DUALITY AND THE EXISTENCE OF WEAKLY COMPLETELY CONTINUOUS ELEMENTS IN A B^*-ALGEBRA

by B. J. TOMIUK

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1. Introduction. Ogasawara and Yoshinaga [9] have shown that a B^*-algebra is weakly completely continuous (w.c.c.) if and only if it is $*$-isomorphic to the $B^*(\infty)$-sum of algebras $LC(H_\lambda)$, where each $LC(H_\lambda)$ is the algebra of all compact linear operators on the Hilbert space H_λ. As Kaplansky [5] has shown that a B^*-algebra is $*$-isomorphic to the $B^*(\infty)$-sum of algebras $LC(H_\lambda)$ if and only if it is dual, it follows that a B^*-algebra A is w.c.c. if and only if it is dual. We have observed that, if only certain key elements of a B^*-algebra A are w.c.c., then A is already dual. This observation constitutes our main theorem which goes as follows. *A B^*-algebra A is dual if and only if for every maximal modular left ideal M there exists a right identity modulo M that is* w.c.c.

To prove our main theorem we use the fact that every closed left (right) ideal of a B^*-algebra A that contains a nonzero w.c.c. element contains a self-adjoint minimal idempotent. In order to prove the existence of minimal idempotents in such ideals we use a group theoretic argument from [2] to show first that every closed left (right) ideal of A that contains a nonzero w.c.c. element contains a w.c.c. self-adjoint idempotent (Lemma 3.1). We next show that a w.c.c. commutative B^*-algebra with identity is finite dimensional (Theorem 3.2). As a consequence of this observation we have the following result due to Ogasawara [8]: A w.c.c. B^*-algebra with identity is finite dimensional. From this and Lemma 3.1 we obtain the existence of self-adjoint minimal idempotents in closed left (right) ideals that contain nonzero w.c.c. elements. We also make use of Theorem 3.2 to give a proof of the following proposition found in [11]: A weakly sequentially complete B^*-algebra is finite dimensional. All of these results are contained in §3. In §4 we prove the main theorem and state some consequences of it.

We are grateful to the referee for suggesting a shorter proof of Theorem 3.2 and for his many comments which have contributed so very much to the presentation of this article.

2. Preliminaries. All algebras and vector spaces under consideration are over the complex field C. A Banach algebra A with involution is called a B^*-algebra if, for any $x \in A$, $\|x^*x\| = \|x\|^2$. By [3, p. 48, Théorème (2.9.5) (iii)], every proper closed left (right) ideal of a B^*-algebra A is the intersection of all maximal modular left (right) ideals containing it; in particular, every maximal closed left (right) ideal of A is modular.

Let A be a Banach algebra and A' its conjugate space. An element $a \in A$ is called weakly completely continuous (w.c.c.) if its left and right multiplication operators are weakly completely continuous, i.e., if they take bounded sets in A into sets that are relatively compact in the weak topology $\sigma(A, A')$ of A. The set of all w.c.c. elements in A is a closed two-sided ideal of A [8, p. 362]. A Banach algebra is called weakly completely continuous (w.c.c.) if each of its elements is w.c.c.

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If X is a Banach space, then every (norm-)closed subspace of X is weakly closed [4, p. 422, Theorem 13]. Thus every closed left (right) ideal of a Banach algebra is weakly closed.

For any subset S of a Banach algebra A, let $l(S)$ and $r(S)$ be the left and right annihilators of S in A, respectively. A Banach algebra A is called dual if $l(r(J)) = J$ and $r(l(R)) = R$ for every closed left ideal J and every closed right ideal R of A. An idempotent e in a Banach algebra A is called minimal if eAe is isomorphic to the complex field. If A is semisimple, then e is a minimal idempotent if and only if Ae (eA) is a minimal left (right) ideal of A [10, p. 46].

For any subset S of a Banach space X, let $\mathscr{B}(S) = \{x \in S: \|x\| \leqq 1)$. If S is a closed subspace of X, then $\mathscr{B}(S)$ is a closed subset of X.

We shall need the following simple result.

LEMMA 2.1. *Let A be a semisimple Banach algebra and let e_1 and e_2 be minimal idempotents in A. If $e_1Ae_2 \neq (0)$, then it is a one-dimensional algebra over the complex field.*

Proof. Suppose that $e_1Ae_2 \neq (0)$. Then there exists an element e_1x in A such that $e_1xe_2 \neq 0$. Since e_1Ae_1 is isomorphic to C and, by [10, p. 45, Lemma (2.1.8)], $e_1Ae_2 = e_1Ae_1xe_2$, it follows that e_1Ae_2 is one-dimensional over C. It is easy to see that, if $e_2e_1 \neq 0$, then e_1Ae_2 is isomorphic to C, and, if $e_2e_1 = 0$, then it is a zero algebra, i.e., the product of any two elements is zero.

