

\wedge -TRANSITIVE DIGRAPHS PRESERVING A CARTESIAN DECOMPOSITION

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Abstract

In this paper, we combine group-theoretic and combinatorial techniques to study \wedge -transitive digraphs admitting a cartesian decomposition of their vertex set. In particular, our approach uncovers a new family of digraphs that may be of considerable interest.

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1. Introduction

One of the most interesting families of highly symmetric graphs is the family of *distance-transitive* graphs. We refer the reader to the survey article [18] for the current status of the project of classifying these graphs. A major step towards this classification is a theorem of Praeger *et al.* [14] which investigates the structure of distance-primitive graphs: distance-transitive graphs admitting a group of automorphisms acting primitively on the vertices. (A graph is distance-transitive if its automorphism group acts transitively on ordered pairs of vertices at each fixed distance.) The main tool in [14] is the O’Nan–Scott theorem for finite primitive permutation groups.

In this paper, by relaxing the conditions required of a distance-primitive graph in three ways, we discover a hitherto unstudied family of digraphs ((iii) from Theorem 1.1 below). These digraphs might well provide nice examples or counter-examples to other interesting problems. Their automorphism groups are not transitive on directed paths between vertices at distance 2 (except when $n = 1$), but are transitive on the arcs of the digraph, with a lot of additional symmetry. Our three relaxations are as follows. First, we allow directed graphs. Second, our (di)graphs admit a kind of 2-distance-transitivity, but we do not require any higher distance-transitivity. More precisely, our (di)graphs are transitive on ordered pairs of vertices that either are adjacent, or

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are nonadjacent but share an out-neighbour. We refer to this as \wedge -transitivity (see Definition 2.2). In the case of graphs, this is the same as 2-distance-transitivity. Third, our (di)graphs do not necessarily admit a group of automorphisms acting primitively on their vertex set. It should be noted that the (di)graphs we consider must admit a group of automorphisms that preserves a cartesian decomposition, and cannot be Cayley (di)graphs on a specified subgroup of their automorphism group (see Section 2 and Definition 2.1 for a precise statement of our hypothesis). Additional examples do arise if we allow Cayley (di)graphs; Lemma 4.1, for example, shows that Paley tournaments are one such class; cycles (directed or undirected) are another.

In the analysis of distance-primitive graphs in [14], the authors use the classification of finite simple groups (CFSG) in two ways: first, in their Proposition 2.4, to deal with primitive groups of product action type; and then (via the Schreier conjecture) to deal with primitive groups of twisted wreath type. Proposition 2.4 of [14] is about digraphs, and while CFSG-free proofs are known in the case where the graphs are undirected (one appears in [17]), they are not readily available. Neither [14] nor [17] claim knowledge of a CFSG-free proof in the directed case. We provide such a proof in this paper. Naturally, since our paper is about product action, we do not consider the use of the classification in the analysis in [14] of twisted wreath type.

Our main result is the following. The notation $H(m, n)$ is used for the Hamming graph that is isomorphic to the cartesian product of the complete graph K_m with itself n times.

THEOREM 1.1. *Let G be a product action type group with base N . Suppose that G acts transitively on the arcs of the digraph Γ , and on the pairs of vertices that are nonadjacent but share an out-neighbour. If Γ is not a Cayley digraph on N then Γ is isomorphic to one of:*

- (i) $H(m, n)$;
- (ii) the complement of $H(m, 2)$; or
- (iii) one of the graphs $X_q(n)$ in Example 3.3. In this case, Γ is a digraph.

In the light of Theorem 1.1 and the proof of the main theorem of [14] we see (without appealing to the CFSG) that if Γ is a distance-primitive graph, then either Γ is as in Theorem 1.1(i) and (ii), or Γ is a Cayley graph over a characteristically simple group.

COROLLARY 1.2. *The graph $H(m, 2)$ and its complement are the only 2-arc-transitive graphs admitting a group of automorphisms of product action type.*

The hypothesis on the decomposition of the vertex set of Γ as a cartesian product is important. In fact, Li and Seress [10] have obtained several intricate examples of 2-distance-transitive graphs Γ with $V\Gamma$ not admitting an $\text{Aut}(\Gamma)$ -invariant cartesian decomposition. In these remarkable examples $V\Gamma$ has a $\text{Aut}(\Gamma)$ -invariant partition \mathcal{B} , and the quotient graph Γ/\mathcal{B} does admit a cartesian decomposition.

Finally, the definition of ‘product action type’ that we use (see Definition 2.1) is inspired by [15], which is a complete treatment of permutation groups that preserve a cartesian decomposition.

2. Notation and basic examples

Let H be a permutation group acting on the set Δ , let T be a transitive normal subgroup of H and let K be a transitive subgroup of the symmetric group $\text{Sym}(n)$ on $\{1, \dots, n\}$ with $n \geq 2$. We let W denote the wreath product $H \text{ wr } K$ acting on the cartesian product $\Omega = \Delta^n$. Thus, for $\sigma \in K$ and $h_1, \dots, h_n \in H$, the group element $g = \sigma(h_1, \dots, h_n)$ of W acts on $(\delta_1, \dots, \delta_n) \in \Omega$ by

$$(\delta_1, \dots, \delta_n)^g = (\delta_{1^{\sigma^{-1}}}^{h_1}, \dots, \delta_{n^{\sigma^{-1}}}^{h_n}).$$

In other words, σ permutes the n coordinates and the n -tuple (h_1, \dots, h_n) acts coordinatewise.

For $i \in \{1, \dots, n\}$, we denote by T_i the i th coordinate subgroup of T^n , that is, $T_i = \{(t_1, \dots, t_n) \in T^n : t_j = 1, \forall j \in \{1, \dots, n\} \setminus \{i\}\}$. As H normalizes T , the group W acts by conjugation on the set $\{T_1, \dots, T_n\}$ and the action of W on $\{T_1, \dots, T_n\}$ is permutation equivalent to the action of K on $\{1, \dots, n\}$. Furthermore, the normal subgroup $N = T_1 \times \dots \times T_n$ of W acts transitively on Ω .

DEFINITION 2.1. We say that $G \leq W$ is of *product action type* with base N if:

- (i) T is not regular on Δ ;
- (ii) $N \leq G$; and
- (iii) the action of G on $\{T_1, \dots, T_n\}$ is transitive.

For each $i \in \{1, \dots, n\}$, consider the group $G_i = N_G(T_i)$. If $g = \sigma(h_1, \dots, h_n) \in G_i$, then $i^\sigma = i$ and the function $\pi_i : G_i \rightarrow H$ mapping g to h_i defines a group homomorphism. As G is transitive on $\{1, \dots, n\}$, for each $i, j \in \{1, \dots, n\}$, the group H^{π_i} is conjugate to H^{π_j} . In particular, replacing H by the image of π_i if necessary, we may assume that each π_i is surjective, for each i .

In this paper, we assume that G is of product action type and is a group of automorphisms of a connected (directed or undirected) graph Γ with vertex set $V\Gamma = \Omega$. We let $A\Gamma$ denote the arcs of Γ and, for a vertex v of Γ , we let $\Gamma^+(v)$ (respectively, $\Gamma^-(v)$) denote the out-neighbours (respectively, in-neighbours) of v and we write

$$A_{\neq}^2\Gamma = \{(u, v) \in V\Gamma \times V\Gamma : u, v \text{ are nonadjacent and } \Gamma^+(u) \cap \Gamma^+(v) \neq \emptyset\}$$

(equivalently, $(u, v) \in A_{\neq}^2\Gamma$ if u and v are nonadjacent and $u, v \in \Gamma^-(w)$, for some $w \in V\Gamma$).

We are concerned with the following kind of action.

DEFINITION 2.2. A group G acts \wedge -transitively on a digraph Γ if G acts transitively on $A\Gamma$ and on $A_{\neq}^2\Gamma$. (We also say that Γ is \wedge -transitive.)

If Γ is undirected, then our definition coincides with the definition of 2-distance-transitive graphs. For digraphs, this is not the most natural definition of 2-distance-transitivity, hence we use the term \wedge -transitive for this action, a term that was suggested to the first author by Peter Neumann during a discussion of this work. In generalizing our arguments from the undirected case to the directed case, this is the transitivity requirement that most naturally arises.

We have two motivations (aside from feasibility) for using this definition. First, this sort of 2-distance-transitivity has been previously studied, although not named. It was investigated by Praeger *et al.* [14] (see, for example, [14, Proposition 2.4]). Their analysis of G and Γ heavily depends upon the CFSG (though they state without proof that they can avoid this in the undirected case). Since our main results generalize theirs in the case of product action, we produce a CFSG-free proof of this part of their result. The combination of arc-transitivity and transitivity on pairs of vertices that share an out-neighbour was also exploited to great effect by Cameron in some of his early work (see, for example, [2–4]), although since he did not require that the pairs of vertices sharing an out-neighbour be nonadjacent, his condition is stronger than ours and his work does not directly apply to ours. It is nonetheless interesting that he did encounter the family of digraphs that we call X_q in his work [3], but immediately discarded it as he was interested only in primitive actions, and did not generalize it to our family $X_q(n)$. Our second motivation is that our definition of \wedge -transitivity covers the special case where G acts transitively on each of the three sorts of pairs of vertices at distance 2, namely $A_+^2\Gamma$,

$$A^2\Gamma = \{(u, v) \in V\Gamma \times V\Gamma : u, v \text{ are nonadjacent and } \Gamma^+(u) \cap \Gamma^-(v) \neq \emptyset\},$$

and

$$A_-^2\Gamma = \{(u, v) \in V\Gamma \times V\Gamma : u, v \text{ are nonadjacent and } \Gamma^-(u) \cap \Gamma^-(v) \neq \emptyset\}$$

(which is a very natural definition of 2-distance-transitivity on digraphs). There is one more remark we wish to make in this direction. (Given a digraph Γ , denote by Γ^{opp} the digraph with $V\Gamma^{\text{opp}} = V\Gamma$ and with $A\Gamma^{\text{opp}} = \{(u, v) : (v, u) \in A\Gamma\}$.) If G acts transitively on $A\Gamma$ and on $\{(u, v) \in V\Gamma \times V\Gamma : u, v \text{ are nonadjacent and } \Gamma^-(u) \cap \Gamma^-(v) \neq \emptyset\}$, then our arguments apply immediately to G and to Γ^{opp} .

We stress that in Definition 2.1 we assume that T does not act regularly on Δ ; this condition is imposed in order to avoid the case where Γ is a Cayley graph on N .

Throughout the rest of the paper, we let Δ, n, Ω, T, H and W be as above. Furthermore, let G be a product action type subgroup of W and let Γ be a connected digraph with $\Omega = V\Gamma$ and with G acting \wedge -transitively on Γ .

REMARK 2.3. The action of a group G of product action type always preserves the Hamming distance between vertices. This is easy to verify, but very important.

We fix, once and for all, δ an element of Δ , $\alpha = (\delta, \dots, \delta) \in V\Gamma$ and $\beta = (\delta_1, \dots, \delta_n) \in \Gamma^-(\alpha)$.

REMARK 2.4. Since N is transitive on $V\Gamma$, we have $G = NG_\alpha$ and, as N acts trivially by conjugation on the set $\{T_1, \dots, T_n\}$ and G acts transitively on $\{T_1, \dots, T_n\}$, we see that G_α acts transitively by conjugation on $\{T_1, \dots, T_n\}$.

2.1. Structure of the paper. Our proof is divided in various cases, depending upon the Hamming distance between α and β . In Section 4 we study the case where α and β are at Hamming distance 1. In Section 5 we study the case where α and β are at Hamming distance at least 2. First we show that $n = 2$ (in particular, α and β are at Hamming distance 2), then in Section 5.1 we study the case where Γ is undirected, and finally in Section 5.2 we study the case where Γ is directed.

3. Examples of \wedge -transitive digraphs with product action

In this section we explain how to construct the graphs and digraphs that are listed in Theorem 1.1. First we give the definition of orbital graph. This will be required in the construction of some of the examples that follow.

