(n \geqslant 1)$ are pairwise perspective so S is a homogeneous sequence.

Now suppose A and B are common relative complements of each of x, a_n in $x + \sum a_n$ and of each of x, b_n in $x + \sum b_n$, respectively.

Then $A + B$ is a common relative complement of each of x, a_n, b_n in $x + \sum a_n + \sum b_n$; for if c is x or a_p then

$$c(A + B) = cA = 0 \quad \text{since} \quad B(x + \textstyle\sum a_n)(x + \sum b_n) = Bx = 0.$$

Similarly $c(A + B) = 0$ if c is b_p. It is clear that $c + A + B = x + \sum a_n + \sum b_n$ if c is x, a_p, or b_p. This proves Lemma 5.3.

Now we shall prove:

THEOREM 5.1. *The following conditions are equivalent for a homogeneous sequence* $\{a_n\}$:

(i) $[0, \sum a_n]$ *is upper* $\aleph_0$*-continuous.*

(ii) $\sum_{i=1}^{n} a_i$ *converges continuously.*

(iii) $\{a_n\}$ *is of type* (A).

Proof. 1. (i) *implies* (ii): this is trivial.

2. (ii) *implies* (iii): By Lemma 5.3 it is sufficient to prove that $\{a_{2n}\}$ is of type (A).

Suppose $(a_{2n}, x_n, a_{2n+2})C$. Then $\{x_n\}$ is a homogeneous sequence with $x_n \sim a_{2n-1}$ for all $n \geqslant 1$ (use (3.1) and (2.2)). Since $(\sum x_n)(\sum a_{2n-1}) \leqslant (\sum a_{2n})(\sum a_{2n-1}) = 0$, Theorem 3.4 shows that there exists a perspective mapping of $[0, \sum x_n]$ onto $[0, \sum a_{2n-1}]$ which maps x_n on a_{2n-1}.

But $\sum_{i=1}^{n} a_{2i-1}$ converges continuously: to see this, observe that for every y,

$$y \sum a_{2i-1} = \Big(y \sum a_{2i-1}\Big)\Big(\sum a_i\Big) = \sum_{n=1}^{\infty} \Big(y \sum a_{2i-1}\Big)\Big(\sum_{j=1}^{2n} a_j\Big)$$
$$= \sum_{n=1}^{\infty} \Big(y \sum_{i=1}^{n} a_{2i-1}\Big).$$

Therefore, $\sum_{i=1}^{n} x_i$ converges continuously. But $a_{2p} \sum_{i=1}^{n} x_i = 0$ for every n, so $x = \sum x_i$ satisfies $a_{2p} x = 0$ for every p. Obviously $a_{2p} + x = \sum a_{2n}$ for every p so the a_{2p} all have x as common relative complement in $\sum a_{2n}$. This proves that $\{a_{2n}\}$ is of type (A) and shows that (ii) implies (iii).

3. (iii) *implies* (i): In Lemma 5.1 use $A = \sum a_{2n}$. Since $\{a_{2n-1}\}$ is of type (A) and $A \sum a_{2n-1} = 0$, $A \sim \sum a_{2n-1}$, therefore Lemma 5.1 applies and shows that $[0, \sum a_{2n}]$ is upper $\aleph_0$-continuous. Similarly $[0, \sum a_{2n-1}]$ is upper $\aleph_0$-continuous.

Now by Theorem 4.3 (the additivity of continuity), $[0, \sum a_n]$ is upper $\aleph_0$-continuous.

COROLLARY 1. *The following are equivalent for a homogeneous sequence* $\{a_n\}$:

(i) $[0, \sum a_n]$ *is lower* $\aleph_0$*-continuous.*

(ii) $\sum_{i=n}^{\infty} a_i$ *converges continuously.*

(iii) $\{a_n\}$ *is of type* (A*).

Proof. Apply Theorem 5.1 to $\{a^*_n\}$ in the lattice dual to $[0, \sum a_n]$.

COROLLARY 2. *If a homogeneous sequence* $\{a_n\}$ *is of type* (A) *and also of type* (A*) *then all* $a_n = 0$.

Proof. Let A be a common relative complement of the a^*_m in $\sum a_n$. Since $A\sum_{i=1}^m a_i \leqslant Aa^*_{m+1} = 0$, and $\sum_{i=1}^m a_i$ converges continuously by Theorem 5.1, therefore $A = 0$. Then $a^*_m = \sum a_n$, $a_m \leqslant a_m a^*_m = 0$, so all a_m are 0.

COROLLARY 3. *If* $\{a_n\}$ *is a homogeneous sequence of type* (A) *and* $[0, a_n]$ *is upper* $\aleph$*-continuous for every* n, *then* $[0, \sum a_n]$ *is also upper* $\aleph$*-continuous.*

Proof. This follows from Theorem 4.4.

Remark. Corollary 3 applies, in particular, if each a_n is an atom.

6. Additivity of finiteness. We assume, as in §§ 4, 5, that L is a complemented $\aleph$-complete modular lattice.

Definition 6.1. L is called *finite* if every independent sequence of pairwise perspective elements has all its elements zero.[12]

THEOREM 6.1. *If* $\{c_n\}$ *is an independent sequence of pairwise perspective elements there exists a homogeneous sequence* $\{d_n\}$ *with* $d_1 = c_1$, $d_m \sim c_n$ *for all* m, n *and* $\sum d_n \leqslant \sum c_n$.

Proof. By Theorem 3.6, applied to our $\sum_{i=1}^n c_i$ with $x = 0$ (no continuity is required in the hypotheses for the case $\aleph = \aleph_0$), there exists a residually independent sequence $\{a_n\}$ with

[12]The following possible definitions of "finiteness" for a modular lattice with zero:

(F_1): as in Definition 6.1,
(F_2): $a \approx b$, $b \leqslant a$, imply $a = b$,
(F_3): $a \sim c$, $c \sim b$, $b \leqslant a$ imply $a = b$

are related as follows:

(i) (F_2) implies (F_3) always.

(ii) (F_1) implies (F_2) if the lattice is also complemented.

(iii) If the lattice is also complemented and countably complete then (F_1), (F_2), and (F_3) are all equivalent.

(iv) If the lattice is not countably complete then (F_3) need not imply (F_1); this is shown by the example of footnote 14 where the lattice is even orthocomplemented and perspectivity is actually transitive.

(i) is trivially true.

To prove (ii): suppose there is a projective mapping ϕ of $[0, a]$ onto $[0, b]$ with $b \leqslant a$ and $b \neq a$. The argument used to prove (2.7) actually shows that for some $0 \neq x_1 \leqslant [a - b]$ we have $x_1 \sim x_2$ where $x_2 = \phi(x_1)$. Let $x_n = \phi(x_{n-1})$ for $n > 1$. Then repeated application of ϕ to the relation $x_1 \sim x_2$ shows that $\{x_n\}$ is independent and pairwise perspective, so (F_1) fails to hold. This proves (ii).

To prove (iii): suppose $\{a_n\}$ pairwise perspective and independent. By Theorem 6.1, with the same a_1, we may assume even strong independence. Then by Theorem 3.4,

$$\Sigma(a_{2n-1}; n \geqslant 1) \sim \Sigma(a_{2n}; n \geqslant 1),\ \Sigma(a_{2n}; n \geqslant 1) \sim \Sigma(a_{2n+1}; n \geqslant 1).$$

Now (F_2) would force $a_1 = 0$, so (F_3) implies (F_1) and (iii) follows from the previous remarks.

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} c_i$$

for every $n \geqslant 1$. We observe that

$$a_1 = c_1 \text{ and for } n > 1, a_n \sim c_n\left(\text{axis } \sum_{i=1}^{n-1} a_i = \sum_{i=1}^{n-1} c_i\right).$$

Now the a_n are pairwise perspective; for if $p > n$ then $c_p(a_n + c_n) = 0$ so (2.2) implies $a_n \sim c_p$. If $m \neq n$ and $p \geqslant m$ and $p \geqslant n$, then $\{a_m, a_n, c_p\}$ is independent, $a_m \sim c_p \sim a_n$, so by (2.2), $a_m \sim a_n$.

If $p < n$ then $a_n(a_p + c_p) = 0$ so $a_p \sim a_n$, $a_p \sim c_p$ yield by (2.2) that $a_n \sim c_p$. Thus $a_n \sim c_p$ for all n, p and $a_1 = c_1$.

Now let y be the residual element of $\{a_n\}$ and let Y be a relative complement of y in $\sum a_n = \sum c_n$ with $Y \geqslant a_1 = c_1$. Let $d_n = (y + a_n)Y$.