3. B*-algebras with w.c.c. elements.

LEMMA 3.1. *Let A be a* B**-algebra. Then every closed left (right) ideal of A that contains a nonzero* w.c.c. *element contains a nonzero* w.c.c. *self-adjoint idempotent.*

Proof. Let J be a closed left ideal of A that contains a nonzero w.c.c. element. Then J contains a self-adjoint w.c.c. element a such that $\|a\| = 1$. We have $\|a^{2^n}\| = 1$ ($n = 0, 1, 2, \ldots$). Consider the sequence $S = \{a^2, a^4, \ldots, a^{2^n}, \ldots\}$ and let $G(a)$ be the set of cluster points of S, i.e., the set of points z such that every weak neighbourhood of each z contains some a^{2^n} for arbitrarily large n. Since S is contained in the set $\{ax: x \in \mathscr{B}(A)\}$ whose weak closure is compact (a being w.c.c.), by [4, p. 430, Theorem 1], $G(a)$ is not empty and every subsequence of S contains a subsequence that converges weakly to an element of $G(a)$. Moreover, it is easy to see that, for every $z \in G(a)$, there is a subsequence of S that converges weakly to z. In fact, let $z \in G(a)$ and let E be the norm-closed linear span of S. Then, by [7, p. 261, (5)], E' is w^*-separable with countable dense subset $\{f_n\}$, say. Hence we may select a subsequence $\{x_{n_k}\}$ of S such that

$$|f_m(x_{n_k} - z)| < \frac{1}{k} \qquad (1 \leqq m \leqq k).$$

Therefore, by the relative weak compactness of S, replacing $\{x_{n_k}\}$ by a subsequence if necessary, there is an element y of E such that $\{x_{n_k}\}$ converges weakly to y. Since $\{f_n\}$ is dense in E', we must have $y = z$. Hence there is a subsequence of S that converges weakly to z.

To show that $G(a)$ contains nonzero elements we argue as follows. Let M be the closed *-subalgebra generated by a and let Ω be the carrier space of M. Then M is isometrically

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*-isomorphic to $C_0(\Omega)$, the algebra of all continuous complex-valued functions on the locally compact space Ω vanishing at infinity. Let ε be given, $0 < \varepsilon < 1$. Then $\{\phi \in \Omega : |\phi(a)| \geqq \varepsilon\}$ is compact and hence there exists $\phi_0 \in \Omega$ such that $|\phi_0(a)| = 1$. Let $f \in A'$ be an extension of ϕ_0 to all of A with $\|f\| = \|\phi_0\|$. Then $f(a^{2^n}) = 1$ $(n = 0, 1, 2, \ldots)$. Hence $G(a)$ contains nonzero elements. Using the argument given in the proof of [**2**, p. 180, Theorem 4], we can show that $G(a)$ is a group. Let u be the identity of $G(a)$. Then $u \neq 0$, $u^2 = u$ and, since $a^* = a$, we also have $u^* = u$. Since J is weakly closed, $u \in J$. Moreover, since the set of all w.c.c. elements of A form a (norm-)closed and hence weakly closed two-sided ideal of A, u is w.c.c. A similar proof holds for a closed right ideal.

THEOREM 3.2. *Let A be a commutative* w.c.c. B*-*algebra with identity. Then A is finite dimensional.*

Proof. Let Ω be the carrier space of A. Then Ω is a compact Hausdorff space and A is isometrically *-isomorphic to $C(\Omega)$, the algebra of continuous complex-valued functions on Ω. Let $a \in \Omega$ and let $\{G_\alpha\}$ be the collection of all open neighbourhoods of a partially ordered by inclusion. Then, for each α, there exists $f_\alpha \in C(\Omega)$ such that $f_\alpha(a) = 1$, $f_\alpha = 0$ off G_α and $0 \leqq f_\alpha \leqq 1$. Since the identity is w.c.c., the net $\{f_\alpha\}$ has a weak adherent point g. Considering the point measures on Ω, we see that the neighbourhood

$$\{f \in C(\Omega) : |f(b) - g(b)| < \varepsilon\} \qquad (b \in \Omega, \varepsilon > 0)$$

contains a cofinal subnet of $\{f_\alpha\}$. On taking $b = a$, this gives $g(a) = 1$, and on taking $b \neq a$ it gives $g(b) = 0$. Since $g \in C(\Omega)$, it follows that Ω is discrete, and being compact it must be finite. Hence A has a finite number of self-adjoint minimal idempotents $e_1, e_2, \ldots, e_n$, say, and $e = e_1 + e_2 + \ldots + e_n$, where e is the identity of A. Thus every $x \in A$ can be expressed in the form $x = \sum_{i=1}^{n} e_i x$ and, since each $e_i A$ is one-dimensional, it follows that A is finite dimensional. This completes the proof.