DEFINITION 3.1. Let G be a transitive permutation group on the set Ω and let α and β be elements of Ω . The *orbital graph* $(\beta, \alpha)^G$ is the graph with vertex set Ω and with arc set $\{(\beta^g, \alpha^g) : g \in G\}$. The group G acts transitively on the arcs of $(\beta, \alpha)^G$, the in-neighbourhood of α is β^{G_α} and the out-neighbourhood of β is α^{G_β} .

In the next example we describe the Hamming distance and the well-known Hamming graphs.

EXAMPLE 3.2. We say that $\omega = (\delta_1, \dots, \delta_n)$ and $\omega' = (\delta'_1, \dots, \delta'_n)$ of $\Omega = \Delta^n$ are at *Hamming distance* k if ω and ω' agree in all but k coordinates, that is, $k = |\{i \in \{1, \dots, n\} : \delta_i \neq \delta'_i\}|$. We denote this by $d_H(\omega, \omega') = k$.

Write $m = |\Delta|$. Let $H(m, n)$ be the graph with vertex set Ω and with ω adjacent to ω' if $d_H(\omega, \omega') = 1$. The group $W = \text{Sym}(\Delta) \text{ wr } \text{Sym}(n)$ acts transitively on the vertices of $H(m, n)$, the stabilizer in W of the vertex $\alpha = (\delta, \dots, \delta)$ of $H(m, n)$ is $\text{Sym}(\Delta \setminus \{\delta\}) \text{ wr } \text{Sym}(n)$ and acts transitively on the neighbourhood of α in $H(m, n)$ and on the vertices at distance 2 from α in $H(m, n)$. Therefore W acts 2-distance-transitively (equivalently, since $H(m, n)$ a graph, \wedge -transitively) on $H(m, n)$. When $n = 2$, $H(m, 2)$ has diameter 2 and so the complement of $H(m, 2)$ is also 2-distance-transitive.

The directed graphs arising in the next example show some remarkable properties which (to the best of our knowledge) have not been noticed previously.

EXAMPLE 3.3 (The directed graphs X_q and $X_q(n)$). Let q be a power of a prime with $q \equiv 3 \pmod{4}$ and $q \geq 7$, and let $H = \text{SL}(2, q)$ be the special linear group. Note that as $q \equiv 3 \pmod{4}$, the element -1 of \mathbb{F}_q is not a square. Let $V = \mathbb{F}_q \times \mathbb{F}_q$ be the vector space of dimension 2 of row vectors over the field \mathbb{F}_q of size q . Let Δ be the set of orbits of the group of diagonal matrices

$$C = \left\{ \begin{pmatrix} x^2 & 0 \\ 0 & x^2 \end{pmatrix} \mid x \in \mathbb{F}_q, x \neq 0 \right\}$$

acting on the set of nonzero vectors $V^* = V \setminus \{(0, 0)\}$. Since C acts semiregularly on V^* and $|C| = (q - 1)/2$, each orbit of C on V^* has size $(q - 1)/2$. As $|V^*| = q^2 - 1$, we obtain that Δ contains $2(q + 1)$ elements. For $(a, b) \in V^*$, we denote by $[a, b]$ the

element of Δ containing (a, b) . Since -1 is not a square, we see that $\Delta = \{[a, \pm 1], [\pm 1, 0] : a \in \mathbb{F}_q\}$.

The only nonidentity proper normal subgroup of H is the centre $Z = \langle z \rangle$ (where z is the scalar matrix with entries -1) and the orbits of Z on Δ are $\{[a, 1], [-a, -1]\}$ (for each $a \in \mathbb{F}_q$) and $\{[1, 0], [-1, 0]\}$. In particular, H acts faithfully on Δ . Furthermore, the action of H on the Z -orbits of Δ is the natural 2-transitive action of $H/Z = \text{PSL}(2, q)$ on the $q + 1$ points of the projective line. The stabilizer in H of the element $[1, 0]$ is the subgroup

$$H_{[1,0]} = \left\{ \begin{pmatrix} x^2 & 0 \\ y & x^{-2} \end{pmatrix} : x, y \in \mathbb{F}_q, x \neq 0 \right\},$$

which has four orbits on Δ , namely $\{[1, 0]\}$, $\{[-1, 0]\}$, $\{[a, 1] : a \in \mathbb{F}_q\}$ and $\{[a, -1] : a \in \mathbb{F}_q\}$, of size $1, 1, q$ and q , respectively.

We let X_q be the H -orbital graph $([1, 0], [0, 1])^H$. A computation (using the fact that -1 is not a square) is required to see that there is no $h \in H$ such that $([1, 0], [0, 1])^h = ([0, 1], [1, 0])$. Therefore X_q is a directed graph of in- and out-valency q . For example, $X_q^+([1, 0]) = \{[a, 1] : a \in \mathbb{F}_q\}$ and $X_q^-([0, 1]) = \{[1, a] : a \in \mathbb{F}_q\}$. By applying

$$\iota = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

to the set $X_q^-([0, 1])$, we obtain $X_q^-([1, 0]) = \{[a, -1] : a \in \mathbb{F}_q\}$. This gives that $VX_q = \{[1, 0], [-1, 0]\} \cup X_q^+([1, 0]) \cup X_q^-([1, 0])$ and $[-1, 0]$ is the unique vertex of X_q not adjacent to $[1, 0]$. Now vertex transitivity shows that, for each vertex v , there exists a unique vertex which is not adjacent to v (namely v^z).

We have

$$(X_q^+([1, 0]))^z = X_q^-([1, 0]),$$

and similarly

$$(X_q^-([1, 0]))^z = X_q^+([1, 0]).$$

Therefore, for each vertex v , we have $X^+(v) = X^-(v^z)$ and $X^-(v) = X^+(v^z)$. Therefore $X^+(v) \cap X^+(v^z) = \emptyset$ and $A_+^2 X = \emptyset$. Consider the matrix

$$o = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the map $\circ : \Delta \rightarrow \Delta$ defined by $v \mapsto v^o$. It is an easy computation to show that \circ determines a graph isomorphism from X_q to X_q^{opp} . So $X_q \cong X_q^{\text{opp}}$.

Let $n \geq 2$, let $W = H \text{ wr } \text{Sym}(n)$ and let $\alpha = ([1, 0], [1, 0], \dots, [1, 0])$ and $\beta = ([0, 1], [1, 0], \dots, [1, 0])$ be in Δ^n . We denote by $X_q(n)$ the orbital graph $(\beta, \alpha)^W$. Clearly, $X_q(1) = X_q$.

In Example 3.3, we exclude the case that $q = 3$. In fact, for $q = 3$, the graphs $X_q(n)$ are still well defined, but, since the socle of the group $H = \text{SL}(2, 3) \cong Q_8 \rtimes C_3$ acts regularly on Δ , we get that $X_q(n)$ is a Cayley graph (recall that we are not concerned with Cayley graphs in this paper).

The graphs $X_q(n)$ are \wedge -transitive, as the following lemma explains.

LEMMA 3.4. *Let $W, X_q(n)$ be as in Example 3.3. Then the action of W on $X_q(n)$ is \wedge -transitive.*

PROOF. The proof consists of routine computations. For a detailed argument, see [12]. □

Since $X_q \cong X_q^{\text{opp}}$, we also have $X_q(n) \cong X_q(n)^{\text{opp}}$. Therefore when $\Gamma = X_q(n)$, $\text{Aut}(\Gamma)$ is transitive not only on $A_+^2\Gamma$, but also on $A^2\Gamma$. Thus, Theorem 1.1 in fact tells us that in the situation we are studying, our definition of \wedge -transitivity for digraphs actually forces transitivity on $A^2\Gamma$ as well. Unfortunately, the automorphism group of $X_q(n)$ is not transitive on $A^2X_q(n)$ when $n > 1$, so the strongest form of 2-distance-transitivity is not forced.

4. The case $d_H(\alpha, \beta) = 1$

In this section we prove Theorem 1.1 when $d_H(\alpha, \beta) = 1$. We start by recalling the definition of a *tournament*, which surprisingly is necessary in our arguments. A tournament is a directed graph obtained by assigning a direction to each edge in an undirected complete graph (that is, every pair of vertices is connected by a single directed edge). A tournament is called *symmetric* if its automorphism group is transitive on the arcs. A finite symmetric tournament \mathcal{T} has an odd number of vertices, say $|V\mathcal{T}| = 1 + 2k$, and every vertex has k in-neighbours and k out-neighbours. The *Paley tournament* \mathcal{T}_q is the tournament with vertices the elements of the finite field \mathbb{F}_q , where $q \equiv 3 \pmod 4$, and with an arc from a to b when $b - a$ is a nonzero square in \mathbb{F}_q (that is, $b - a = x^2$ for some $x \in \mathbb{F}_q \setminus \{0\}$).

LEMMA 4.1. *Let \mathcal{T} be a finite symmetric tournament and let H be a group of automorphisms acting transitively on the arcs of \mathcal{T} . Then $\mathcal{T} \cong \mathcal{T}_q$, for some $q \equiv 3 \pmod 4$, and the socle of H acts regularly on the vertices of \mathcal{T} .*

PROOF. Berggren [1] shows that if \mathcal{T} is a finite symmetric tournament, then \mathcal{T} is isomorphic to \mathcal{T}_q for some $q \equiv 3 \pmod 4$. In particular, we may assume that $\mathcal{T} = \mathcal{T}_q$. Moreover, [1, Theorem A] gives that the automorphism group $\text{Aut}(\mathcal{T}_q)$ of \mathcal{T}_q is the group of all affine permutations of \mathbb{F}_q of the form $\tau_{\sigma, x^2, c} : a \mapsto x^2 a^\sigma + c$, where $c \in \mathbb{F}_q$, $x \in \mathbb{F}_q \setminus \{0\}$, and $\sigma \in \text{Gal}(\mathbb{F}_q)$. Using this description of $\text{Aut}(\mathcal{T}_q)$, it is easy to see that if H acts transitively on the arcs of \mathcal{T}_q , then $A = \{\tau_{\text{id}, x^2, c} : x, c \in \mathbb{F}_q, x \neq 0\}$ is a subgroup of H (where id denotes the identity Galois automorphism of \mathbb{F}_q). Now the socle of A is $\{\tau_{\text{id}, 1, c} : c \in \mathbb{F}_q\}$ and coincides with the socle of $\text{Aut}(\mathcal{T}_q)$. Clearly $T = \{\tau_{\text{id}, 1, c} : c \in \mathbb{F}_q\}$ acts regularly on the vertices of \mathcal{T}_q . □

Before proceeding, we need the following definition. (The normal quotient technique is a very important idea introduced in [13] which has proven useful in the investigation of many graphs [8, 11].)

DEFINITION 4.2. Let G be a group acting transitively on the digraph Γ , and let C be a normal subgroup of G . Let α^C denote the C -orbit containing $\alpha \in V\Gamma$. The *normal quotient* Γ_C is the graph whose vertices are the C -orbits on $V\Gamma$, with an arc between

distinct vertices α^C and β^C if and only if there is an arc of Γ between α' and β' , for some $\alpha' \in \alpha^C$ and $\beta' \in \beta^C$.

The following proposition is the most substantial result of this section. The proof is quite long and involved, however, it is elementary and we do not make use of the CFSG. The corollary that follows it will complete the proof of Theorem 1.1 in the case where $d_H(\alpha, \beta) = 1$.

PROPOSITION 4.3. *Let X be a connected H -orbital graph (where H is as defined at the start of Section 2) and let δ' be an arbitrary vertex of X (so $\delta' \in \Delta$). Assume that any two vertices of $X^-(\delta')$ are adjacent. Then either X is the complete graph, or $X = X_q$, for some $q \equiv 3 \pmod 4$.*

PROOF. Fix δ_0 a vertex of X . Suppose that X is undirected. As $X^-(\delta_0) = X(\delta_0)$ is a complete graph and as every vertex of $X(\delta_0)$ is adjacent to δ_0 , we obtain that $\{\delta_0\} \cup X(\delta_0)$ is a connected component of X . Since X is connected, we see that X is complete. Therefore, in the rest of the proof we may assume that X is a digraph. Let q be the in-valency of X .