Then Theorem 3.3 shows that $\{d_n\}$ is strongly independent. Now d_n is the map of a_n in a perspective mapping of

$$\left[0, \sum_{i=1}^{m} a_i\right] \text{ onto } \left[0, \sum_{i=1}^{m} d_i\right]$$

with axis y, for any $m \geqslant n$. Hence the d_n are pairwise perspective, along with the a_n. Thus $\{d_n\}$ is a homogeneous sequence.

The definitions show that $d_1 = a_1 = c_1$ and for every n,

$$y + \sum_{i=1}^{n} d_i = y + \sum_{i=1}^{n} a_i = y + \sum_{i=1}^{n} c_i.$$

If $n > 1$, then $d_n \sim c_n$ with axis $y + \sum_{i=1}^{n-1} c_i$ (use: $y\sum_{i=1}^{n} c_i = y\sum_{i=1}^{n} a_i = 0$ and $y\sum_{i=1}^{n} d_i \leqslant yY = 0$).

But then $d_m \sim c_n$ for all $m \neq n$; for $d_m \sim d_1 = c_1 \sim c_n$ and $\{d_m, c_1, c_n\}$ is independent, so (2.2) yields $d_m \sim c_n$.

Since $\sum d_n \leqslant \sum c_n$ obviously, Theorem 6.1 is proved.

COROLLARY. *If every homogeneous sequence has all its elements zero then the lattice is finite.*

LEMMA 6.1. *Suppose L is $\aleph'$-continuous for all $\aleph' < \aleph$ and suppose L can be doubled. If $\{a_\beta; \beta < \Omega\}$ is strongly independent and $x \leqslant \sum a_\beta$ but $x\sum(a_\gamma; \gamma \leqslant \beta) = 0$ for all $\beta < \Omega$ then x is a member of a homogeneous sequence.*

Proof. Let $X = [(\sum a_\beta) - x]$ and define $\bar{a}_\beta = (a_\beta + x)X$. Then $\{\bar{a}_\beta\}$ is obviously an independent family and $a_\beta \sim \bar{a}_\beta$ for each β. Now Corollary 2 to Theorem 3.6, applied to $[0, X]$, gives a strongly independent family $\{b_\beta\}$ with $b_\beta \leqslant X$ and $\bar{a}_\beta \approx b_\beta$.

Since $\{a_\beta\}$, $\{b_\beta\}$ are both strongly independent and $a_\beta \approx b_\beta$ for every β, and $\sum b_\beta \leqslant \sum a_\beta$, therefore (ii) of the Corollary to Theorem 3.4, together with Theorem 6.1, show that $[\sum a_\beta - \sum b_\beta]$ is a member of a homogeneous sequence. But the relative complement $[\sum a_\beta - \sum b_\beta]$ could be chosen $\geqslant x$ so (use Remark 2 following Definition 5.3) x itself is a member of a homogeneous sequence, as stated.

THEOREM 6.2. *The following properties are equivalent:*

(i) *every homogeneous sequence in L is of type* (A),

(ii) *for every strongly independent sequence $\{a_n\}$ for which $[0, \sum a_n\}$ can be doubled, $\sum_{i=1}^n a_i$ converges continuously.*

Proof. 1. (ii) implies (i): Let $\{x_n\}$ be a homogeneous sequence. Then each of $\{x_{2n}\}$, $\{x_{2n-1}\}$ is strongly independent (in fact, a homogeneous sequence) and each of $\sum x_{2n}$, $\sum x_{2n-1}$ can be doubled (in fact, $(\sum x_{2n})(\sum x_{2n-1}) = 0$ and $(\sum x_{2n}) \sim (\sum x_{2n-1})$ by (i) of the Corollary to Theorem 3.4).

Now if (ii) holds, then each of

$$\sum_{i=1}^n x_{2i}, \sum_{i=1}^n x_{2i-1}$$

converges continuously so by Theorem 5.1, each of $[0, \sum x_{2n}]$, $[0, \sum x_{2n-1}]$ is upper $\aleph_0$-continuous; hence $[0, \sum x_n]$ is upper $\aleph_0$-continuous by Theorem 4.3. Finally $\{x_n\}$ is of type (A) by Theorem 5.1. So (ii) implies (i).

2. (i) *implies* (ii). Suppose (i) holds. We may suppose that $\{a_n\}$ is strongly independent and that the lattice $[0, \sum a_n]$ can be doubled and we need only prove that $\sum_{i=1}^n a_i$ converges continuously.

We may suppose that there exists an element $x \neq 0$ such that $x \leqslant \sum a_n$ and $x\sum_{i=1}^n a_i = 0$ for all n and we need only derive a contradiction.

3. By replacing each a_n by $a_n(x + a^*_n)$ we may even suppose that a_n is perspective to a subelement of x (observe: $\sum a_n(x + a^*_n) \leqslant \sum a_n$, so $[0, \sum a_n(x + a^*_n)]$ can be doubled and has property (i) along with $[0, \sum a_n]$; also $\{a_n(x + a^*_n)\}$ is strongly independent, along with $\{a_n\}$; finally, $a_n(x + a^*_n) \sim [x - xa^*_n]$ with axis a^*_n).

4. We shall show now that $[0, \sum a_n]$ is upper $\aleph_0$-continuous; this implies that $x = 0$ and gives the desired contradiction.

5. In the present situation, Lemma 6.1 applies, with $[0, \sum a_n]$ in place of L, and shows that there exists a homogeneous sequence $\{x_n\}$ in $[0, \sum a_n]$ with $x = x_1$. The validity of (i) in $[0, \sum a_n]$ then implies that $\{x_n\}$ is of type (A).

6. Since $[0, \sum a_n]$ can be doubled we may (and shall) assume that $[0, \sum a_n]$ is identified with $[0, v]$ in some modular lattice with zero, L_1, in such a way that there exists a perspective mapping ϕ of $[0, \sum a_n]$ onto $[0, u]$ for some u in L_1 with $u\sum a_n = 0$ (we do not know that L_1 is complemented and has property (i) but $[0, \sum a_n]$, but so also $[0, u]$, does have these properties). Then $\{\phi(x_n)\}$ is a homogeneous sequence of type (A) along with $\{x_n\}$.

Since a_n is perspective to a subelement of x, and x is perspective to x_n and $(a_n + x + x_n)\phi(x_n) = 0$, (2.2) and $x_n \sim \phi(x_n)$ imply that a_n is perspective to a subelement of $\phi(x_n)$.

By (i) of the Corollary to Theorem 3.4, $\sum a_n$ is perspective to a subelement of $\sum\phi(x_n)$. But $[0, \sum\phi(x_n)]$ is upper $\aleph_0$-continuous by Theorem 5.1, hence $[0, \sum a_n]$ is also upper $\aleph_0$-continuous. This completes the proof of Theorem 6.2.

LEMMA 6.2. *If* $\{c_n\}$ *is a strongly independent sequence and* a *is an arbitrary element, there exists a decomposition* $c_n = c_n' \oplus c_n''$ *with the properties:*

(i) $c_n' \sim d_n$ *for some strongly independent* $\{d_n\}$ *with* $\sum d_n \leqslant a(\sum c_n)$,

(ii) $a\sum c_n'' = 0$.

Proof. 1. Put

$$c_n' = c_n\Big(a + \sum_{m>n} c_n\Big), \qquad c_n'' = [c_n - c_n'].$$

2. (ii) is immediate since

$$\begin{aligned} a\Big(\sum_{m=n}^{\infty} c_m''\Big) &= a\Big(c_n''\Big(a + \sum_{m>n} c_m''\Big) + \sum_{m>n} c_m''\Big) \\ &= a \sum_{m>n} c_m''; \end{aligned}$$

$$\begin{aligned} a(\sum c_m'') &= a\sum(c_m''; m \geqslant 2) = \ldots = a\sum(c_m''; m \geqslant n) \\ &= a\textstyle\prod_n \sum(c_m''; m > n) \leqslant \prod_n c_n^* = 0. \end{aligned}$$

3. To prove (i) we note that

$$c_n' \oplus \sum_{m>n} c_m \leqslant a + \sum_{m>n} c_m$$

hence

$$c_n' \oplus \sum_{m>n} c_m = d_n \oplus \sum_{m>n} c_m \tag{6.1}$$

for suitable $d_n \leqslant a$.

Now (6.1) shows that $d_n \sim c_n'$ (axis $\sum_{m>n} c_m$) and $\{d_n\}$ is strongly independent by (3.1).

Since each $d_m \leqslant \sum c_n$, and $\leqslant a$, therefore $\sum d_n \leqslant a\sum c_n$ and Lemma 6.2 is proved.

THEOREM 6.3. *If in* $[0, a]$ *and in* $[0, b]$ *every homogeneous sequence is of type* (A) *then this is true in* $[0, a + b]$.