As a consequence of Theorem 3.2 we obtain the following result due to Ogasawara [**8**, p. 362, Theorem 3]:

COROLLARY 3.3. *Let A be a* w.c.c. B*-*algebra with identity. Then A is finite dimensional.*

Proof. Let B be a maximal commutative *-subalgebra of A. Then B is a commutative w.c.c. B^*-algebra with identity and hence, by Theorem 3.2, it is finite dimensional. Let $\{e_1, e_2, \ldots, e_n\}$ be the set of all self-adjoint minimal idempotents in B. It is easily shown that $\{e_1, e_2, \ldots, e_n\}$ is a maximal orthogonal family of self-adjoint minimal idempotents in A (see [**9**, p. 21]). Since $e = e_1 + e_2 + \ldots + e_n$, where e is the identity of A, every $x \in A$ can be written in the form $x = \sum_{i,j=1}^{n} e_i x e_j$. Hence, by Lemma 2.1, A is finite dimensional.

COROLLARY 3.4. *Let A be a* B*-*algebra. Then every closed left* (*right*) *ideal of A that contains a nonzero* w.c.c. *element contains a self-adjoint minimal idempotent that is* w.c.c.

Proof. Let J be a closed left ideal that contains a nonzero w.c.c. element. Then, by Lemma 3.1, J contains a w.c.c. self-adjoint idempotent $u \neq 0$. Since $B = uAu$ is a w.c.c.

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B^*-algebra with identity, it contains a self-adjoint minimal idempotent e by Corollary 3.3. Since $eAe = euAue = eBe$, it follows that e is also a self-adjoint minimal idempotent of A; clearly $e \in J$. A similar proof holds for a closed right ideal. This completes the proof.

We can also use Theorem 3.2 to prove the following result due to Sakai [**11**, p. 661, Proposition 2].

THEOREM 3.5. *Let A be a weakly sequentially complete* B*-*algebra. Then A is finite dimensional.*

Proof. We may clearly assume that A has identity. Let M be a maximal commutative $*$-subalgebra of A. Then M is weakly sequentially complete and isometrically $*$-isomorphic to $C(\Omega)$, the algebra of all continuous complex-valued functions on a compact Hausdorff space Ω. Hence, for each $a \in C(\Omega)$, the multiplication operator T_a on $C(\Omega)$ is weakly compact [**4**, p. 494, Theorem 6]. Thus M is w.c.c. and so, by Theorem 3.2, it is finite dimensional. The proof of Corollary 3.3 now shows that A is finite dimensional.

4. Main theorem.

THEOREM 4.1. *Let A be a* B*-*algebra. Then the following statements are equivalent*:

(i) *A is a dual algebra.*
(ii) *For every maximal modular left ideal M of A there exists a right identity modulo M that is* w.c.c.

Proof. (i) $\Rightarrow$ (ii). If A is dual, then every maximal modular left ideal $M = \{x - xe : x \in A\}$, where e is a self-adjoint minimal idempotent. (See [**1**, p. 155, Theorem 1] and [**10**, p. 261, Lemma (4.10.1)].) Now Ae is a minimal left ideal of A and the scalar-valued function (x, y) given by $(x, y)e = y^*x$ $(x, y \in Ae)$, is an inner product on Ae with $\|x\| = |x|_0 = (x, x)^{1/2}$ for all $x \in Ae$. Thus Ae is a Hilbert space and so $\mathscr{B}(Ae)$ is a weakly compact subset of Ae and hence of A. Therefore every $x \in Ae$ is w.c.c., and in particular e is w.c.c.

(ii) $\Rightarrow$ (i). Suppose that (ii) holds. Let u be a right identity modulo the maximal modular left ideal M and suppose that u is w.c.c. Since $u \neq 0$, $Au \neq (0)$. Let $J = \{x - xu : x \in A\}$; $J \subset M$ and $J + Au = A$. If $M \cap Au = (0)$, then clearly $M = J$ and Au is a minimal left ideal and hence closed [**10**, p. 45–46]. Hence $u \in Au$. As $u - u^2 \in M \cap Au = (0)$, $u^2 = u$, i.e., u is an idempotent and so M is an annihilator ideal.