Suppose that $|VX| = 1 + 2q$, that is, $VX = \{\delta_0\} \cup X^+(\delta_0) \cup X^-(\delta_0)$. Since H acts transitively on the vertices of X , we see that any two vertices of X are adjacent. In particular, X is a tournament. Since we are assuming that T does not act regularly on VX , from Lemma 4.1 we obtain a contradiction. In particular, in the rest of the proof we may assume that $|VX| > 1 + 2q$. If $q = 1$, then X is a directed cycle and its automorphism group is a cyclic group. The socle of $\text{Aut}(X)$ acts regularly on VX which again contradicts our hypothesis on T . Thus $q > 1$.

As H_{δ_0} acts transitively on $X^-(\delta_0)$ and as any two vertices of $X^-(\delta_0)$ are adjacent, we see that the induced subgraph of X on $X^-(\delta_0)$ is a regular tournament. In particular, q is odd. We now prove eight claims from which the result will follow.

Claim 1. The induced subgraph of X on $X^+(\delta_0)$ is a regular tournament.

Let δ' be in $X^+(\delta_0)$, let δ'' be in $X^-(\delta_0)$ and let $Y = X^+(\delta'') \cap X^-(\delta_0)$ be the out-neighbours of δ'' in $X^-(\delta_0)$. Since the induced subgraph of X on $X^-(\delta_0)$ is a regular tournament, we have $|Y| = (q - 1)/2$. As (δ'', δ_0) and (δ_0, δ') are arcs of X and H is transitive on AX , there exists $h \in H$ with $(\delta'', \delta_0)^h = (\delta_0, \delta')$. We obtain $Y^h = X^+(\delta_0) \cap X^-(\delta')$ and δ' has $|Y^h| = (q - 1)/2$ in-neighbours in $X^+(\delta_0)$. Since H_{δ_0} is transitive on $X^+(\delta_0)$, the induced subgraph of X on $X^+(\delta_0)$ has out-valency $(q - 1)/2$ and in-valency $(q - 1)/2$ and hence it is a regular tournament.

Claim 2. Let δ' be in $X^+(\delta_0)$. Then δ' has exactly $(q - 1)/2$ in-neighbours in $X^+(\delta_0)$ and exactly $(q - 1)/2$ in-neighbours in $X^-(\delta_0)$.

As $X^+(\delta_0)$ is a regular tournament, δ' has $(q - 1)/2$ in-neighbours in $X^+(\delta_0)$, that is, $|X^-(\delta') \cap X^+(\delta_0)| = (q - 1)/2$. Set $Y = X^-(\delta') \setminus (\{\delta_0\} \cup X^+(\delta_0))$ and let δ'' be in Y . As $X^-(\delta')$ is a regular tournament and as $\delta_0, \delta'' \in X^-(\delta')$, we have that δ_0 and δ'' are adjacent. Since $\delta'' \notin X^+(\delta_0)$, we get $\delta'' \in X^-(\delta_0)$. As δ'' is an arbitrary element of Y , we have $Y \subseteq X^-(\delta_0)$ and δ' has $|Y| = (q - 1)/2$ in-neighbours in $X^-(\delta_0)$.

Claim 2 shows that for each $\delta' \in X^+(\delta_0)$, we have $X^-(\delta') \subseteq (\{\delta_0\} \cup X^+(\delta_0) \cup X^-(\delta_0))$. If, for every $\delta' \in X^+(\delta_0)$, we also have $X^+(\delta') \subseteq (\{\delta_0\} \cup X^+(\delta_0) \cup X^-(\delta_0))$, then (using the transitivity of H on VX together with a connectedness argument) we obtain $VX = \{\delta_0\} \cup X^+(\delta_0) \cup X^-(\delta_0)$, which contradicts $|VX| > 1 + 2q$. This shows that there exists $\delta' \in X^+(\delta_0)$ and $\delta_0^* \in X^+(\delta')$ with $\delta_0^* \notin (\{\delta_0\} \cup X^+(\delta_0) \cup X^-(\delta_0))$.

Claim 3. $X^-(\delta_0^*) = X^+(\delta_0)$.

Let δ'' be an out-neighbour of δ' in $X^+(\delta_0)$. Since $\delta_0^*, \delta'' \in X^+(\delta')$, by Claim 1 and by vertex transitivity, we obtain that δ_0^* and δ'' are adjacent. If $\delta_0^* \in X^-(\delta'')$, then by Claim 2 applied to δ'' , we see that $\delta_0^* \in (\{\delta_0\} \cup X^+(\delta_0) \cup X^-(\delta_0))$, which contradicts our choice of δ_0^* . Therefore $\delta_0^* \in X^+(\delta'')$. Since δ'' is an arbitrary out-neighbour of δ' in $X^+(\delta_0)$, we see that every out-neighbour of δ' in $X^+(\delta_0)$ is an in-neighbour of δ_0^* . Since $X^+(\delta_0)$ is a regular tournament, it is strongly connected, so we can iterate this argument with vertices from $X^+(\delta') \cap X^+(\delta)$ taking the role of δ' to conclude that $X^+(\delta_0) \subseteq X^-(\delta_0^*)$. Since $q = |X^+(\delta_0)| = |X^-(\delta_0^*)|$, we have $X^+(\delta_0) = X^-(\delta_0^*)$.

Claim 4. $X^+(\delta_0^*) = X^-(\delta_0)$.

We first show that δ_0^* has at least one out-neighbour in $X^-(\delta_0)$. Fix an element w in $X^+(\delta_0)$ and write $U = X^-(w) \cap X^+(\delta_0)$ and $V = X^-(w) \cap X^-(\delta_0)$. From Claim 2, $|U| = |V| = (q - 1)/2$. As $X^-(\delta_0)$ is a regular tournament, by H -transitivity, we have that $X^-(w) = \{\delta_0\} \cup U \cup V$ is a regular tournament. Let δ'' be in U . Now, as δ'' has $(q - 1)/2$ out-neighbours in $X^-(w)$ and as δ_0 is not an out-neighbour of δ'' (because $\delta'' \in U \subseteq X^+(\delta_0)$), we obtain by the pigeon-hole principle that δ'' has an out-neighbour δ'_0 in V , that is, $\delta'_0 \in X^+(\delta'') \cap V$. As $\delta'' \in X^+(\delta_0) = X^-(\delta_0^*)$, we see that $\delta'_0 \in X^+(\delta'')$. Therefore, δ'_0 and δ_0^* are both in $X^+(\delta'')$. From Claim 1 applied to δ'' we get that δ'_0 and δ_0^* are adjacent. Since $\delta'_0 \in V \subseteq X^-(\delta_0)$ and since $X^-(\delta_0^*) = X^+(\delta_0)$, we have $\delta'_0 \notin X^-(\delta_0^*)$. Therefore we must have that $\delta'_0 \in X^+(\delta_0^*)$.

Now that we have shown that δ_0^* has one out-neighbour δ'_0 in $X^-(\delta_0)$, using an argument similar to the argument in the proof of Claim 3, we obtain that every element of $X^-(\delta_0)$ is an out-neighbour of δ_0^* . So $X^+(\delta_0^*) = X^-(\delta_0)$.

Claim 5. $VX = \{\delta_0, \delta_0^*\} \cup X^+(\delta_0) \cup X^-(\delta_0)$ and $|VX| = 2(1 + q)$.

We show that every vertex v in the induced subgraph of X on $V = \{\delta_0, \delta_0^*\} \cup X^+(\delta_0) \cup X^-(\delta_0)$ has in-valency q , from which the claim follows by connectedness. If $v = \delta_0$, then there is nothing to prove. If $v = \delta_0^*$, then from Claim 3 we have $X^-(\delta_0^*) = X^+(\delta_0) \subseteq V$. Also, if $v \in X^+(\delta_0)$, then from Claim 2 we have $X^-(v) \subseteq V$. It remains to consider $v \in X^-(\delta_0)$. Applying the argument in Claim 2 with $v = \delta'$ and with $\delta_0^* = \delta_0$, we obtain that v has $(q - 1)/2$ in-neighbours in $X^+(\delta_0^*)$ and $(q - 1)/2$ in-neighbours in $X^-(\delta_0^*)$. As δ_0^* is also an in-neighbour of v , we obtain $X^-(v) \subseteq V$ from Claims 3 and 4. □

Since H acts transitively on the vertices of X , we have that for every vertex v , there exists a *unique* vertex v^* with $X^+(v) = X^-(v^*)$ and $X^-(v) = X^+(v^*)$. In particular, the set $\mathcal{B} = \{v, v^*\} : v \in VX$ is a system of imprimitivity for the action of H on VX . Let C be the kernel of the action of H on \mathcal{B} .

Claim 6. Let v be in VX and let v' be in $VX \setminus \{v, v^*\}$. Then the induced subgraph of X on $\{v, v^*, v', (v')^*\}$ is a directed cycle.

From Claim 5, we see that v and v' are adjacent, so replacing (v, v') by (v', v) if necessary, we may assume that $v' \in X^+(v)$. As $v' \in X^+(v) = X^-(v^*)$, (v', v^*) is an arc of X . Since $(v')^* \neq v^*$ and since v is adjacent to every element different from v^* , we obtain that either $(v')^* \in X^+(v)$ or $(v')^* \in X^-(v)$. If $(v')^* \in X^+(v)$, then $v', (v')^* \in X^+(v)$ and so from Claim 1, v' and $(v')^*$ are adjacent, a contradiction. Therefore $((v')^*, v)$ is an arc of X . As $(v')^* \in X^-(v) = X^+(v^*)$, $(v^*, (v')^*)$ is also an arc of X .

Claim 7. H contains a unique element of order 2 and $|C| = 2$.

As $|VX| = 2(q + 1)$ is even and H acts transitively on X , the group H contains an element h of order 2. Assume that $h \in H \setminus C$. As $h \notin C$, there exists $v \in VX$ with $\{v, v^*\}^h \neq \{v, v^*\}$. Set $v' = v^h$. From Claim 6, the induced subgraph of X on $\{v, v^*, v', (v')^*\}$ is a directed cycle which is h -invariant because $h^2 = 1$. As the automorphism group of a directed cycle of length 4 is a cyclic group whose generator squares to an involution mapping v to v^* , we obtain $v' = v^h = v^*$, a contradiction. Therefore, every involution of H lies in C .

Since the blocks of \mathcal{B} have size 2, we have that C is an elementary abelian 2-group. Let h be an element of $C \setminus \{1\}$ and assume that h fixes a vertex, v say, of X . Let v' be any vertex of X with $v' \notin \{v, v^*\}$. Now, from Claim 6, the induced subgraph of X on $\{v, v^*, v', (v')^*\}$ is a directed cycle which is h -invariant. As the automorphism group of a directed cycle is a cyclic group acting regularly and as h fixes v , we obtain that h fixes $v, v^*, v', (v')^*$. Since v' is an arbitrary element of VX with $v' \notin \{v, v^*\}$, we obtain $h = 1$. This shows that $|C| = 2$.

We let z denote the generator of C . We have $v^z = v^*$, for each $v \in VX$. Since the group C acts transitively on $\{v, v^*\}$ (for each $v \in VX$), the system of imprimitivity \mathcal{B} consists of the orbits of C on VX . In particular, the quotient graph X_C is a normal quotient. From Claim 5, X_C is an undirected complete graph with $q + 1$ vertices. Write $\overline{H} = H/C$. Since H acts arc-transitively on X , the group \overline{H} acts arc-transitively on X_C . In particular, \overline{H} is a 2-transitive group of degree $q + 1$. Let $\overline{T} = T/C$ be the socle of \overline{H} with $C \leq T \trianglelefteq H$. By a celebrated theorem of Burnside [7, Theorem 4.1B], \overline{T} is either a regular elementary abelian p -group (for some prime p), or a nonregular nonabelian simple group.