Proof. We may suppose $a \oplus b = 1$. By Theorem 6.2 we need only prove: if $\{c_n\}$ is strongly independent and $[0, \sum c_n]$ can be doubled, then $\sum_{i=1}^{n} c_i$ converges continuously.

We shall use the decomposition of c_n, $c_n = c_n' \oplus c_n''$ and the d_n, provided by Lemma 6.2 for the present a. We shall show:

(i) $\sum_{i=1}^{n} c_i'$ converges continuously,

(ii) $\sum_{i=1}^{n} c_i''$ converges continuously.

It will then follow from Theorem 4.1 that $\sum_1^n c_n$ converges continuously, proving the theorem.

To prove (ii): We note that $[0, \sum c_n'']$ is mapped by a perspective mapping (axis a) on a sublattice of $[0, b]$. Since this sublattice can be doubled (it is lattice isomorphic to $[0, \sum c_n'']$ and $\sum c_n'' \leqslant \sum c_n$), (ii) follows from the assumed properties of $[0, b]$.

To prove (i), we observe that (ii) of the Corollary to Theorem 3.4 applies to the lattice $[0, \sum c_n]$ since $\{d_n\}$, $\{c_n'\}$ are each strongly independent, $d_n \sim c_n''$ for each n, $d_n \leqslant \sum c_m$, $c_n'' \leqslant \sum c_m$ and $[0, \sum c_n]$ can be doubled. Thus $[0, \sum c_n'']$ is lattice isomorphic to $[0, \sum d_n]$.

Since $[0, \sum d_n]$ can be doubled (along with $[0, \sum c_n'']$, along with $[0, \sum c_n]$) and since $\sum d_n \leqslant a$, it follows from the hypothesis that $\sum_{i=1}^{n} d_i$ converges continuously. Hence $\sum_{i=1}^{n} c_i''$ also converges continuously.

This proves (i) and completes the proof of Theorem 6.3.

Corollary 1. *If in* $[0, a]$ *and in* $[0, b]$ *every homogeneous sequence is of type* (A*) *then this is also true in* $[0, a + b]$.

Proof. We may suppose $a \oplus b = 1$. Now Theorem 6.3 (for L') implies: if in each of $[a, 1]'$, $[b, 1]'$ *considered as sublattices of* L', every homogeneous sequence is of type (A), then this is true in $[ab, 1]'$, that is, $[0, 1]'$.

But $[a, 1]'$, $[b, 1]'$ are anti-isomorphic to $[0, b]$, $[0, a]$ respectively, by (2.1). Thus, if every homogeneous sequence in $[0, a]$ or $[0, b]$ is of type (A*) then every homogeneous sequence in $[b, 1]'$ or $[a, 1]'$ is of type (A) (use the Remark 1 following Definition 5.3); hence every homogeneous sequence in $[0, 1]'$ is of type (A); finally every homogeneous sequence in $[0, 1]$ is of type (A*) (again using Remark 1 following Definition 5.3).

This proves Corollary 1.

Corollary 2. *Additivity of finiteness. If each of* $[0, a]$, $[0, b]$ *is finite, so is* $[0, a + b]$.

Proof. If $\{a_n\}$ is a homogeneous sequence in $[0, a + b]$, then $\{a_n\}$ is of type (A), and also of type (A^*) by Theorem 6.3 and its Corollary 1. Then, by Corollary 2 to Theorem 5.1, all a_n are 0.

Then, by the Corollary to Theorem 6.1, $[0, a + b]$ is finite.

7. Unrestricted additivity of continuity in finite lattices.

We assume that L is a complemented $\aleph$-complete modular lattice.

Lemma 7.1. *Suppose L is upper $\aleph$-continuous. Then for every family* $\{a_\beta; \beta < \Omega\}$ *there exists a strongly independent family* $\{\bar{a}_\beta\}$ *such that* $\bar{a}_\beta \leqslant a_\beta$ *and* $\sum \bar{a}_\beta = \sum a_\beta$.

Proof. Put $\bar{a}_\beta = [a_\beta - a_\beta \sum (a_\gamma; \gamma < \beta)]$. Obviously $\{\bar{a}_\beta\}$ is independent, $\bar{a}_\beta \leqslant a_\beta$ and by transfinite induction on γ, $\sum(\bar{a}_\beta; \beta < \gamma) = \sum(a_\beta; \beta < \gamma)$ for all $\gamma < \Omega$.

Strong independence of $\{\bar{a}_\beta\}$ is equivalent to independence of $\{\bar{a}_\beta\}$ since L is upper $\aleph$-continuous (see the last sentence preceding Theorem 3.1).

COROLLARY. *If x is an $\aleph$-residual element and $[0, x]$ is upper $\aleph$-continuous, then x can be doubled in L.*

Proof. By the Corollary to Theorem 3.2 there exists a residual C-system $\{(x_\beta, c_\beta, b_\beta)\}$ with $x = \sum x_\beta$.

By Lemma 7.1, $\bar{x}_\beta = [x_\beta - x_\beta \sum(x_\gamma; \gamma < \beta)]$ has the properties: $\{\bar{x}_\beta\}$ is strongly independent, $\bar{x}_\beta \leqslant x_\beta$ and $x = \sum \bar{x}_\beta$.

Let $\bar{b}_\beta = b_\beta(\bar{x}_\beta + c_\beta)$. Then $\{\bar{b}_\beta\}$ is strongly independent (along with $\{b_\beta\}$), $\bar{x}_\beta \sim \bar{b}_\beta$ for each β and $(\sum \bar{x}_\beta)(\sum \bar{b}_\beta) \leqslant x \sum b_\beta = 0$. Now by (i) of the Corollary to Theorem 3.4, $x \sim \sum \bar{b}_\beta$. Since $x \sum \bar{b}_\beta = 0$ this proves that x can be doubled in L.

LEMMA 7.2. *Suppose $x \leqslant y$, $y \oplus Y = 1$ with $[0, Y]$ upper $\aleph$-continuous. If there exists an increasing family $\{c_\beta\}$ with $xc_\beta = 0$ for every β and $x \leqslant \sum c_\beta$, then there exists such an increasing family with $\sum c_\beta \leqslant y$.*

Proof. Let $c_\beta' = c_\beta + [1 - \sum c_\gamma]$. Then $\sum c_\beta' = 1$. Now yc_β' has the properties specified:

$$\sum (yc_\beta') = y \geqslant x$$

(observe that $[y, 1]$ is lattice isomorphic to $[0, Y]$ by (2.1), hence upper $\aleph$-continuous, and use Theorem 4.2);

$$x(yc_\beta') = xc_\beta' = x(\sum c_\gamma)c_\beta' = x(c_\beta + 0) = 0.$$

COROLLARY. *Suppose x is an $\aleph$-residual element in L and L is $\aleph'$-continuous for every $\aleph' < \aleph$. If $y \oplus Y = 1$ with $x \leqslant y$ and $[0, Y]$ upper $\aleph$-continuous then x is an $\aleph$-residual element in $[0, y]$.*

Proof. By hypothesis, x is the residual element of some residually independent family $\{a_\beta; \beta < \Omega\}$.

Define $c_\beta = \sum(a_\gamma; \gamma \leqslant \beta)$. Then $xc_\beta = 0$ for each β since L is upper $\aleph'$-continuous for all $\aleph' < \aleph$. And $x \leqslant \sum c_\beta$ since $\sum c_\beta = \sum a_\beta \geqslant x$.

Hence Lemma 7.2 shows that an increasing family $\{\bar{c}_\beta\}$ exists with $\sum \bar{c}_\beta \leqslant y$, $x\bar{c}_\beta = 0$ for every β and $x \leqslant \sum \bar{c}_\beta$. By Theorem 3.6, applied to $[0, y]$, $x \leqslant t$ for some t which is an $\aleph$-residual element in $[0, y]$; hence x itself has this property, by Theorem 3.3.

THEOREM 7.1. *Suppose x is an $\aleph$-residual element in L with $[0, x]$ upper $\aleph$-continuous. If L is $\aleph'$-continuous for all $\aleph' < \aleph$ then x is a member of a homogeneous sequence.*

Proof. 1. By Theorem 6.1 it is sufficient to show that x is a member of an independent sequence of pairwise perspective elements.

2. It is sufficient therefore to prove that if $\{x, x_1, \ldots, x_m\}$ is an independent family of pairwise perspective elements and $m \geqslant 0$, then there exists some $x_{m+1} \sim x$ such that $\{x, x_1, \ldots, x_{m+1}\}$ is independent.