Now suppose that $M \cap Au \neq (0)$. Then M contains nonzero w.c.c. elements and therefore, by Corollary 3.4, M contains self-adjoint minimal idempotents. Let $\{e_\alpha\}$ be a maximal orthogonal family of self-adjoint minimal idempotents in M. Then, for any finite number of elements $e_{\alpha_1}, e_{\alpha_2}, \ldots, e_{\alpha_n}$ from $\{e_\alpha\}$, $u - u(e_{\alpha_1} + \ldots + e_{\alpha_n})$ is a right identity modulo M. Let Q be the set of all elements in A that are finite sums of elements from $\{e_\alpha\}$. Since u is w.c.c. and the net $Q \subset \mathscr{B}(A)$, uQ has a weak adherent point u', say, and it is easy to show that u' is unique. Since uQ is contained in M and M is weakly closed, $u' \in M$. Hence $u - u' \neq 0$ and is a right identity modulo M; moreover $(u - u')e_\alpha = 0$ for all α. Let $J = \mathrm{cl}(A(u - u'))$. Then $J \neq 0$ and $Je_\alpha = 0$ for all α. If $J \cap M \neq (0)$, then $J \cap M$ contains a self-adjoint minimal idempotent f and $fe_\alpha = 0$ for all α. As $f \in M$, this means that $\{e_\alpha\}$ is not a maximal orthogonal family in M, a contradiction. Hence $J \cap M = (0)$. It is now easy to conclude that $u - u'$ is an idempotent, $J=A(u-u')$ and $M=\{x-x(u-u'): x\in A\}$ (see the paragraph above). Hence, if (ii) holds, then every maximal modular left ideal of A has a nonzero right annihilator. By the continuity of the involution, every maximal modular right ideal has a nonzero left annihilator. Since every proper closed left (right) ideal is the intersection of all maximal modular left (right) ideals containing it, A is an annihilator algebra and consequently, by [**1**, p. 157, Corollary], A is dual.

COROLLARY 4.2. *Let A be a* B**-algebra. Then A is dual if and only if, for every maximal modular left ideal M, there is a right identity modulo M that belongs to the closure of the socle of A.*

Proof. If A is dual, the proof of (i) $\Rightarrow$ (ii) of Theorem 4.1 shows that, for every maximal modular left ideal M, there is a right identity modulo M which belongs to the socle of A. Since every element of the closure of the socle is w.c.c. the converse is an immediate consequence of Theorem 4.1.

COROLLARY 4.3. *A* B**-algebra A is dual if and only if it is* w.c.c.

Proof. If A is dual, then the socle of A is dense [**10**, p. 100, Theorem (2.8.15)] and consequently A is w.c.c. The converse is clear from Theorem 4.1.

COROLLARY 4.4. *A* B**-algebra A is dual if and only if its socle is dense.*

For other proofs of Corollaries 4.3 and 4.4, see [**9**] and [**6**] respectively.

REFERENCES

1. F. F. Bonsall and A. W. Goldie, Annihilator algebras, *Proc. London Math. Soc.* (3) **4** (1954), 154–167.

2. F. F. Bonsall and B. J. Tomiuk, The semi-algebra generated by a compact linear operator, *Proc. Edinburgh Math. Soc.* (2) **14** (1965), 177–196.

3. J. Dixmier, *C*-algèbres et leurs représentations* (Paris, 1964).

4. N. Dunford and J. T. Schwartz, *Linear operators*, Part I (New York, 1958).

5. I. Kaplansky, Normed algebras, *Duke Math. J.* **16** (1949), 399–418.

6. I. Kaplansky, The structure of certain operator algebras, *Trans. Amer. Math. Soc.* **70** (1951), 219–255.

7. G. Köthe, *Topologische lineare Räume* (Springer-Verlag, 1960).

8. T. Ogasawara, Finite dimensionality of certain Banach algebras, *J. Sci. Hiroshima Univ. Ser. A* **17** (1954), 359–364.

9. T. Ogasawara and K. Yoshinaga, Weakly completely continuous Banach*-algebras, *J. Sci. Hiroshima Univ. Ser. A* **18** (1954), 15–36.

10. C. E. Rickart, *General theory of Banach algebras* (New York, 1960).

11. S. Sakai, Weakly compact operators on operator algebras, *Pacific J. Math.* **14** (1964), 659–664.

UNIVERSITY OF OTTAWA
OTTAWA, CANADA

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