Assume that \overline{T} is abelian (we show that this is impossible). Since $|\overline{T}| = q + 1$, we have $|T| = 2(q + 1)$ and T acts regularly on VX , that is, $H = TH_v$ and $T \cap H_v = 1$. As z is the unique element of T of order 2, p is odd and $T = \mathbf{O}_2(T) \times \mathbf{O}_p(T) = C \times \mathbf{O}_p(T)$. Clearly, H_v centralizes C and hence $H = TH_v = C \times (\mathbf{O}_p(T)H_v)$. In particular, $\mathbf{O}_p(T)H_v$ has odd order and so does H_v . As \overline{H} is 2-transitive, \overline{H}_v has order divisible by q , which is even, a contradiction. This shows that \overline{T} is a nonregular nonabelian simple group.

As \overline{H} is primitive, \overline{T} is transitive on VX_C and T is transitive on VX . Let S be a Sylow 2-subgroup of T . Since H has a unique involution, the group S has a unique

involution (namely z). It follows from [16, Proposition (4.4), page 59] that S is either a cyclic group of order 2^a (for $a \geq 1$), or a quaternion group of order 2^a (for $a \geq 3$). It follows that either S/C is a cyclic group of order 2^{a-1} , or a dihedral group of order 2^{a-1} (if $a \geq 4$), or an elementary abelian 2-group of order 4 (if $a = 3$).

This shows that \bar{T} is a nonregular nonabelian simple group whose Sylow 2-subgroup S/C is either cyclic, or dihedral or elementary abelian of order 4.

Claim 8. $T = \text{SL}(2, r)$ for some odd r .

From [16, Corollary 2, page 144], we see that the Sylow 2-subgroup of a simple group is not cyclic. If C splits over T (that is, C has a complement, L say, in T), then $T = L \times C$ for some finite nonabelian simple group L . As L has even order by the odd order theorem, the group T has more than one involution, which contradicts Claim 7. Thus C does not split over T . Therefore T is a quotient of the universal covering group U of \bar{T} , that is, $T \cong U/Z$ for some central subgroup Z of U . We now show that $\bar{T} \cong \text{PSL}(2, r)$, for some odd r .

Suppose that S/C is a dihedral group. From the classification of Gorenstein and Walter [9] of the nonabelian simple groups with a dihedral Sylow 2-subgroup, we see that either $\bar{T} \cong \text{Alt}(7)$ or $\bar{T} \cong \text{PSL}(2, r)$ for some odd r with $r \equiv 1 \pmod{8}$ or $r \equiv 7 \pmod{8}$. From [6], we see that $\text{Alt}(7)$ has only two 2-transitive permutation representations, one of degree 7 and one of degree 15. As $q + 1$ is even, we obtain that $\bar{T} \not\cong \text{Alt}(7)$.

We may now assume that S/C is an elementary abelian group of order 4. From the classification of Walter [19] of the nonabelian simple groups with an abelian Sylow 2-subgroup, we see that either $\bar{T} \cong \text{PSL}(2, r)$ (for $r = 2^b$, or for some odd r with $r \equiv 3 \pmod{8}$ or $r \equiv 5 \pmod{8}$), or $\bar{T} = J_1$, or $\bar{T} = {}^2G_2(3^\ell)$ (for some odd $\ell > 1$). From [6, Table 5 and page 36], we see that the universal covering groups of ${}^2G_2(3^\ell)$ and of J_1 are simple. Hence $T = U = \bar{T}$, a contradiction. Since a Sylow 2-subgroup of $\text{PSL}(2, 2^b)$ has order 2^b , we obtain that $b = 2$. Moreover, since $\text{PSL}(2, 4) \cong \text{PSL}(2, 5)$, we can include this case in the odd characteristic.

From [6, Table 5], the universal covering group of $\text{PSL}(2, r)$, with r odd, is $\text{SL}(2, r)$, whose centre has order 2. Since $|T| = 2|\bar{T}|$, we get $T = U = \text{SL}(2, r)$ and C is the centre of T .

We are now ready to conclude the proof. From [5, Table 7.4, page 197], the group $\text{PSL}(2, r)$ has only one 2-transitive permutation representation of even degree, namely the natural action of degree $r + 1$ on the points of the projective line. In particular, $r = q$. Moreover, there exists $\{v, v^*\} \in \mathcal{B}$ such that the stabilizer $T_{\{v, v^*\}}$ is the Borel subgroup

$$\left\{ \begin{pmatrix} x & 0 \\ b & x^{-1} \end{pmatrix} : x, b \in \mathbb{F}_q, x \neq 0 \right\}.$$

Since $|\{v, v^*\}| = 2$, the stabilizer T_v has index 2 in $T_{\{v, v^*\}}$. As $T_{\{v, v^*\}} \cong \mathbb{F}_q \rtimes \mathbb{F}_q^*$, we see that $T_{\{v, v^*\}}$ has a unique subgroup of index 2 and hence

$$T_v = \left\{ \begin{pmatrix} x^2 & 0 \\ b & x^{-2} \end{pmatrix} : x, b \in \mathbb{F}_q, x \neq 0 \right\}.$$

Therefore the action of T on VX can be identified with the action of T on the right cosets of T_v .

If $q \equiv 1 \pmod 4$, then the centre C of T is contained in T_v and the action of T on VX is unfaithful, a contradiction. So $q \equiv 3 \pmod 4$. Moreover, T_v has four orbits on T/T_v of size 1, 1, q and q respectively. Therefore, T gives rise to only two orbital graphs of out-valency q , namely X_q and X_q^{opp} . As $X_q \cong X_q^{\text{opp}}$, the proof is complete. \square

As a consequence of Proposition 4.3 we obtain the following corollary.

COROLLARY 4.4. *If $d_H(\alpha, \beta) = 1$, then either $\Gamma = H(m, n)$ or $\Gamma = X_q(n)$.*

PROOF. Since $d_H(\alpha, \beta) = 1$ and since the action of G_α on $\{T_1, \dots, T_n\}$ is transitive, replacing β by a suitable conjugate under G_α if necessary, we may assume that $\beta = (\delta', \delta, \dots, \delta)$, for some $\delta' \in \Delta \setminus \{\delta\}$. In particular,

$$\begin{aligned} \Gamma^-(\alpha) = \beta^{G_\alpha} &= (\delta'^{H_\delta} \times \{\delta\} \times \dots \times \{\delta\}) \cup (\{\delta\} \times \delta'^{H_\delta} \times \{\delta\} \times \dots \times \{\delta\}) \\ &\cup \dots \cup (\{\delta\} \times \dots \times \{\delta\} \times \delta'^{H_\delta}). \end{aligned} \tag{4.1}$$

We denote by X_1, \dots, X_n the n sets on the right-hand side of β^{G_α} (for instance, $X_1 = \delta'^{H_\delta} \times \{\delta\} \times \dots \times \{\delta\}$). For each $i \in \{1, \dots, n\}$, any two distinct vertices in X_i are at Hamming distance 1. Furthermore, for each $i, j \in \{1, \dots, n\}$ with $i \neq j$, and for each $x \in X_i, y \in X_j$, we have that $d_H(x, y) = 2$. As α and β are adjacent and $d_H(\alpha, \beta) = 1$ and as G is transitive on $A_+^2\Gamma$, this shows that if $(u, v) \in A_+^2\Gamma$, then $d_H(u, v) = 2$. Furthermore, for each $i \in \{1, \dots, n\}$, every two vertices u, v of X_i are adjacent because $d_H(u, v) = 1$ and $u, v \in \Gamma^-(\alpha)$.

Let X be the H -orbital graph $(\delta', \delta)^H$ and let X' be the connected component of X containing δ . Observe that $(\delta, \dots, \delta, \nu, \delta, \dots, \delta) \in \Gamma^-(\alpha)$ if and only if $(\nu, \delta) \in AX$. Set $Y = X' \times \dots \times X'$ (seen as a subset of $V\Gamma = \Delta^n$). Let $u = (\varepsilon_1, \dots, \varepsilon_n)$ be an element of Y . With a computation similar to the case of the vertex α , we have

$$\begin{aligned} \Gamma^-(u) &= \{(\nu, \varepsilon_2, \varepsilon_3, \dots, \varepsilon_n) : (\nu, \varepsilon_1) \in AX\} \cup \{(\varepsilon_1, \nu, \varepsilon_3, \dots, \varepsilon_n) : (\nu, \varepsilon_2) \in AX\} \\ &\cup \dots \cup \{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}, \nu) : (\nu, \varepsilon_n) \in AX\}. \end{aligned}$$

For each $i \in \{1, \dots, n\}$ and $\nu \in VX$ with $(\nu, \varepsilon_i) \in AX$, we have $\nu \in X'$ because $\varepsilon_i \in X'$ and X' is a connected component of X . Therefore we obtain $\Gamma^-(u) \subseteq Y$. As u is an arbitrary vertex in Y and Γ is connected, we have $Y = \Delta^n$. Hence $VX' = \Delta$, that is, X is connected.

Let x and y be in $X^-(\delta) = \delta'^{H_\delta}$. From (4.1), the vertices $\gamma_x = (x, \delta, \dots, \delta)$ and $\gamma_y = (y, \delta, \dots, \delta)$ are in X_1 and hence are adjacent in Γ . In particular, it can be shown that any $g \in G_\alpha$ that takes γ_x to γ_y , must fix the first coordinate, and hence that x and y must be adjacent in X . Since x and y are arbitrary elements of $X^-(\delta)$, we have that any two vertices in $X^-(\delta)$ are adjacent.

As X is connected, from Proposition 4.3 we have that either X is complete (and so $\Gamma = H(m, n)$) or $X = X_q$ (and so $\Gamma = X_q(n)$). \square

5. The case $d_H(\alpha, \beta) \geq 2$

In this section, we start our analysis of the case in which $d_H(\alpha, \beta) \geq 2$, by showing that if this occurs, then in fact $n = 2$ (so $d_H(\alpha, \beta) = 2$) as well as some other restrictions.

LEMMA 5.1. *Assume that $d_H(\alpha, \beta) = k \geq 2$. If $(u, v) \in A_+^2\Gamma$, then $d_H(u, v) = 1$. Furthermore, $n = 2$ and neither δ_1 nor δ_2 is fixed by T_δ .*

PROOF. Note that since $d_H(\alpha, \beta) = k \geq 2$, $\beta \in \Gamma^-(\alpha)$, and G is arc-transitive, Remark 2.3 implies that any pair of adjacent vertices must be at Hamming distance $k \geq 2$. Suppose that N_α fixes $\Gamma^-(\alpha)$ pointwise. Since $N \trianglelefteq G$, we have that N_γ fixes $\Gamma^-(\gamma)$ pointwise for each vertex γ , and so by connectedness, $N_\alpha = 1$ and N acts regularly on the vertices of Γ . As we are assuming that N is not regular, we have a contradiction. Therefore N_α does not fix $\Gamma^-(\alpha)$ pointwise. Since $N_\alpha \trianglelefteq G_\alpha$, $\beta \in \Gamma^-(\alpha)$, and G_α is transitive on $\Gamma^-(\alpha)$, the group $N_\alpha = T_\delta^n$ does not fix β and hence there exists a coordinate δ_i of β with T_δ not fixing δ_i , for some $i \in \{1, \dots, n\}$. Using Remark 2.4, we see that, replacing β by a suitable conjugate under G_α if necessary, we may assume that $i = 1$. Let t be in $T_\delta \setminus T_{\delta_1}$, that is, $\delta^t = \delta$ and $\delta_1^t \neq \delta_1$. Now $g = (t, 1, \dots, 1) \in N_\alpha \subseteq G_\alpha$ and $\gamma = \beta^g \in \Gamma^-(\alpha)$. Since $d_H(\beta, \gamma) = 1$ but any pair of adjacent vertices is at Hamming distance $k \geq 2$, we see that β and γ are not adjacent in Γ . Therefore since $\beta, \gamma \in \Gamma^-(\alpha)$, we have that $(\beta, \gamma) \in A_+^2\Gamma$. Since G acts transitively on $A_+^2\Gamma$, Remark 2.3 implies that all the pairs in $A_+^2\Gamma$ are at Hamming distance 1, which proves the first part of this lemma.