3. We choose $Y = x_1 + \ldots + x_m$ and $y = [1 - Y]$ with $y \geqslant x$.

Since each $[0, x_i]$ is upper $\aleph$-continuous, along with $[0, x]$, so is $[0, Y]$, by Theorem 4.3. Hence, by the Corollary to Lemma 7.2, x is a residual element in $[0, y]$. Now the Corollary to Lemma 7.1, applied to $[0, y]$ shows that $x \sim x_{m+1}$ for some $x_{m+1} \leqslant y$ with $xx_{m+1} = 0$. Then

$$x_{m+1}(x + x_1 + \ldots + x_m) = x_{m+1}y(x + x_1 + \ldots + x_m) = x_{m+1}(x + 0) = 0$$

so $\{x, x_1, \ldots, x_{m+1}\}$ is independent.

This x_{m+1} satisfies our requirements and this completes the proof of Theorem 7.1.

COROLLARY 1. *If L is finite and locally $\aleph$-continuous then L is $\aleph$-continuous.*

Proof. We prove this by transfinite induction. Hence we can suppose L is $\aleph'$-continuous for all $\aleph' < \aleph$.

By Corollary 7 to Theorem 3.6 it is sufficient to show that every $\aleph$-residual element t must be 0.

But if $t \neq 0$ then for some non-zero x with $x \leqslant t$, $[0, x]$ is $\aleph$-continuous (*a fortiori*, upper $\aleph$-continuous). Then, by Theorem 3.3, x is also an $\aleph$-residual element. Now Theorem 7.1 shows that x is a member of a homogeneous sequence.

But L is finite, so $x = 0$. This gives a contradiction and shows that $t \neq 0$ is impossible. Thus Corollary 1 must be valid.

Remark. The proof of Theorem 7.1 shows that if L is finite and locally upper $\aleph$-continuous and $\aleph'$-continuous for all $\aleph' < \aleph$, then L is upper $\aleph$-continuous. When $\aleph = \aleph_0$ this becomes: if L is finite and locally upper $\aleph_0$-continuous then L is upper $\aleph_0$-continuous.

COROLLARY 2. *If in L every homogeneous sequence is of type* (A) *and L is locally upper $\aleph_0$-continuous then L is upper $\aleph_0$-continuous.*

(Note: the Remark following Corollary 1 to Theorem 7.1 uses the stronger condition that L be *finite*.)

Proof. Suppose, if possible, that L is not upper $\aleph_0$-continuous. Then there exists some $t \neq 0$ with t an $\aleph_0$-residual element in L.

By the hypotheses, there exists an $x \neq 0$ with $x \leqslant t$ and $[0, x]$ upper $\aleph_0$-continuous. By Theorem 3.3, x itself is also an $\aleph_0$-residual element in L.

Now Theorem 7.1 shows that x is a member of a homogeneous sequence $\{x, y_0, y_1, \ldots\}$, by the hypotheses necessarily of type (A).

Choose $Y = \sum y_n$ and $y = [1 - Y]$ with $y \geqslant x$. Then $[0, Y]$ is upper $\aleph_0$-continuous by Theorem 5.1 so x is an $\aleph_0$-residual element in $[0, y]$, by the Corollary to Lemma 7.2.

Now by the Corollary to Theorem 3.2 there exists a residual C-sequence $\{(x|c_n, b_n)\}$ with $x + \sum b_n \leqslant y$. Since b_n is perspective to a subelement of x and x is perspective to y_n and $(b_n + x)y_n \leqslant yY = 0$ therefore (2.2) shows that b_n is perspective to a subelement of y_n.

Since $x \sim y_0$ and $(x + \sum b_n)(\sum y_n) \leqslant yY = 0$, therefore (i) of the Corollary to Theorem 3.4 shows that $x + \sum b_n$ is perspective to a subelement of $\sum y_n$.

But $[0, \sum y_n] = [0, Y]$ is upper $\aleph_0$-continuous so $[0, x + \sum b_n]$ has the same property. Hence $x = x\sum c_n = \sum_{n=1}^{\infty}(x\sum_{i=1}^{n} c_i) = \sum(0) = 0$, a contradiction.

Thus Corollary 2 must be vaild.

8. Homogeneous sequences (continued). In this section we assume that L is a complemented countably complete modular lattice.

LEMMA 8.1. *Suppose* $\{a_n\}$ *is a homogeneous sequence with* $(a_n, x_n, a_{n+1})C$ *for every* n. *Then*

(i) $a_1(\sum x_n) = 0$ *implies* $\{a_n\}$ *is of type* (A);

(ii) $a_1 \leqslant \sum x_n$ *implies* $\{a_n\}$ *is of type* (A*).

Proof of (i): We shall show that $\sum x_i$ is a complement of every a_n in $\sum a_i$. Clearly $a_n + \sum x_i = \sum a_i$ so we need only prove $a_n \sum x_i = 0$ for each n. But the axis $x_1 + \ldots + x_{n-1}$ gives a perspective mapping of $[0, a_n]$ onto $[0, a_1]$ and by this mapping $a_n \sum x_i$ is mapped on $(a_n \sum x_i + x_1 + \ldots + x_{n-1})a_1 \leqslant (\sum x_i)a_1 = 0$, so $a_n \sum x_i$ itself must be 0.

Proof of (ii): We need only show that $\{x_n\}$ is a homogeneous sequence of type (A*). For the Corollary 1 to Theorem 5.1 will show that $[0, \sum x_n]$ is lower $\aleph_0$-continuous; then $[0, \sum a_n]$ will also be lower $\aleph_0$-continuous since the hypothesis implies that $\sum a_n \leqslant \sum x_n$; then $\{a_n\}$ will be of type (A*), again by Corollary 1 to Theorem 5.1.

To show that $\{x_n\}$ is homogeneous of type (A*) it is sufficient (by Remark 1 following Definition 5.3) to prove:

(8.1) $\{x_n\}$ is strongly independent,

(8.2) a_1 is a complement of every x^*_n in $\sum x_i$.

Now (8.1) follows from (3.1).

To prove (8.2) we verify:

$$a_1 x_n^* = a_1\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n-1} x_i + \sum_{i=n+1}^{\infty} x_i\right)$$

$$= a_1 \sum_{i=1}^{n-1} x_i + 0 = 0;$$

$$a_1 + x_n^* = a_1 + x_1 + \ldots + x_{n-1} + \sum_{i=n+1}^{\infty} x_i$$

$$= a_1 + \ldots + a_{n-1} + \sum_{i=n+1}^{\infty} x_i$$

so we need only show

(8.3) $$a_n \leqslant \sum_{i=n+1}^{\infty} x_i \qquad \text{for every } n.$$

But (8.3) holds for $n = 1$, by hypothesis. Also, (8.3) holds for $n = 2$ since

$$a_2 \leqslant \left(\sum_{i=2}^{\infty} a_i\right)(a_1 + x_1) \leqslant \left(\sum_{i=2}^{\infty} a_i\right)\left(\sum_{i=2}^{\infty} x_i + x_1\right)$$
$$= \left(\sum_{i=2}^{\infty} x_i\right) + \left(\sum_{i=2}^{\infty} a_i\right)x_1(a_1 + a_2)$$
$$= \left(\sum_{i=2}^{\infty} x_i\right) + 0 \qquad \text{since } x_1 a_2 = 0,$$
$$a_2 \leqslant \sum_{i=2}^{\infty} x_i.$$

By repetition of this calculation, (8.3) can be proved for all n. This shows that (ii) holds and completes the proof of Lemma 8.1.

THEOREM 8.1. (i) *If* $\{b_n\}$, $\{c_n\}$ *are homogeneous sequences of types* (A) *and* (A*) *respectively then* $\sum b_n$ *and* $\sum c_n$ *are completely disjoint.*

(ii) *If* $\{a_n\}$ *is a homogeneous sequence there is a unique decomposition* $a_n = b_n \oplus c_n$ *such that* $\{b_n\}$ *and* $\{c_n\}$ *are homogeneous sequences of types* (A), (A*) *respectively.*

Proof of (i): Let $b = \sum b_n$, $c = \sum c_n$. We may suppose $(b, c)P$ false and we need only derive a contradiction. Clearly we may suppose (replacing b_n, c_n by suitable subelements) $b_1 \sim c_1 \neq 0$. Then $x \leqslant c$, $(x, b)P$ together imply $x = 0$.

Let $d = [c - bc]$. Lemma 5.2 (with our d in place of A in Lemma 5.2) shows that $[0, d]$ is upper $\aleph_0$-continuous. And $[0, bc]$ is upper $\aleph_0$-continuous since $bc \leqslant b$ and $[0, b]$ is upper $\aleph_0$-continuous. Then Theorem 4.3 shows that $[0, c]$ is upper $\aleph_0$-continuous.

Hence, by Theorem 5.1, $\{c_n\}$ is of type (A). Since $\{c_n\}$ is also of type (A*), all c_n are 0 (by Corollary 2 to Theorem 5.1). This contradicts $c_1 \neq 0$ and (i) is therefore established.

Proof of (ii): We need only obtain one decomposition as described, since uniqueness will follow from (i).

Since $\{a_n\}$ is a homogeneous sequence, $(a_n, x_n, a_{n+1})C$ holds for certain x_n. Put $x = \sum x_n$ and let

$$c_1 = a_1 x, \qquad b_1 = [a_1 - a_1 x]$$

and for $n \geqslant 1$,

$$c_{n+1} = (c_n + x_n)a_{n+1} \qquad b_{n+1} = (b_n + x_n)a_{n+1}.$$

Thus, c_{n+1}, b_{n+1} obtain from c_n, b_n respectively by the perspective mapping of $[0, a_n]$ onto $[0, a_{n+1}]$ with axis x_n.

It follows that $\{b_n\}$, $\{c_n\}$ are homogeneous sequences and $a_n = b_n \oplus c_n$.

Now let

$$y_n = x_n(b_n + b_{n+1}), \qquad z_n = x_n(c_n + c_{n+1}).$$

Then as the reader can verify easily, $(b_n, y_n, b_{n+1})C$, $(c_n, z_n, c_{n+1})C$ hold and $x_n = y_n \oplus z_n$.

Since $b_1\sum y_n \leqslant b_1a_1\sum x_n = 0$, Lemma 8.1 (i) shows that $\{b_n\}$ is a homogeneous sequence of type (A).

Since

$$\begin{aligned} c_1 = c_1a_1x &= c_1(\sum y_n + \sum z_n) \\ &= c_1(\sum y_n + (c_1 + \sum y_n)\sum z_n) = c_1\sum y_n \end{aligned}$$

(observe: $(c_1 + \sum y_n)\sum z_n \leqslant (\sum c_n)(\sum b_n) = 0$), therefore Lemma 8.1 (ii) shows that $\{c_n\}$ is of type (A*).

This completes the proof of Theorem 8.1.

COROLLARY. *If L is complete then L is a direct sum $L_1 + L_2 + L_3$ where $L_i = [0, a_i]$ with all a_i in the centre of L, L_1, is upper $\aleph_0$-continuous, L_2 is lower $\aleph_0$-continuous and L_3 is finite (a_1, a_2, a_3 are unique if L_3 is maximal with the finiteness property).*

Proof. Let $a_1 = \sum x$ where x varies over all elements perspective to members of homogeneous sequences of type (A), let $a_2 = \sum y$ where y varies over all elements perspective to members of homogeneous sequences of type (A*). Then a_1, a_2 are in the centre of L.

Let $a_3 = [1 - (a_1 + a_2)]$. Then $[0, a_3]$ is clearly finite.

Now every homogeneous sequence in $[0, a_1]$ is of type (A). For otherwise $a_1 \neq 0$ and some $y \neq 0$ with $y \leqslant a_1$ would be a member of a homogeneous sequence of type (A*), by Theorem 8.1 (ii). Also $(y, x)P$ would be false for some x perspective to a member of a homogeneous sequence of type (A) hence for some x which is itself a member of a homogeneous sequence of type (A). But Theorem 8.1 (i) shows that $(y, x)P$ holds in such circumstances. This contradiction proves that every homogeneous sequence in $[0, a_1]$ is of type (A).

Also $[0, a_1]$ is locally upper $\aleph_0$-continuous. For if $y \neq 0$, $y \leqslant a_1$ holds, then as in the preceding paragraph $(y, x)P$ is false for some x which is a member of a homogeneous sequence of type (A), so $[0, x]$ is upper $\aleph_0$-continuous (using Theorem 5.1). Then for some $y_1 \neq 0$, $y_1 \leqslant y$, the lattice $[0, y_1]$ is perspective to $[0, x_1]$ for some $x_1 \leqslant x$ so $[0, y_1]$ is also upper $\aleph_0$-continuous. This proves $[0, a_1]$ to be locally upper $\aleph_0$-continuous.

Then Corollary 2 to Theorem 7.1 shows that $[0, a_1]$ is upper $\aleph_0$-continuous. Similarly, using the dual to Corollary 2 to Theorem 7.1, $[0, a_2]$ is lower $\aleph_0$-continuous.

Remark 1. If L is $\aleph$-complete but not necessarily complete we can show that L is a sublattice of such a direct sum $L_1 + L_2 + L_3$.

Remark 2. If L is complete and irreducible then L must be upper $\aleph_0$-continuous, or lower $\aleph_0$-continuous or finite.[13] (See the Note added at end of this paper.)

9. Kaplansky's theorem.

THEOREM 9.1. *Suppose L is a complemented countably complete modular lattice.*[14] *Then L is finite if it has the property:*

(9.1) *for every $a \neq 0$ there exists an anti-automorphism ϕ of L such that $(a, b)P$ is false for every complement b of $\phi(a)$.*[15]

Proof.[16] By the Corollary to Theorem 6.1 it is sufficient to show that if $\{a_n\}$ is a homogeneous sequence in L then $a_1 = 0$. By Theorem 8.1 (ii) we may suppose that $\{a_n\}$ is of type (A) or (A*).

Suppose if possible that $a_1 \neq 0$. Let ϕ be an anti-automorphism of L (as provided by (9.1)) such that $(a_1, b)P$ is false whenever $b \oplus \phi(a_1) = 1$. Then $\{\phi(a_n)\}$, considered in L', is homogeneous of type (A) or (A*).

Hence every dual sequence $\{b_n\}$ of $\{\phi(a_n)\}$ is homogeneous of type (A*) or (A) respectively, in L. Therefore $(a_1, b_1)P$ holds by Theorem 8.1 (i).

But b_1 is a complement of $\phi(a_1)$ by the definition of dual sequence, so this gives a contradiction to the property assumed for ϕ.

Thus $a_1 \neq 0$ is not possible and Theorem 9.1 is proved.

Remark 1. (9.1) is obviously implied by the property:

(9.2) *for every $a \neq 0$ there exists an anti-automorphism ϕ of L such that $a \leqslant \phi(a)$ is false.*

Hence, *if L is a countably complete, orthocomplemented modular lattice, then L must be finite* (see the Appendix for a direct proof of this result).

Remark 2. If L is a complemented, *complete* modular lattice, then (9.1) is implied by the property:

(9.2)′ *for every $z \neq 0$ with z in the centre of L, there exists an anti-automorphism ϕ of L such that $z \leqslant \phi(z)$ is false.*

To derive (9.1) *from:* (9.2)′ suppose $a \neq 0$ and let z be the least central element with $z \geqslant a$. Then there exists an anti-automorphism ϕ of L (as provided by (9.2)′) such that $z \leqslant \phi(z)$ is false.

[13] Irreducibility for a lattice L means: $L = L_1 + L_2$ (direct sum) only if L_1 or L_2 consists of one element. If L is complemented and modular, this is equivalent to: 0, 1 are the only elements in the centre of L (it was shown first by von Neumann (**3**, Part I, Theorems 5.2, 5.3) that for complemented modular lattices, irreducibility in the above sense is equivalent to: 0, 1 are the only elements with unique complements).

[14] Theorem 9.1 (and also its Corollary) may fail to hold if L is not countably complete. This failure occurs in the orthocomplemented modular lattice consisting of all the linear subspaces of finite dimension and their orthogonal complements in Hilbert space.

[15] If $(a, b)P$ is false for *one* complement b of $\phi(a)$ then $(a, b)P$ is necessarily false for every complement b of $\phi(a)$.

[16] See footnote 23.

Since $z \leqslant z_1$ for every central element $z_1 \geqslant a$ the anti-automorphism ϕ yields: $\phi(z) \geqslant z_1$ for every central element $z_1 \leqslant \phi(a)$. Since $\phi(z) \geqslant z$ is false, therefore $z \leqslant \phi(a)$ must be false, also. Hence $zb \neq 0$ for some complement b of $\phi(a)$.

Using (2.8) we have: $(a, b)P$ is false for at least one complement b of $\phi(a)$, hence for every complement b of $\phi(a)$.[17] Thus (9.1) has been derived from (9.2)$'$.

COROLLARY.[18] *A complete complemented modular lattice L is necessarily finite if it possesses an anti-automorphism ϕ which is an orthocomplementation on the centre (that is, $\phi(z) \oplus z = 1$ for every central element z), in particular if L is irreducible*[19] *and possesses at least one anti-automorphism.*

Definition 9.1. For a lattice L the property $(SI)_{\aleph}$ shall mean:

$(SI)_{\aleph}$: For every increasing family $\{c_\beta; \beta < \Omega\}$ there exists a strongly independent family $\{a_\beta; \beta < \Omega\}$ such that $c_\beta = \sum(a_\gamma; \gamma \leqslant \beta)$ for all $\beta < \Omega$.