Suppose that β has only one entry not fixed by T_δ . Using Remark 2.4, we see that, replacing β by a suitable conjugate if necessary, we may assume that δ_1 (the first coordinate of β) is not fixed by T_δ and $\delta_2, \dots, \delta_n$ are pointwise fixed by T_δ . Therefore $T_\delta = T_{\delta_i}$ for each $2 \leq i \leq n$ and

$$N_\beta = T_{\delta_1} \times T_\delta \times \dots \times T_\delta \quad \text{with } T_\delta \neq T_{\delta_1}. \tag{5.1}$$

Since N is transitive on the vertices of Γ and since $N_\alpha \trianglelefteq G_\alpha$, we have that for every vertex γ of Γ and for every $v \in \Gamma^-(\gamma)$, the group N_γ acts nontrivially only on one coordinate of v . Since G_α acts transitively on $\{T_1, \dots, T_n\}$, there exists $x = \tau(h_1, \dots, h_n)$ in $G_\alpha \subseteq W_\alpha = H_\delta \text{ wr } K$ with $1^\tau = 2$. Consider the vertex

$$\begin{aligned} \gamma = \beta^x &= (\delta_{1^{\tau-1}}^{h_1}, \delta_{2^{\tau-1}}^{h_2}, \delta_{3^{\tau-1}}^{h_3}, \dots, \delta_{n^{\tau-1}}^{h_n}) \\ &= (\delta_{1^{\tau-1}}^{h_1}, \delta_1^{h_2}, \delta_{3^{\tau-1}}^{h_3}, \dots, \delta_{n^{\tau-1}}^{h_n}) \in \Gamma^-(\alpha). \end{aligned} \tag{5.2}$$

Since $x \in G_\alpha$, we have $\gamma \in \Gamma^-(\alpha)$, so T_δ acts nontrivially on only one coordinate of γ . Since T_δ does not fix δ_1 , $T_\delta \trianglelefteq H_\delta$ and $h_2 \in H_\delta$, we obtain that T_δ does not fix $\delta_1^{h_2}$. As T_δ fixes δ_2 , we have $\delta_2 \neq \delta_1^{h_2}$. Moreover, as T_δ fixes $\delta_{1^{\tau-1}}$, we have $\delta_1 \neq \delta_{1^{\tau-1}}^{h_1}$. So $d_H(\beta, \gamma) \geq 2$ (the first two coordinates of β and γ are distinct). Since $\beta, \gamma \in \Gamma^-(\alpha)$, we obtain from the previous paragraph that $(\beta, \gamma) \notin A_+^2\Gamma$. So β and γ are adjacent in Γ , that is, either $\gamma \in \Gamma^-(\beta)$ or $\beta \in \Gamma^-(\gamma)$. Suppose that $\gamma \in \Gamma^-(\beta)$. Thus N_β acts nontrivially on only one coordinate of γ . Now, as T_δ does not fix $\delta_1^{h_2}$, we obtain from (5.1) that the

second coordinate of γ is the only coordinate not fixed by N_β . Therefore, from (5.1) and (5.2), we have

$$T_{\delta_1} = T_{\delta_{1^{\tau^{-1}}}}^{h_1}, \quad T_\delta \neq T_{\delta_1^{h_2}} \quad \text{and} \quad T_\delta = T_{\delta_i^{h_i}} \quad \text{for } i \in \{3, \dots, n\}.$$

Since $1^{\tau^{-1}} \neq 1$, we see from (5.1) that T_δ fixes $\delta_{1^{\tau^{-1}}}$ and, since $T_\delta \triangleleft H_\delta$ and $h_1 \in H_\delta$, we have

$$T_{\delta_1} = T_{\delta_{1^{\tau^{-1}}}}^{h_1} = (T_{\delta_1^{\tau^{-1}}})^{h_1} = T_\delta^{h_1} = T_\delta.$$

Thus T_δ fixes δ_1 , contradicting (5.1). A similar contradiction (along these lines) is obtained by supposing that $\beta \in \Gamma^-(\gamma)$. Therefore β must have at least two entries not fixed by T_δ . Now, to conclude the proof, it suffices to show that $n = 2$.

Replacing G by a suitable conjugate in H wr $\text{Sym}(n)$ if necessary, we may assume that the first two coordinates of β (that is, δ_1 and δ_2) are not fixed by T_δ . Let $t' \in T_\delta \setminus T_{\delta_2}$ so $\delta_2^{t'} \neq \delta_2$, and set $n' = (t, t', 1, \dots, 1) \in N_\alpha$. As $\gamma' = \beta^{n'}$ has exactly two entries different from β (namely, in the first two coordinates), we see that $(\beta, \gamma') \notin A_+^2\Gamma$ (the pairs in $A_+^2\Gamma$ are at Hamming distance 1). Since $\beta, \gamma' \in \Gamma^-(\alpha)$, we see that β and γ' are adjacent and hence, by Remark 2.3, since G acts arc-transitively, $k = 2$. In particular, as β is adjacent to α , we have $\beta = (\delta_1, \delta_2, \delta, \dots, \delta)$.

Suppose that $n \geq 3$. Since G_α acts transitively on $\{1, \dots, n\}$, there exists $x = \tau(h_1, \dots, h_n) \in G_\alpha \leq W_\alpha = H_\delta$ wr K with $3^x = 1$. Since the third coordinate of β is δ , we obtain that the first coordinate of $\gamma = \beta^x$ is δ . Since β and γ each have $n - 2$ coordinates equal to δ and since the first coordinate of β is $\delta_1 \neq \delta$, we obtain that $d_H(\beta, \gamma) \geq 2$. Therefore $(\beta, \gamma) \notin A_+^2\Gamma$. As $\beta, \gamma \in \Gamma^-(\alpha)$, the vertices β and γ are adjacent in Γ and, in particular, $d_H(\beta, \gamma) = k = 2$. Since β and γ differ in the first coordinate and in the i th coordinate for some $i \in \{3, \dots, n\}$ such that the i th coordinate of γ is not δ , we obtain that $\gamma = (\delta, \delta_2, \dots)$, that is, the second coordinate of γ equals the second coordinate of β . Set $n'' = (1, t', 1, \dots, 1) \in N_\alpha$. The vertex $\gamma^{n''}$ is adjacent to α and $d_H(\gamma^{n''}, \beta) = 3$, thus $(\beta, \gamma^{n''}) \notin A_+^2\Gamma$ and β and $\gamma^{n''}$ are adjacent, contradicting that $k = 2$. This yields $n = 2$. □

From Lemma 5.1, we have $n = 2$, $\alpha = (\delta, \delta)$, $\beta = (\delta_1, \delta_2)$ and δ_1, δ_2 are not fixed by T_δ . We start our analysis with a rather technical lemma. This tells us that if two rows (or columns) of Δ^2 are in the same H_δ -orbit, then there is an element of G that fixes α and takes the first row (or column) to the second (without exchanging the coordinates).

LEMMA 5.2. *For $\varepsilon_1, \varepsilon_2 \in \Delta$ and for $i \in \{1, 2\}$, we have $\pi_i(G_i \cap G_{(\varepsilon_1, \varepsilon_2)}) = H_{\varepsilon_i}$.*

PROOF. Fix $i \in \{1, 2\}$ and write $R = \pi_i(G_i \cap G_{(\varepsilon_1, \varepsilon_2)})$. Since $G_{(\varepsilon_1, \varepsilon_2)} \leq W_{(\varepsilon_1, \varepsilon_2)}$, we see that $R \leq H_{\varepsilon_i}$. Also, as $T_{\varepsilon_1} \times T_{\varepsilon_2} \leq G_{(\varepsilon_1, \varepsilon_2)}$, we see that $T_{\varepsilon_i} \leq R$.

Since N is transitive on the vertices of Γ , we have $G = NG_{(\varepsilon_1, \varepsilon_2)}$. Furthermore, since $N \leq G_i$, from the ‘modular law’, we obtain $G_i = NG_{(\varepsilon_1, \varepsilon_2)} \cap G_i = N(G_i \cap G_{(\varepsilon_1, \varepsilon_2)})$. Applying π_i on both sides of this equality, we get $H = \pi_i(G_i) = \pi_i(T)\pi_i(G_i \cap G_{(\varepsilon_1, \varepsilon_2)}) = TR$. Again using the ‘modular law’, we see that $H_{\varepsilon_i} = H_{\varepsilon_i} \cap TR = (H_{\varepsilon_i} \cap T)R = T_{\varepsilon_i}R = R$. □

In what follows we use Lemma 5.2 with $\varepsilon_1 = \varepsilon_2 = \delta$ (except in the proof of Lemma 5.14, where we need it in its full generality).

The following two facts hardly deserve to be called lemmas, but will be used several times.

LEMMA 5.3. *If $\gamma_1, \gamma_2 \in \Gamma^-(\gamma)$ and $d_H(\gamma_1, \gamma_2) = 2$, then γ_1 and γ_2 are adjacent.*

PROOF. Either $(\gamma_1, \gamma_2) \in A_+^2\Gamma$, or γ_1 is adjacent to γ_2 . But Lemma 5.1 says that if $(\gamma_1, \gamma_2) \in A_+^2\Gamma$, then $d_H(\gamma_1, \gamma_2) = 1$, a contradiction. \square

LEMMA 5.4. *If $d_H(\gamma, \gamma') = 1$, then γ_1 and γ_2 are not adjacent.*

PROOF. This is simply a reminder of Remark 2.3. \square

We require some information about neighbourhoods. These will be used in our proofs of both the undirected and directed cases. Since each of these results applies (with the same proof) to any one of $\Gamma(\alpha')$, $\Gamma^+(\alpha')$ and $\Gamma^-(\alpha')$ for the appropriate choice of α' , we introduce the notation $\Gamma^*(\alpha')$. This notation will be used to indicate that the result holds when ‘ Γ^* ’ is replaced by any one of ‘ Γ ’, ‘ Γ^+ ’, or ‘ Γ^- ’.

We have $\beta = (\delta_1, \delta_2) \in \Gamma^-(\alpha)$ (in the undirected case, this is $\Gamma(\alpha)$). Let $\gamma = (\delta'_1, \delta'_2) \in \Gamma^+(\alpha)$. Clearly,

$$\begin{aligned} \Gamma(\alpha) &= \beta^{G_\alpha} \subseteq \beta^{W_\alpha} = (\delta_1^{H_\delta} \times \delta_2^{H_\delta}) \cup (\delta_2^{H_\delta} \times \delta_1^{H_\delta}), \\ \Gamma^-(\alpha) &= \beta^{G_\alpha} \subseteq \beta^{W_\alpha} = (\delta_1^{H_\delta} \times \delta_2^{H_\delta}) \cup (\delta_2^{H_\delta} \times \delta_1^{H_\delta}), \\ \Gamma^+(\alpha) &= \gamma^{G_\alpha} \subseteq \gamma^{W_\alpha} = ((\delta'_1)^{H_\delta} \times (\delta'_2)^{H_\delta}) \cup ((\delta'_2)^{H_\delta} \times (\delta'_1)^{H_\delta}). \end{aligned}$$

Thus $\Gamma(\alpha)$ (in the undirected case) or $\Gamma^-(\alpha)$ (in the directed case) is the disjoint union of N_α -orbits each of which is a ‘rectangle’ with $|\delta_1^{H_\delta}|$ rows and $|\delta_2^{H_\delta}|$ columns or with $|\delta_2^{H_\delta}|$ rows and $|\delta_1^{H_\delta}|$ columns. If δ_1 and δ_2 are in the same H_δ -orbit then these two rectangles will instead be a single square. Similarly, $\Gamma^+(\alpha)$ has the same structure.