THEOREM 9.2. *An orthocomplemented $\aleph$-complete modular lattice has the property $(SI)_{\aleph}$.*

Proof. Suppose $a \rightarrow \phi(a)$ denotes the orthocomplementation. If $\{c_\beta\}$ is an increasing family, choose $a_1 = c_1$, and for $1 < \beta < \Omega$, $a_\beta = c_\beta \prod(\phi(c_\gamma); \gamma < \beta)$.

Then $\{a_\beta\}$ is strongly independent; for $\gamma < \beta$ implies that a_γ is orthogonal to a_β; hence a_γ is orthogonal to a^*_γ, $\prod a^*_\gamma$ is orthogonal to every a_β, hence to $\sum a_\beta$. Since $\prod a^*_\gamma \leqslant \sum a_\beta$ this implies $\prod a^*_\gamma = 0$, so $\{a_\beta\}$ is indeed strongly independent.

By transfinite induction on β it is easy to show that $c_\beta = \sum(a_\gamma; \gamma \leqslant \beta)$ for all $\beta < \Omega$.

This proves Theorem 9.2.

THEOREM 9.3. *Suppose L is a complemented $\aleph$-complete modular lattice with the property $(SI)_{\aleph}$. If L is finite and can be doubled then L is upper $\aleph$-continuous.*

Proof 1. We may suppose $xc_\beta = 0$ for all $\beta < \Omega_1$ for some $\Omega_1 \leqslant \Omega$ and $c_\beta \uparrow 1$ and we need only prove $x = 0$.

Let X be a complement of x and let $c_\beta' = (x + c_\beta)X$. Then $[0, c_\beta]$ is mapped onto $[0, c_\beta']$ by the perspective mapping with axis x.

Since L is assumed to have the property $(\mathrm{SI})_{\aleph}$, there exist strongly independent families $\{a_\beta\}$, $\{a_\beta'\}$ such that $c_\beta = \sum(a_\gamma; \gamma \leqslant \beta)$, $c_\beta' = \sum(a_\gamma'; \gamma \leqslant \beta)$ for all $\beta < \Omega_1$.

Then $a_\beta \sim (x + a_\beta)X$ (with axis x). Since $a_\beta \oplus \sum(a_\gamma; \gamma < \beta) = c_\beta$, that is, $a_\beta \oplus \sum(c_\gamma; \gamma < \beta) = c_\beta$ therefore (by the perspective mapping with axis x),

[17]See footnote 15.
[18]See footnote 14.
[19]See footnote 13.

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$$(x + a_\beta)X \oplus \sum(c'_\gamma; \gamma < \beta) = c'_\beta$$

so $(x + a_\beta)X \sim a_\beta'$ (with axis $\sum(a_\gamma'; \gamma < \beta) = \sum(c_\gamma'; \gamma < \beta)$).

Therefore $a_\beta \approx a_\beta'$ for each $\beta < \Omega_1$, $\sum a_\beta = \sum c_\beta = 1 > \sum a_\beta'$ and L can be doubled.

If $x \neq 0$, then $\sum a_\beta' \leqslant X < 1$ so $[\sum a_\beta - \sum a_\beta'] \neq 0$ and by (ii) of the Corollary to Theorem 3.4, $[\sum a_\beta - \sum a_\beta']$ is a member of an independent sequence of pairwise perspective elements. This would contradict the assumed finiteness of L.

Hence $x = 0$ as required and Theorem 9.3 is proved.

THEOREM 9.4. *If L is orthocomplemented $\aleph$-complete and modular then L is $\aleph$-continuous.*[20]

Proof. L is finite by Remark 1 following Theorem 9.1. Thus, by Corollary 1 to Theorem 7.1 it is sufficient to prove local $\aleph$-continuity of L. Since L possesses an anti-automorphism it is sufficient to prove that L is locally upper $\aleph$-continuous.

We may suppose x is an element of L with $x \neq 0$ and we need only show that for some element y, with $0 \neq y \leqslant x$, the lattice $[0, y]$ is upper $\aleph$-continuous

Now if there exists an element $0 \neq y \leqslant x$ such that $[0, y]$ can be doubled then, since $[0, y]$ has the property $(\mathrm{SI})_\aleph$, Theorem 9.3 shows that $[0, y]$ is upper $\aleph$-continuous.

On the other hand, if $0 \neq y \leqslant x$ implies that $[0, y]$ cannot be doubled then $(y, z)P$ holds whenever $y \leqslant x$, $z \leqslant x$ with $yz = 0$.[21] Now if $\{a_\beta\}$ is an increasing family in $[0, x]$ and $ya_\beta = 0$ for every β, then $(y, a_\beta)P$ holds for every β, so $(y, \sum a_\beta)P$ holds, hence $y\sum a_\beta = 0$. This proves that $[0, x]$ is itself upper $\aleph$-continuous.

This completes the proof of Theorem 9.4.

THEOREM 9.5. *Suppose L is a complemented $\aleph$-complete modular finite lattice which possesses an anti-automorphism ϕ of period two with the following continuity property: $\phi(x_\beta) \oplus x_\beta = 1$, $x_\beta \uparrow x$ together imply $\phi(x) \oplus x = 1$. Then L is $\aleph$-continuous.*

Remark. Such ϕ generalize orthocomplementation.

Proof 1. We prove this theorem by transfinite induction on $\aleph$ so we may suppose L is $\aleph'$-continuous for all $\aleph' < \aleph$.

[20]This is a strengthened form of Kaplansky's theorem **(1)**. In a letter to one of us dated June 13, 1957, Kaplansky conjectured that any complemented complete modular lattice is continuous if it possesses an anti-automorphism of period two which is an orthocomplementation on the centre. Our Theorem 9.1 establishes finiteness under even weaker conditions but our Theorem 9.5 establishes continuity only under conditions somewhat more restrictive than those of Kaplansky's conjecture.

[21]In this case $[0, x]$ is a complemented modular lattice in which every element has a unique complement, that is, a Boolean algebra.

2. It is sufficient to prove that L is locally upper $\aleph$-continuous, for the anti-automorphic character of L will show then that L is locally lower $\aleph$-continuous, hence locally $\aleph$-continuous. Since L is assumed to be finite, Corollary 1 to Theorem 7.1 will then show that L is $\aleph$-continuous.

3. Thus we may suppose $a \neq 0$ and we need only prove that $[0, b]$ is upper $\aleph$-continuous for some $0 \neq b \leqslant a$. We shall prove below:

(i) If $\phi(x) \geqslant x$ is false for every $0 \neq x \leqslant a$ then $[0, a]$ possesses an orthocomplementation $u \to a\phi(u)$ (Theorem 9.4 then shows that $[0, a]$ itself is $\aleph$-continuous).

(ii) If $\phi(x) \geqslant x$ holds for some $0 \neq x \leqslant a$ then $[0, x]$ is upper $\aleph$-continuous.

4. *To prove* (i): We note that

$$\phi(x\phi(x)) = \phi(x) + x \geqslant x\phi(x).$$

Since $x\phi(x) \leqslant x$ the assumption of (i) implies that $x\phi(x) = 0$ for all $x \leqslant a$. Then also $x + \phi(x) = 1$. Thus $x + a\phi(x) = a$, $a\phi(a\phi(x)) = a(\phi(a) + x) = x + a\phi(a) = x$ so $x \to a\phi(x)$ is an orthocomplementation on $[0, a]$. This proves (i).

5. *To prove* (ii): We may suppose $a_\beta \uparrow x (\beta < \Omega)$, $ya_\beta = 0$ for all β and $y \leqslant x$; we need only prove $y = 0$.

We can choose b_β by transfinite induction on β so that $b_1 = [x - a_1]$, and for $\beta > 1$,

$$b_\beta = [\Pi(b_\gamma; \gamma < \beta) - a_\beta \Pi(b_\gamma; \gamma < \beta)],$$

and so that $b_\beta \geqslant y$ for all β.[22] Then $a_\beta \oplus b_\beta = x$ for all β since $[0, x]$ is lower $\aleph'$-continuous for all $\aleph' < \aleph$.

Now let X be a complement of x. Set $c_\beta = X\phi(a_\beta)$. Then $c_\beta \downarrow$, $c_\beta x \leqslant Xx = 0$ and $c_\beta \oplus x = X\phi(a_\beta) + x = \phi(a_\beta)(X + x) = \phi(a_\beta)$ for all β. Hence

$$(b_\beta \oplus c_\beta) \oplus a_\beta = \phi(a_\beta), \qquad \phi(b_\beta + c_\beta)\phi(a_\beta) = a_\beta,$$
$$(b_\beta + c_\beta)\phi(b_\beta + c_\beta) = (b_\beta + c_\beta)\phi(a_\beta)\phi(b_\beta + c_\beta) = (b_\beta + c_\beta)a_\beta = 0,$$

so

$$(b_\beta + c_\beta)\phi(b_\beta + c_\beta) = 0$$

for all β.

Suppose $\phi(b_\beta + c_\beta) \uparrow d$. Then $d\phi(d) = 0$ and $(b_\beta + c_\beta) \downarrow \phi(d)$. Since $y \leqslant b_\beta$ for every β, therefore $y \leqslant \Pi(b_\beta + c_\beta)$, that is, $y \leqslant \phi(d)$; but also

$$\phi(d) \leqslant \Pi(b_\beta + c_\beta) \leqslant \Pi\phi(a_\beta) = \phi(\sum a_\beta) = \phi(x)$$

so $d \geqslant x$. Thus $y \leqslant x\phi(d) \leqslant d\phi(d) = 0$ as required.

This proves that $[0, x]$ is upper $\aleph$-continuous and completes the proof of Theorem 9.5.

---

[22]Choose
$$b_1 = y + [x - (y + a_1)],$$
$$b_\beta = y + [\Pi(b_\gamma; \gamma < \beta) - (y + a_\beta)\ \Pi(b_\gamma; \gamma < \beta)].$$

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Appendix on Finiteness

Theorem.[23] *In an orthocomplemented countably complete modular lattice every independent*[24] *sequence of pairwise perspective elements must have all its elements zero.*

Proof. 1. We call an infinite sequence $\{x_n; n \geqslant 1\}$ *residually independent* if $x_n\sum(x_i; i \neq n) = 0$ for all n, *strongly independent* if $\prod_n\sum(x_i; i \neq n) = 0$. Since, for every p,

$$x_p\sum(x_i; i \neq p) \leqslant \prod_n\sum(x_i; i \neq n)$$

strong independence implies residual independence.

We note that if the x_n are pairwise orthogonal then $\{x_n\}$ is strongly independent. Also, if $\{x_n\}$ is strongly independent and for every n, $y_n \leqslant \sum(x_i; i \geqslant n)$, $y_n\sum(x_i; i > n) = 0$ then $\{y_n\}$ is also strongly independent. If $\{x_n\}$ is strongly independent then so is $\{(x^*_n)^\perp\sum x_n\}$ where x^*_n denotes $\sum(x_i; i \neq n)$ and $x^\perp$ denotes the orthogonal complement of the element x.

2. We may suppose $\{a_n\}$ is an independent sequence of pairwise perspective elements with $a_1 \neq 0$ and we need only derive a contradiction. By replacing a_n for $n > 1$ by $(a_1 + \ldots + a_n)(a_1 + \ldots + a_{n-1})^\perp$ we may even suppose that $\{a_n\}$ is strongly independent.

3. By using suitable replacements for the a_n we may even assume that they have a common relative complement A, that is, $a_n + A = \sum a_m$, $a_nA = 0$ for all n.[25] To see this, suppose $a_n \sim a_{n+1}$ (axis x_n), that is, $a_n + x_n = a_{n+1} + x_n = a_n + a_{n+1}$, $a_nx_n = a_{n+1}x_n = 0$. Let $x = \sum x_n$. We must consider two cases: $xa_1 \neq a_1$ and $xa_1 = a_1$.

---

[23]This theorem (first proved by Kaplansky (**1**, Theorem 1), see footnote 25) is contained in our Theorem 9.1 (see Remark 1 following Theorem 9.1) but we give here a direct (lattice-theoretic) proof for this orthocomplemented case which can be read independently of the rest of this paper provided the reader has some slight familiarity with complemented modular lattices.

With slight modification this direct proof actually establishes Theorem 9.1 in full generality.

[24]A family $\{x_\lambda; \lambda \in \Lambda\}$ is called independent if for every *finite* subset $F \subset \Lambda$, $x_\mu\Sigma(x_\lambda; \lambda \in F) = 0$ whenever $\mu \notin F$.

[25]Kaplansky constructs a common relative complement A for *every* sequence $\{a_n\}$ of pairwise orthogonal and perspective elements (of course, the Theorem will show finally that all a_n must be 0).

Kaplansky's method is as follows: first, he replaces L by $[0, \Sigma a_n[$. Then he shows that $\{\Sigma(a_{4n+i}; n \geqslant 1); i = 0, 1,2,3\}$ is a homogeneous basis of order 4 in the sense of von Neumann (**3**, Part II, Definition 3.1). Therefore L can be identified with the lattice of principal right ideals of a suitable regular ring $\mathfrak{R}$, by the coordinatization theorem of von Neumann (**3**, Part II, Theorem 14.1).

Since L is orthocomplemented, there exists a conjugation operation in $\mathfrak{R}$: $x \to x^*$ (that is, $(x + y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ and $x^{**} = x$) such that every lattice element a in L is of the form $(e)_r$ with e a unique idempotent which is Hermitian (that is, $e^* = e$) and then $a^\perp = (1\text{-}e)_r$ (all proved in von Neumann (**3**, chapter II, Theorem 4.5)).

If $xa_1 \neq a_1$, let a_1' be a relative complement of xa_1 in a_1, let a_{n+1}' be obtained from a_n' by the perspective mapping with axis x_n, that is, $a_{n+1}' = (a_n' + x_n)a_{n+1}$ and let $x_n' = x_n(a_n' + a_{n+1}')$, $x' = \sum x_n'$. Then $\{a_n'\}$ is strongly independent, pairwise perspective and the a_n' have x as a common relative complement.

If $xa_1 = a_1$, then $\{x_n\}$ is strongly independent, pairwise perspective and the $x^*_n = \sum(x_i; i \neq n)$ have a_1 as common relative complement. Hence $\{(x^*_n)^\perp\}$ is strongly independent, pairwise perspective and $a_1^\perp$ is a common relative complement.

4. Suppose now that our strongly independent sequence of pairwise perspective elements is written as two sequences a_n, b_n, and let A be their common relative complement. The strong independence of the set $\{$all a_n, all $b_n\}$ implies $(\sum a_n)(\sum b_n) = 0$.

Then $a_n \sim a_{n+1}$ with axis $c_n = A(a_n + a_{n+1})$ and $a_1 \sum c_n = 0$.

5. We shall show below that there exists a *residually* independent sequence of pairwise perspective elements $\{b_n'\}$ with $b_1' \sim b_1$, $(\sum b_n')(\sum a_n) = 0$ and axes of perspectivity d_n between b_n' and b_{n+1}' such that $b_1 \leqslant \sum d_n$.

6. We will then derive a contradiction as follows: $a_1 \sim b_1$, $b_1 \sim b_1'$ with $a_1(b_1 + b_1') = 0$ implies $a_1 \sim b_1'$. Let t_1 be an axis of perspectivity for a_1, b_1' and define t_{n+1} for $n \geqslant 1$, by induction as follows:

$$t_{n+1} = (t_n + c_n + d_n)(a_n + b_n').$$

Let $t = \sum t_n$. Then $t + \sum d_n = t + \sum c_n$ and (use: for each n, $\{a_n, \sum(a_m; m > n)$, b_n, $\sum(b_m; m > n)\}$ is independent since $\{a_m\}$ and $\{b_m\}$ are both residually independent):

$$t\sum a_n = \left(t_1 + \sum_{i=2}^{\infty} t_i\right)\left(a_1 + \sum_{i=2}^{\infty} a_i\right) = t_1 a_1 + \left(\sum_{i=2}^{\infty} t_i\right)\left(\sum_{i=2}^{\infty} a_i\right)$$
$$= 0 + \left(\sum_{i=2}^{\infty} t_i\right)\left(\sum_{i=2}^{\infty} a_i\right) = \ldots \leqslant \Pi_n\left(\sum_{i=2}^{\infty} a_i\right) = 0.$$

Thus the given a_n must be of the form $(e_n)_r$ with all e_n idempotent, Hermitian and $e_n e_m = 0$ for $n \neq m$.

Kaplansky now constructs elements in $\mathfrak{R}$, namely w, e_{1i}, e_{i1} (for $i \geqslant 1$) such that: $e_{11} = e_1$; for all i, $e_{1i} = e_1 e_{1i} e_i$, $e_{1i} e_{i1} = e_1$ and $e_{i1} e_{1i} = e_i$; $w = e_1 w$ and $we_i = e_{1i}$ for all i (see Kaplansky (**1**, Lemma 21).

Now $(w)^r$, the set of all u such that $wu = 0$, is a principal right ideal, as shown by von Neumann (**3**, Part II, Lemma 2.2).