LEMMA 5.5. *Fix any $\delta' \in \Delta$. For each $\varepsilon \in \delta'^{H_\delta}$, the number a of $v \in \Delta$ with $(\varepsilon, v) \in \Gamma^*(\alpha)$ is equal to the number of $v \in \Delta$ with $(v, \varepsilon) \in \Gamma^*(\alpha)$ and depends only on the H_δ -orbit δ'^{H_δ} (and not on the element ε).*

Furthermore, if $(\varepsilon_1, \varepsilon_2) \in \Gamma^(\alpha)$ and $\varepsilon_1^{H_\delta} = \varepsilon_2^{H_\delta}$, then $|\Gamma^*(\alpha)| = a|\varepsilon_1^{H_\delta}|$.*

PROOF. Fix ε in $(\delta')^{H_\delta}$ and let h_1 be in H_δ with $\varepsilon^{h_1} = \delta'$. From Lemma 5.2, there exists $h_2 \in H_\delta$ such that $g = (h_1, h_2) \in G_\alpha$. In particular, applying the automorphism g , we see that if v_1, \dots, v_a are the elements of Δ with $(\varepsilon, v_i) \in \Gamma^*(\alpha)$, then $v_1^{h_2}, \dots, v_a^{h_2}$ are exactly the elements of Δ with $(\delta', v_i^{h_2}) \in \Gamma^*(\alpha)$. This shows that the number a does not depend on the choice of ε in $(\delta')^{H_\delta}$.

We now show that there are exactly a elements v in Δ with $(v, \delta') \in \Gamma^*(\alpha)$. Let v_1, \dots, v_a be the elements of Δ with $(\delta', v_i) \in \Gamma^*(\alpha)$. Since G_α is transitive on $\{T_1, T_2\}$, there exists $x = (1 \ 2)(t_1, t_2) \in G_\alpha$. As $t_2 \in H_\delta$, from Lemma 5.2 there exists $h_1 \in H_\delta$ such that $y = (h_1, t_2^{-1}) \in G_\alpha$. Now $z = xy = (1 \ 2)(t_1 h_1, 1) \in G_\alpha$ and $(\delta', v_i)^z = (v_i^{t_1 h_1}, \delta')$, for $i \in \{1, \dots, a\}$, are exactly the elements in $\Gamma^*(\alpha)$ with second coordinate δ' .

If $(\varepsilon_1, \varepsilon_2) \in \Gamma^*(\alpha)$ and $\varepsilon_1^{H_\delta} = \varepsilon_2^{H_\delta}$, then $\Gamma^*(\alpha) \subseteq \varepsilon_1^{H_\delta} \times \varepsilon_1^{H_\delta}$, and by Lemma 5.2, there are elements of $\Gamma^*(\alpha)$ with every possible first coordinate from $\varepsilon_1^{H_\delta}$. By the earlier part of this lemma, there are exactly a such elements for every possible first coordinate, making $|\Gamma^*(\alpha)| = a|\varepsilon_1^{H_\delta}|$, as claimed. \square

5.1. Γ is undirected. We confine our attention to the undirected case first, which will prove easiest to complete. We reserve the letters a and b to denote the numbers defined in Lemma 5.5 that come from choosing $\delta' = \delta_1$ and $\delta' = \delta_2$, respectively. As T_δ does not fix either δ_1 or δ_2 , we have $a, b \geq 2$.

A subset $X \subseteq V\Gamma$ is said to be *independent* if any two elements of X are nonadjacent.

LEMMA 5.6. *If γ is any vertex of Γ and $(\varepsilon_1, \varepsilon_2) = \gamma' \in \Gamma(\gamma)$, then $\{(v_1, v_2) \in \Gamma(\gamma) : v_1 = \varepsilon_1\}$ and $\{(v_1, v_2) \in \Gamma(\gamma) : v_2 = \varepsilon_2\}$ are the only maximal independent sets in $\Gamma(\gamma)$ containing γ' . Moreover, the cardinalities of these sets are a and b (respectively).*

PROOF. Write $X_1 = \{(v_1, v_2) \in \Gamma(\gamma) : v_1 = \varepsilon_1\}$ and $X_2 = \{(v_1, v_2) \in \Gamma(\gamma) : v_2 = \varepsilon_2\}$. Lemma 5.4 shows that X_1 and X_2 are independent sets (both containing γ'), and Lemma 5.3 implies that no other vertex of $\Gamma(\gamma)$ is independent of γ' , so these independent sets are maximal and there are no others. The cardinality follows from Lemma 5.5. \square

COROLLARY 5.7. *If $b \neq a$, then for each vertex γ , the neighbourhood $\Gamma(\gamma)$ can be uniquely decomposed into a disjoint union of independent sets of cardinality b .*

PROOF. Since $b \neq a$, Lemma 5.6 says that every neighbour of γ lies in a unique maximal independent set of cardinality b . The uniqueness means that these sets must be disjoint. The result follows. \square

LEMMA 5.8. *The vertices of $\Gamma(\beta) \setminus \Gamma(\alpha)$ are: (δ, v) for every v such that $(\delta_1, v) \in \Gamma(\alpha)$, and (v, δ) for every v such that $(v, \delta_2) \in \Gamma(\alpha)$.*

PROOF. From Lemmas 5.3 and 5.4, we see that the elements of $\Gamma(\beta) \setminus (\Gamma(\alpha) \cup \{\alpha\})$ are of the form (δ, v) or (v, δ) , for some $v \in \Delta \setminus \{\delta\}$. Let v be in $\Delta \setminus \{\delta\}$ with $(\delta, v) \in \Gamma(\beta)$. We need to show that $(\delta_1, v) \in \Gamma(\alpha)$. We argue by contradiction, so we assume that $(\delta_1, v) \notin \Gamma(\alpha)$. Write $X_\alpha = \{\eta \in \Delta : (\eta, v) \in \Gamma(\alpha)\}$ and $X_\beta = \{\eta \in \Delta : (\eta, v) \in \Gamma(\beta)\}$.

Lemma 5.6 shows that $|X_\beta|$ is either a or b . Since a and b are each at least 2, $|X_\beta| \geq 2$, so Lemma 5.3 implies that $X_\alpha \neq \emptyset$. Hence $|X_\alpha|$ is also either a or b . Now, replacing a by b if necessary, we may assume that $b \geq a$. Since α and β are adjacent, Lemmas 5.3 and 5.4 yield that $X_\alpha \setminus \{\delta_1\} \subseteq X_\beta$ and $X_\beta \setminus \{\delta\} \subseteq X_\alpha$. (Because if $(\mu, v) \in \Gamma(\alpha)$ and $\mu \neq \delta_1$, then since $(\delta, v) \in \Gamma(\beta)$ we have $v \neq \delta_2$, so $\beta \in \Gamma(\alpha)$ implies $(\mu, v) \in \Gamma(\beta)$.) As we are assuming that $\delta_1 \notin X_\alpha$, we get $X_\beta = X_\alpha \cup \{\delta\}$, so $|X_\beta| = |X_\alpha| + 1$. Since $b \geq a$, we obtain $|X_\beta| = b$, $|X_\alpha| = a$ and $b = a + 1$.

Write $X_\alpha = \{x_1, \dots, x_a\}$, $\gamma = (\delta, v)$ and $\gamma_i = (x_i, v)$, for $i \in \{1, \dots, a\}$. From the previous paragraph, $\{\gamma_1, \dots, \gamma_a\}$ is a maximal independent set of $\Gamma(\alpha)$ of size a and $\{\gamma, \gamma_1, \dots, \gamma_a\}$ is a maximal independent set of $\Gamma(\beta)$ of size b . So, from Lemma 5.6 applied to α and γ_i (for each $i \in \{1, \dots, a\}$), we see that there exists $Y_i \subseteq \Delta$ of size b

such that $V_i = \{(x_i, y) : y \in Y_i\}$ is a maximal independent set of $\Gamma(\alpha)$. If $\delta_2 \notin Y_i$, then by Lemmas 5.3 and 5.4 the set V_i is contained in $\Gamma(\beta)$. Therefore V_i and $\{\gamma, \gamma_1, \dots, \gamma_a\}$ are both independent sets of $\Gamma(\beta)$ of size b containing γ_i , which contradicts Lemma 5.6. Thus $\delta_2 \in Y_i$ for each $i \in \{1, \dots, a\}$, so $(x_i, \delta_2) \in \Gamma(\alpha)$ for every $i \in \{1, \dots, a\}$.

Now V_1 and $\{\beta, (x_1, \delta_2), \dots, (x_a, \delta_2)\}$ are independent sets of $\Gamma(\alpha)$ of size b both containing (x_1, δ_2) , again contradicting Lemma 5.6. This final contradiction gives that $(\delta_1, \nu) \in \Gamma(\alpha)$.

The proof for the neighbours of β of the form (ν, δ) is entirely symmetric. \square

LEMMA 5.9. Γ has diameter 2.

PROOF. Towards a contradiction, suppose that γ is a vertex at distance 3 from α . Then γ has a neighbour α' that is at distance 2 from α . By Lemma 5.1, $d_H(\alpha', \alpha) = 1$.

Let σ be an arbitrary mutual neighbour of α and α' . Then σ can take the role of β in Lemma 5.8, which means that since $\alpha' \in \Gamma(\sigma) \setminus \Gamma(\alpha)$, it must be the case that the unique vertex that lies at Hamming distance 1 from both σ and α' but at Hamming distance 2 from α , is in $\Gamma(\alpha)$. Thus, the mutual neighbours of α and α' can be found in the following manner: first, choose any vertex $\tau \in \Gamma(\alpha)$ with $d_H(\tau, \alpha') = 1$; then (by Lemma 5.8), any vertex $\sigma \in \Gamma(\alpha)$ with $d_H(\sigma, \tau) = 1$ but $d_H(\sigma, \alpha') = 2$, will be a mutual neighbour of α and α' .

Since γ is at distance 3 from α , and any such σ is adjacent to α and to α' , it must be the case that σ is at distance 2 from γ . Then by Lemma 5.1, for any such σ , $d_H(\sigma, \gamma) = 1$.

We have either a or b choices for τ (vertices that are in $\Gamma(\alpha)$ at Hamming distance 1 from α'). For each choice of τ , we have either $b - 1$ or $a - 1$ choices for σ (vertices that are in $\Gamma(\alpha)$, at Hamming distance 1 from τ and at Hamming distance 2 from α'). Thus, there are either $b(a - 1)$ or $a(b - 1)$ possible choices for σ . However every such σ is in $\Gamma(\alpha')$ together with γ . By Lemmas 5.4 and 5.6, there can be at most $a - 1 + b - 1 = a + b - 2$ choices for σ . The inequality $b(a - 1) \leq a + b - 2$ can be solved only if $a = 2$, while the inequality $a(b - 1) \leq a + b - 2$ can be solved only if $b = 2$. So, replacing a by b if necessary, we may assume that $a = 2$. In particular, $b \geq a$. Furthermore, without loss of generality, we can assume that there are b choices for τ , and for each of these there is a unique choice for σ . (We denote by σ_τ the choice of σ determined by τ .)

Using \wedge -transitivity, we may assume that $\alpha' = (\delta, \delta')$ (recall that $\alpha = (\delta, \delta)$). Notice that the b choices for τ all have the same value in their second entry because they are at Hamming distance 2 from α and at Hamming distance 1 from α' . So they have b distinct values in their first entry. However, since $d_H(\sigma_\tau, \alpha') = 2$ and $d_H(\sigma_\tau, \tau) = 1$, we obtain that τ and σ_τ have the same first entry. Moreover, for each σ_τ , $d_H(\sigma_\tau, \gamma) = 1$ only if σ_τ has the same second entry as γ , for each τ . But in this case, these b choices for σ_τ , together with γ , form an independent set of cardinality $b + 1$ in $\Gamma(\alpha')$, contradicting Lemma 5.6. \square

The proof of Theorem 1.1 when Γ is undirected is now easy.