This $(w)^r$ is a common complement of the a_n. For $w(e_{i1} - e_1) = 0$ for every i; so for every j, $(e_j)_r + (w)^r$ contains $e_j e_{j1} + (e_{j1} - e_1)(-e_1) = e_1$, hence it contains also, for every i, $(e_{i1} - e_1) + e_1 = e_{i1}$, hence also $e_{i1} e_{1i} = e_i$. Thus

$$(e_j)_r + (w)^r = 1 \text{ for all } j.$$

Finally, if u is in both $(e_j)_r$ and $(w)^r$ then $u = e_j u$ and $wu = 0$; that is, $we_j u = e_{1j} u = 0$, so $u = e_{j1}(e_{1j} u) = 0$. This means the meet of $(e_j)_r$ and $(w)^r$ is 0, and proves that $(w)^r$ is a common complement of all $a_n = (e_n)_r$.

Then

$$b_1' = b_1'\sum d_n = b_1'(t + \sum c_n)(t + a_1) = b_1'(t + (\sum c_n)a_1)$$
$$= b_1'(t + 0) = b_1't = b_1't_1 = 0.$$

This implies $a_1 = 0$ and gives the desired contradiction.

7. Thus we need only to construct $\{b_n'\}$ as described in 5 above. Since $A\sum b_m$ is a common relative complement of the b_n we may suppose $\sum b_m = 1$ and write A again in place of $A\sum b_m$. Now let a denote $A^\perp$ and let $\bar{b}_n$ denote $(b^*_n)^\perp$.

We shall prove that the family $\{a, \bar{b}_1, \bar{b}_2, \ldots\}$ are pairwise perspective and independent, $\bar{b}_1 \leqslant a + \sum(\bar{b}_i; i > 1)$, and $\bar{b}_1 \sim b_1$.

Indeed:

$$\text{(i)} \qquad (a\bar{b}^*_n)^\perp = A + \left(\sum_{m\neq n} \bar{b}_m\right)^\perp = A + \prod_{m\neq n} b^*_m = A + b_n = 1$$

since $\{b_n\}$ strongly independent implies that $\prod_{m\neq n} b^*_m = b_n$, and so $a\bar{b}^*_n = 0$.

$$\text{(ii)} \qquad (a + \bar{b}^*_n)^\perp = Ab_n = 0, \text{ so } a + \bar{b}^*_n = 1.$$

Since $\{\bar{b}_n\}$ is strongly independent, (i) shows that $\{a, \bar{b}_1, \bar{b}_2, \ldots\}$ is independent. Then (i) and (ii) show that $a \sim \bar{b}_n$ with axis $\bar{b}^*_n$ so all of $\{a, \bar{b}_1, \bar{b}_2, \ldots\}$ are pairwise perspective, and $\bar{b}_1 \leqslant a + \bar{b}^*_1 = 1$. Finally, $\bar{b}_1 \sim b_1$ with axis b^*_1.

Thus the b_n' will be available, as described in 5. if we prove the following "orthogonalization" lemma.

LEMMA.[26] *Suppose $\{b_0, f_1, f_2, \ldots\}$ is independent and pairwise perspective. Then there exists a sequence $b_0, b_1, b_2, \ldots$ such that $\{b_n; n \geqslant 0\}$ is residually independent, and for $n \geqslant 0$, $b_{n-1} \sim b_n$ with axis d_n, so that $f_1 + \ldots + f_m = d_1 + \ldots + d_m$ for every m (in particular if $b_0 \leqslant \sum(f_m; m \geqslant 1)$ then $b_0 \leqslant \sum(d_m; m \geqslant 1)$).*

Proof. 1. Choose $d_1 = f_1$. Then b_0 and f_1 have some axis of perspectivity u and we choose $b_1 = (b_0 + f_1)u$. Then [27]

$$b_0 \oplus d_1 = b_1 \oplus d_1 = b_0 \oplus b_1.$$

Hence we can choose B_0, a complement of b_0, so that

$$B_0 = b_1 + [1 - (b_1 + d_1)]. \qquad \text{Let } B_{-1} = 1.$$

[26]This "orthogonalization" lemma is proved here for every complemented countably complete modular lattice. Even countable completeness need not be assumed if residual independence of $\{x_n\}$ is defined to mean: for every n there exists an element X_n such that $x_n X_n = 0$ and $X_n \geqslant x_m$ for all $n < m$.

[27]$x \oplus y$ denotes the lattice union $x + y$ but implies that the meet xy is 0. When $x \leqslant y$ the symbol $[x - y]$ denotes an arbitrary but fixed relative complement of y in x, that is, $[x - y] \oplus y = x$.

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2. We may suppose that for some $r \geqslant 1$, the following statements hold:

$$
(W)_r \quad \left\{ \begin{array}{l} d_1, \ldots, d_r, b_1, \ldots, b_r, B_0, \ldots, B_{r-1} \\ \text{have all been defined so that:} \\ b_{n-1} \sim b_n \text{ with axis } d_n \text{ for } n = 1, \ldots, r; \\ d_1 + \ldots + d_s = f_1 + \ldots + f_s \text{ for } s = 1, \ldots, r; \\ 1 = b_0 \oplus B_0; B_0 = b_1 \oplus B_1; \ldots ; B_{r-2} = b_{r-1} \oplus B_{r-1}; \\ B_{r-1} \geqslant b_r, \end{array} \right.
$$

and we need only show how to define d_{r+1}, b_{r+1}, B_r so that $(W)_{r+1}$ holds. (Observe that $(W)_1$ does hold for the d_1, b_1, B_0 defined in 1 above.)

3. Choose $d_{r+1} = B_{r-1}(f_1 + \ldots + f_{r+1})$.

Then

$$
\begin{aligned}
\text{(i)} \quad d_1 + \ldots + d_{r+1} &= (d_1 + \ldots + d_r + B_{r-1})(f_1 + \ldots + f_{r+1}) \\
&= (d_1 + \ldots + d_r + b_r + B_{r-1})(f_1 + \ldots + f_{r+1}) \\
&= (d_1 + \ldots d_r + b_{r-1} + B_{r-1})(f_1 + \ldots + f_{r+1}) \\
&= (d_1 + \ldots d_{r-1} + B_{r-2})(f_1 + \ldots + f_{r+1}) \\
&= \ldots = B_{-1}(f_1 + \ldots + f_{r+1}) = f_1 + \ldots + f_{r+1}.
\end{aligned}
$$

$$
\begin{aligned}
\text{(ii)} \quad & d_{r+1}(b_0 + b_1 + \ldots + b_r) = d_{r+1}b_r \\
& = d_{r+1}(f_1 + \ldots + f_{r+1})b_r(b_0 + f_1 + \ldots + f_r) \\
& = d_{r+1}b_r(f_1 + \ldots + f_r) = d_{r+1}b_r(d_1 + \ldots + d_r) = 0, \\
& d_{r+1}(f_1 + \ldots + f_r) \leqslant d_{r+1}(b_0 + b_1 + \ldots + b_r) = 0,
\end{aligned}
$$

so $d_{r+1} \sim f_{r+1}$ (axis $f_1 + \ldots + f_r = d_1 + \ldots + d_r$).

But $f_{r+1} \sim b_0$ and $b_0(f_{r+1} + d_{r+1}) \leqslant b_0(f_1 + \ldots + f_{r+1}) = 0$ so $d_{r+1} \sim b_0$.

Now $\{b_0, b_1, \ldots, b_r\}$ is independent, $b_{n-1} \sim b_n$ for $n \leqslant r$, and $d_{r+1}(b_0 + \ldots + b_r) = 0$, so $d_{r+1} \sim b_r$.

4. Since $d_{r+1} \sim b_r$, there exists an axis u such that

$$b_r \oplus u = d_{r+1} \oplus u.$$

We choose $b_{r+1} = (b_r \oplus d_{r+1})u$. Then we have

$$b_r \oplus d_{r+1} = b_{r-1} \oplus d_{r+1} = b_r \oplus b_{r+1}$$

so $b_r \sim b_{r+1}$ with axis d_{r+1}.

5. Since $b_r + d_{r+1} \leqslant B_{r-1}$ we can choose

$$B_r = b_{r+1} + [B_{r-1} - (b_r + d_{r+1})].$$

Then $B_r \geqslant b_{r+1}$, and $B_{r-1} = b_r \oplus B_r$.

Thus $(W)_{r+1}$ is satisfied and so the Lemma is proved and hence the Theorem is proved.

Note added in proof. A recent paper by Ornstein (*Dual vector spaces*, Ann. Math., *69* (1959), 520–34) obtains the following result (his Corollary 5.1):

Suppose L is a complete, atomic, centreless, complemented, modular lattice in which 1 is the union of a countable number of atoms and O is the intersection of a countable number of co-atoms; then L is either isomorphic or anti-isomorphic to the lattice of all subspaces of a vector space of countable dimension.

Ornstein's result can be deduced also from Remark 2 at the end of our § 8.

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