COROLLARY 5.10. Γ is isomorphic to the complement of $H(m, 2)$.

PROOF. From Lemma 5.9, we conclude that there are no vertices at distance 3 from α . Since Γ is connected, every vertex is at distance 1 or 2 from α . Since G is \wedge -transitive and preserves Hamming distance, every vertex at Hamming distance 2 from α is adjacent to α in Γ , and every vertex at Hamming distance 1 from α is at distance 2 from α in Γ . But then Γ is isomorphic to the complement of the Hamming graph $H(m, 2)$. \square

5.2. Γ is directed. In this subsection we assume that Γ is directed and we conclude the proof of Theorem 1.1. We let k denote the number of out-neighbours of any vertex (so the total valency of a vertex is $2k$). Note that for any vertex v , we have $\Gamma^+(v) \cap \Gamma^-(v) = \emptyset$, since otherwise, by arc-transitivity, Γ is undirected. Furthermore, we fix $\gamma = (\delta'_1, \delta'_2) \in \Gamma^+(\alpha)$.

LEMMA 5.11. For η and v in Δ , we have $|\eta^{H_v}| = |v^{H_\eta}|$.

PROOF. A double counting gives

$$|\Delta||v^{H_\eta}| = |(\eta, v)^H| = |\Delta||\eta^{H_v}|$$

(see [20, Theorem 16.3]). Since Δ is finite and nonempty, this concludes our proof. \square

We will need the following fact in a few places.

LEMMA 5.12. We have $\delta_1^{H_\delta} = \delta_2^{H_\delta}$ if and only if $\delta_1^{H_\delta} = \delta_2^{H_\delta}$.

PROOF. If $\delta_1^{H_\delta} = \delta_2^{H_\delta}$, then there is some $h \in H_\delta$ with $\delta_1^h = \delta_2$. Arc-transitivity means that there is some $g \in G$ with $(\beta, \alpha)^g = (\alpha, \gamma)$. Now, $g = \sigma(h_1, h_2)$ with $h_1, h_2 \in H$ and $\sigma = 1$ or $\sigma = (1\ 2)$. As $\alpha^g = \gamma$, we must have $\delta^{h_1} = \delta'_1$ and $\delta^{h_2} = \delta'_2$.

If $\sigma = 1$, from $\beta^g = \alpha$ we have $\delta_1^{h_1} = \delta$ and $\delta_2^{h_2} = \delta$. It is now clear that $h_2^{-1}h^{-1}h_1 \in H_\delta$, and $(\delta'_2)^{h_2^{-1}h^{-1}h_1} = \delta'_1$, completing the proof in this case. If $\sigma = (1\ 2)$, from $\beta^g = \alpha$ we have $\delta_2^{h_1} = \delta$ and $\delta_1^{h_2} = \delta$. It is clear that $h_1^{-1}h^{-1}h_2 \in H_\delta$, and $(\delta'_1)^{h_1^{-1}h^{-1}h_2} = \delta'_2$, completing the proof.

The converse is analogous. \square

The next result nicely limits the cases that we need to consider.

LEMMA 5.13. If $(\varepsilon_1, \delta), (\delta, \varepsilon_2) \in \Gamma^-(\gamma)$ with $\varepsilon_1, \varepsilon_2 \neq \delta$, then $\varepsilon_1^{H_\delta} = \varepsilon_2^{H_\delta}$. Moreover, $\delta_1^{H_\delta} = \delta_2^{H_\delta}$, and either $\varepsilon_1^{H_\delta} = \delta_1^{H_\delta}$, or $\varepsilon_1^{H_\delta} = \delta_1^{H_\delta}$.

PROOF. We start by proving that if $\alpha_1 = (\varepsilon_1, \delta)$, $\alpha_2 = (\delta, \varepsilon_2)$ and $\alpha_1, \alpha_2 \in \Gamma^-(\gamma)$, then $\varepsilon_1^{H_\delta} = \varepsilon_2^{H_\delta}$. As $d_H(\alpha_1, \alpha) = d_H(\alpha_2, \alpha) = 1$, we see from Lemma 5.4 that α_1 and α_2 are not adjacent to α , that is, $(\alpha, \alpha_1), (\alpha, \alpha_2) \in A_+^2\Gamma$. So, by \wedge -transitivity, there must be some element $g \in G_\alpha$ with $\alpha_1^g = \alpha_2$. It is not hard to see that $g = (1\ 2)(h_1, h_2)$, $h_1, h_2 \in H_\delta$ and $\varepsilon_1^{h_2} = \varepsilon_2$. Hence ε_1 and ε_2 are in the same H_δ -orbit, that is, $\varepsilon_1^{H_\delta} = \varepsilon_2^{H_\delta}$.

Certainly,

$$\Gamma^-(\gamma) \supseteq \alpha^{N_\gamma} = \delta^{T_{\delta'_1}} \times \delta^{T_{\delta'_2}}. \tag{\dagger}$$

Let $\varepsilon_1 \in \delta^{T_{\delta'_1}} \setminus \{\delta\}$ and $\varepsilon_2 \in \delta^{T_{\delta'_2}} \setminus \{\delta\}$. (Note that ε_1 and ε_2 are well defined because $T_{\delta'_i}$ does not fix δ .) As $(\varepsilon_1, \delta), (\delta, \varepsilon_2) \in \Gamma^-(\gamma)$, from the previous paragraph, we have $\varepsilon_1^{H_\delta} = \varepsilon_2^{H_\delta}$. We also have from (\dagger) that $\alpha' = (\varepsilon_1, \varepsilon_2) \in \Gamma^-(\gamma)$ has $d_H(\alpha', \alpha) = 2$. So by Lemma 5.3, α and α' are adjacent. If $\alpha' \in \Gamma^+(\alpha)$, then by arc-transitivity, $\Gamma^+(\alpha) = \alpha'^{G_\alpha} \subseteq \alpha'^{W_\alpha} = \varepsilon_1^{H_\delta} \times \varepsilon_2^{H_\delta}$ (since ε_1 and ε_2 are in the same H_δ -orbit). As $\gamma \in \Gamma^+(\alpha)$ and $\gamma = (\delta'_1, \delta'_2)$, we obtain $\delta'_1, \delta'_2 \in \varepsilon_1^{H_\delta}$ and $\delta_1^{H_\delta} = \delta_2^{H_\delta}$. On the other hand, if $\alpha' \in \Gamma^-(\alpha)$, then an analogous argument yields $\Gamma^-(\alpha) \subseteq \varepsilon_1^{H_\delta} \times \varepsilon_2^{H_\delta}$, so $\delta_1^{H_\delta} = \delta_2^{H_\delta} = \varepsilon_1^{H_\delta}$, and Lemma 5.12 completes the proof. \square

We can now obtain an extension of Lemma 5.5.

LEMMA 5.14. *Let $\gamma' = (\delta'_1, \delta'_1)$. For each $\varepsilon_1 \in \delta_1^{H_\delta}$ and $\varepsilon_2 \in \delta^{H_{\delta'_1}}$, the cardinalities of the following sets are equal and do not depend on ε_1 or ε_2 :*

- (a) $\{v \in \Delta : (\varepsilon_1, v) \in \Gamma^+(\alpha)\}$;
- (b) $\{v \in \Delta : (v, \varepsilon_1) \in \Gamma^+(\alpha)\}$;
- (c) $\{v \in \Delta : (\varepsilon_2, v) \in \Gamma^-(\gamma')\}$;
- (d) $\{v \in \Delta : (v, \varepsilon_2) \in \Gamma^-(\gamma')\}$.

Furthermore, $k = a|\delta_1^{H_\delta}|$, where a is the cardinality of each of these sets, and $a \geq 2$.

PROOF. The equality of the cardinalities of the sets in (a) and (b) follows immediately from Lemma 5.5. Replacing α by γ' in Lemma 5.5 shows the equality of the cardinalities of the sets in (c) and (d). Call the first of these cardinalities a , and the second a' . By Lemma 5.13 we have $\delta_1^{H_\delta} = \delta_2^{H_\delta}$, so since $\gamma = (\delta'_1, \delta'_2) \in \Gamma^+(\alpha)$, Lemma 5.5 tells us that $k = a|\delta_1^{H_\delta}|$.

We will now find an in-neighbour of γ' whose second entry is δ , and whose first entry is in $\delta^{H_{\delta'_1}}$. Then Lemma 5.5 applied to this in-neighbour shows that $k = a'|\delta^{H_{\delta'_1}}|$. Since $|\delta_1^{H_\delta}| = |\delta^{H_{\delta'_1}}|$ by Lemma 5.11, this will show that $a = a'$.

Let $t \in H_\delta$ such that $(\delta'_2)^t = \delta'_1$ (we can do this since $(\delta'_2)^{H_\delta} = (\delta'_1)^{H_\delta}$ by Lemma 5.13). Let $\alpha'' = (\delta'_1, \delta)$. By Lemma 5.2, there must be some $g = (t', t) \in G_{\alpha''}$. Applying g to α and γ , we get $\gamma^g = (\delta'_1, \delta'_1)$ and $\alpha^g = (\delta', \delta)$. This shows that α^g is an in-neighbour of γ' with the desired form.

That $a \geq 2$ follows from the fact that $T_{\delta'_i}$ does not fix δ . \square

For the rest of this subsection, we let a denote the constant defined in Lemma 5.14.

LEMMA 5.15. *The sets $\Gamma^+(\beta') \cap \Gamma^-(\alpha)$ and $\Gamma^-(\beta') \cap \Gamma^-(\alpha)$ each have cardinality $(k - 2a + 1)/2$ for any $\beta' \in \Gamma^-(\alpha)$.*

PROOF. If $\beta' \in \Gamma^-(\alpha)$ with $d_H(\beta', \beta) = 1$, then by Lemma 5.4, there cannot be an arc between β and β' . However, if $\beta' \in \Gamma^-(\alpha)$ with $d_H(\beta', \beta) = 2$, then by Lemma 5.3, there must be an arc between β and β' . Using the fact that there cannot be arcs in both directions between β and any other vertex, we conclude that β has precisely $k - 2a + 1$ arcs to or from other vertices in $\Gamma^-(\alpha)$. Since G is arc-transitive, every vertex of $\Gamma^-(\alpha)$ has the same number of out-neighbours in $\Gamma^-(\alpha)$ as every other vertex; also, every

vertex of $\Gamma^-(\alpha)$ has the same number of in-neighbours in $\Gamma^-(\alpha)$ as every other vertex. This shows that the total number of arcs both of whose endpoints lie within $\Gamma^-(\alpha)$ is $k(k - 2a + 1)/2$ (we divide by 2 since each arc has been counted at both ends). Our conclusions are immediate. \square

We can now generalize Lemma 5.3 to vertices that share an in-neighbour.

LEMMA 5.16. *If γ_1 and γ_2 share an in-neighbour and $d_H(\gamma_1, \gamma_2) = 2$, then they must be adjacent.*

PROOF. Call the shared in-neighbour α' . We will show that γ_1 has $(k - 2a + 1)/2$ out-neighbours and $(k - 2a + 1)/2$ in-neighbours in $\Gamma^+(\alpha')$. Since there are $2a - 2$ vertices in $\Gamma^+(\alpha')$ that are at Hamming distance 1 from γ_1 (by Lemma 5.14), and $d_H(\gamma_1, \gamma_1) \neq 2$, there must be $k - 2a + 1$ vertices in $\Gamma^+(\alpha')$ that are at Hamming distance 2 from γ_1 , so this count will show that all of these vertices are adjacent to γ_1 , which yields the conclusion.

By Lemma 5.15 and arc-transitivity, $|\Gamma^+(\alpha') \cap \Gamma^-(\gamma_1)| = (k - 2a + 1)/2$. Consider the induced subgraph on $\Gamma^+(\alpha')$. Since $G_{\alpha'}$ is transitive on this set, the in-valency and out-valency of every vertex is constant in this subgraph, so every vertex has in-valency and out-valency $(k - 2a + 1)/2$, since γ_1 has this in-valency. \square

LEMMA 5.17. *For any $\gamma' \in \Gamma^+(\alpha)$, the sets $\Gamma^-(\gamma') \cap \Gamma^+(\alpha)$ and $\Gamma^+(\gamma') \cap \Gamma^+(\alpha)$ have cardinality $(k - 2a + 1)/2$.*

PROOF. Replacing $\Gamma^-(\alpha)$ by $\Gamma^+(\alpha)$ and β by γ' throughout the proof of Lemma 5.15, with Lemma 5.3 replaced by Lemma 5.16, yields the desired conclusion. \square

COROLLARY 5.18. *If $\beta' \in \Gamma^-(\alpha)$, then $\Gamma^+(\beta') \subset \{\alpha\} \cup \Gamma^+(\alpha) \cup \Gamma^-(\alpha) \cup HD_1(\alpha)$, where $HD_1(\alpha)$ is the set of vertices at Hamming distance 1 from α .*

Also, if $\gamma' \in \Gamma^+(\alpha)$, then $\Gamma^-(\gamma') \subset \{\alpha\} \cup \Gamma^+(\alpha) \cup \Gamma^-(\alpha) \cup HD_1(\alpha)$.

PROOF. We have $|\Gamma^+(\beta')| = k$. The sets $\Gamma^+(\alpha)$ and $\Gamma^-(\alpha)$ are disjoint since we are in the directed case, and Lemmas 5.15 and 5.17 together with arc-transitivity tell us that $|\Gamma^+(\beta') \cap \Gamma^+(\alpha)| = |\Gamma^+(\beta') \cap \Gamma^-(\alpha)| = (k - 2a + 1)/2$, so this accounts for all but $2a - 1$ of the out-neighbours of β' . But $\alpha \in \Gamma^+(\beta')$, and by Lemma 5.14 with vertex-transitivity, we see that for $i = 1, 2$ there must be precisely $a - 1$ other out-neighbours of β' that have the same entry as α in coordinate i . By Lemma 5.4, none of these vertices is in either $\Gamma^+(\alpha)$ or $\Gamma^-(\alpha)$, so these together with α itself form the remaining $2a - 1$ out-neighbours of β' .

The proof for γ' is analogous. \square

LEMMA 5.19. *Suppose that $\delta_1^{H_\delta} = \delta_1^{H_\delta}$. Then for any $\delta' \in \delta_1^{H_\delta}$, we have $\delta^{H_{\delta'}} \setminus \{\delta\} = \delta'^{H_\delta} \setminus \{\delta'\}$.*

PROOF. We will show that $\delta^{H_{\delta'}} = (\delta^{H_\delta} \cup \{\delta\}) \setminus \{\delta'\}$. Clearly $\delta \in \delta^{H_{\delta'}} \setminus \delta'^{H_\delta}$ and $\delta' \in \delta'^{H_\delta} \setminus \delta^{H_{\delta'}}$, and the cardinalities of the two orbits are equal (by Lemma 5.11), so if we can show that $\delta^{H_{\delta'}} \subset \delta'^{H_\delta} \cup \{\delta\}$, that will be sufficient.

Since $\gamma = (\delta'_1, \delta'_2) \in \Gamma^+(\alpha)$ and $\delta' \in \delta_1^{H_\delta}$, Lemma 5.2 tells us that there is some element of G_α that fixes the first coordinate (so fixes each of the two coordinates) and takes γ to some vertex γ' whose first entry is δ' . Clearly, $\gamma' \in \Gamma^+(\alpha)$.

A similar argument shows that there is some in-neighbour of α whose first entry is any fixed element of $\delta_1^{H_\delta}$. Thus, there exist some in-neighbour of α and some out-neighbour of α whose first entries are any fixed element of $\delta_1^{H_\delta}$. In fact, Lemma 5.14 tells us that there exist a in-neighbours and a out-neighbours of α in each of these columns. These are in fact all of the in- and out-neighbours of α , since $k = a|\delta_1^{H_\delta}|$.

By Corollary 5.18, $\Gamma^-(\gamma') \subset \Gamma^+(\alpha) \cup \Gamma^-(\alpha) \cup HD_1(\alpha)$. Now, we have just concluded that any neighbour of α must have its first entry in the set $\delta_1^{H_\delta}$. The in-neighbours of γ' , therefore, must have their first entries in the set $(\delta_1^{H_\delta} \cup \{\delta\}) \setminus \{\delta'\}$. But since α is an in-neighbour of γ' , Lemma 5.2 tells us that there is an element of G that fixes γ' , fixes the coordinates, and takes the column containing δ to the column indexed by any element of $\delta^{H_{\delta'}}$. So these indices must be elements of $(\delta_1^{H_\delta} \cup \{\delta\}) \setminus \{\delta'\}$, meaning that we must have $\delta^{H_{\delta'}} \subset \delta^{H_\delta} \cup \{\delta\}$, as desired. \square

LEMMA 5.20. *Suppose that $\delta_1^{H_\delta} = \delta_2^{H_\delta} = \delta_1^{H_{\delta'}} = \delta_2^{H_{\delta'}}$. Then H is 2-transitive on Δ .*

PROOF. We claim that if $\Delta \neq \delta_1^{H_\delta} \cup \{\delta\}$, then Γ is disconnected. This will be a contradiction, so we conclude that $\Delta = \delta_1^{H_\delta} \cup \{\delta\}$, which forces H to be 2-transitive on Δ , completing the proof.

If we can prove that whenever ε is in $(\delta_1^{H_\delta} \cup \{\delta\}) \times (\delta_1^{H_{\delta'}} \cup \{\delta\})$ and has some in- or out-neighbour in this set, all of its in- and out-neighbours must be in this set, this will establish the claim we made in the preceding paragraph, and so complete the proof. Let $\varepsilon = (\delta_3, \delta_4)$ and $\mu = (\delta'_3, \delta'_4)$, where $\delta_3, \delta_4, \delta'_3, \delta'_4 \in \delta_1^{H_\delta} \cup \{\delta\}$, and suppose that μ is either an in-neighbour or an out-neighbour of ε . Let us suppose that $\mu \in \Gamma^-(\varepsilon)$. By arc-transitivity, we have $\Gamma^-(\varepsilon) = \mu^{G_\varepsilon} \subseteq ((\delta'_3)^{H_{\delta_3}} \times (\delta'_4)^{H_{\delta_4}}) \cup ((\delta'_4)^{H_{\delta_3}} \times (\delta'_3)^{H_{\delta_4}})$. Using Lemma 5.19, it is straightforward to verify that this is a subset of $((\delta'_1)^{H_\delta} \cup \{\delta\}) \times ((\delta'_1)^{H_{\delta'}} \cup \{\delta\})$. It remains to show that $\Gamma^+(\varepsilon)$ is also in this set. Notice that a similar argument shows that $\Gamma^+(\mu) \subseteq (\delta_1^{H_\delta} \cup \{\delta\}) \times (\delta_1^{H_{\delta'}} \cup \{\delta\})$ since μ has one out-neighbour (namely, ε) in this set. But by Lemma 5.17 and arc-transitivity, $\Gamma^+(\mu) \cap \Gamma^+(\varepsilon)$ has cardinality $(k - 2a + 1)/2$. Since Lemma 5.14 shows that a divides k and $a \geq 2$, we must have $(k - 2a + 1)/2 > 0$, so ε has at least one out-neighbour in $(\delta_1^{H_\delta} \cup \{\delta\}) \times (\delta_1^{H_{\delta'}} \cup \{\delta\})$, from which a similar argument shows that all out-neighbours of ε are in this set.

The case in which $\mu \in \Gamma^+(\varepsilon)$ is precisely analogous to the above case. \square

LEMMA 5.21. *It is not possible to have $\delta_1^{H_\delta} = \delta_2^{H_\delta} = \delta_1^{H_{\delta'}} = \delta_2^{H_{\delta'}}$.*

PROOF. Towards a contradiction, suppose that $\delta_1^{H_\delta} = \delta_2^{H_\delta} = \delta_1^{H_{\delta'}} = \delta_2^{H_{\delta'}}$. By Lemma 5.15 we have $(k - 2a + 1)/2 \in \mathbb{Z}$, so k must be odd. But $k = a|\delta_1^{H_\delta}|$ (by Lemma 5.14), so a and $|\delta_1^{H_\delta}|$ are both odd. Then by Lemma 5.20, $|\Delta| = |\delta_1^{H_\delta}| + 1$, so $|\Delta|$ must be even.

Now, T is a transitive group acting on Δ , so T must have even order. Hence T contains an involution t . Without loss of generality, we can assume that $\delta' = \delta' \neq \delta$,

and $\delta'' = \delta$. Since $\Delta = \delta_1^{H_\delta} \cup \{\delta\}$, we have $\delta' \in \delta_1^{H_\delta} = (\delta'_1)^{H_\delta}$. Now by Lemma 5.14, α has a out-neighbours whose first entry is δ' ; we choose one of these, (δ', δ'') .

Since H is 2-transitive on Δ (by Lemma 5.20), there exists $h \in H_\delta$ such that $\delta'^h = \delta''$. Now since $T \triangleleft H$, we have $h^{-1}th \in T$. We know that $T \times T = N \leq G$; consider the action of $g = (t, h^{-1}th) \in G$ on α and on (δ', δ'') . We have $(\delta, \delta)^{(t, h^{-1}th)} = (\delta', \delta'')$ since $h \in H_\delta$. And $(\delta', \delta'')^{(t, h^{-1}th)} = (\delta, (\delta')^{th}) = (\delta, \delta^h) = (\delta, \delta)$. So g reverses this arc, contradicting the fact that we are in the directed case. \square

By Lemma 5.13 we have $(\delta'_1)^{H_\delta} = (\delta'_2)^{H_\delta}$, so, by Lemma 5.12, $\delta_1^{H_\delta} = \delta_2^{H_\delta}$ and the hypothesis eliminated in our next lemma is the only remaining possibility.

THEOREM 5.22. *It is not possible to have $\delta_1^{H_\delta} \neq \delta_1^{H_\delta}$.*

PROOF. By Corollary 5.18, $\Gamma^-(\gamma) \subset \Gamma^+(\alpha) \cup \Gamma^-(\alpha) \cup HD_1(\alpha)$. So by Lemma 5.13, we can conclude that either

$$\begin{aligned} \Gamma^-(\gamma) \cap HD_1(\alpha) &\subseteq (\delta_1^{H_\delta} \times \{\delta\}) \cup (\{\delta\} \times \delta_1^{H_\delta}), \quad \text{or} \\ \Gamma^-(\gamma) \cap HD_1(\alpha) &\subseteq (\delta_1^{H_\delta} \times \{\delta\}) \cup (\{\delta\} \times \delta_1^{H_\delta}). \end{aligned}$$

We assume that the first of these possibilities is true; the other proof is analogous.

Since $\Gamma^-(\gamma) \subset \Gamma^+(\alpha) \cup \Gamma^-(\alpha) \cup HD_1(\alpha)$, our assumption forces the rows and columns of $\delta_1^{H_\delta} \times \delta_1^{H_\delta}$ to be disjoint from the rows and columns of all other in-neighbours of γ . Now $\Gamma^+(\alpha) \subseteq \delta_1^{H_\delta} \times \delta_1^{H_\delta}$, and γ has either 0 or a in-neighbours in any of these rows or columns (Lemma 5.14 together with vertex-transitivity yield this conclusion), all of which must also be out-neighbours of α ; and these are all of the in-neighbours of γ that are also out-neighbours of α . Hence we must have $|\Gamma^-(\gamma) \cap \Gamma^+(\alpha)| = ja$ for some j . But Lemma 5.17 tells us that $|\Gamma^-(\gamma) \cap \Gamma^+(\alpha)| = (k - 2a + 1)/2$. So we have $k - 2a + 1 = 2ja$, but a divides each of these values with the exception of 1, and we know that $a \geq 2$ (by Lemma 5.14), a contradiction. \square

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