Bull. Aust. Math. Soc. **98** (2018), 481–493
doi:10.1017/S0004972718000746

A SHORT NOTE ON THE FRAME SET OF ODD FUNCTIONS

MARKUS FAULHUBER

(Received 2 February 2018; accepted 11 June 2018; first published online 15 August 2018)

Abstract

We give a simple argument which shows that Gabor systems consisting of odd functions of d variables and symplectic lattices of density 2^d cannot constitute a Gabor frame. In the one-dimensional, separable case, this follows from a more general result of Lyubarskii and Nes [‘Gabor frames with rational density’, *Appl. Comput. Harmon. Anal.* **34**(3) (2013), 488–494]. We use a different approach exploiting the algebraic relation between the ambiguity function and the Wigner distribution as well as their relation given by the (symplectic) Fourier transform. Also, we do not need the assumption that the lattice is separable and, hence, new restrictions are added to the full frame set of odd functions.

2010 *Mathematics subject classification*: primary 42C15; secondary 81S30.

Keywords and phrases: ambiguity function, Feichtinger’s algebra, frame set, Gabor frame, Wigner distribution.

1. Introduction and main result

In this note we show that the full frame set of any odd function of d variables in Feichtinger’s algebra cannot contain symplectic lattices of density 2^d. In the one-dimensional, separable case, this follows from a more general result derived by Lyubarskii and Nes [16], who showed that no odd window function $g \in S_0(\mathbb{R})$ can produce a separable Gabor frame of redundancy $1 + 1/n$ for $n \in \mathbb{N}$ by studying the vector-valued Zak transform and Zebulski–Zeevi matrices. For an alternative proof of this result, see the survey article by Gröchenig [10].

Our arguments are somewhat simpler and hold for symplectic lattices in arbitrary dimension d, which makes up for the drawback that we do not derive more general results. The key argument is that the Wigner distribution is the symplectic Fourier transform of the ambiguity function and that they also fulfil a simple algebraic relation. Moreover, our arguments show that, after a proper scaling, the cross Wigner distribution of any function in Feichtinger’s algebra and any even function in Feichtinger’s algebra is an eigenfunction of the symplectic Fourier transform with

The author was supported by the Erwin Schrödinger program of the Austrian Science Fund (FWF): J4100-N32.

eigenvalue 1 and the pairing with any odd function in Feichtinger's algebra is an eigenfunction with eigenvalue -1.

This work concerns the fine structure of Gabor frames as described in [10], that is, relations between the properties of a fixed window and its frame set. For a (window) function $g \in L^2(\mathbb{R}^d)$ and an index set $\Lambda \subset \mathbb{R}^{2d}$, we denote the resulting Gabor system by $\mathcal{G}(g, \Lambda)$. The (full) frame set of the window g is given by

$$\mathfrak{F}_{full}(g) = \{\Lambda \subset \mathbb{R}^{2d}, \Lambda \text{ a lattice} \mid \mathcal{G}(g, \Lambda) \text{ is a frame}\}.$$

Inspired by the work of Lemvig [15], the original intention was to find simple restrictions for the full frame set of odd (one-dimensional) Hermite functions by showing that certain sums vanish, but the restriction to this very special class of functions turned out to be unnecessary. Unfortunately, we do not get any new insights into the frame set of even (Hermite) functions. Among other counterexamples, Lemvig showed that the square lattice of density 2 does not generate a Gabor frame for the second Hermite function (the Gaussian being indexed as the 0th Hermite function), which was the first known obstruction to the frame set of the second Hermite function. Numerical inspections suggest that, for the second Hermite function, among all separable lattices of density 2, the square lattice is the only lattice which does not yield a Gabor frame. In particular, in the case of the square lattice, the lower frame bound is zero and it yields the global minimum of the lower frame bound seen as a function of the lattice parameters. This example stands in sharp contrast to the results given in [3], where it was shown that under the same assumptions, but using the Gaussian instead of the second Hermite function, the square lattice gives the global maximum of the lower frame bound seen as a function of the lattice parameters. The common theme, however, is that in both cases the highest possible symmetry of the lattice leads to extremal frame bounds. It was proven in [2] that, for a Gabor frame of even redundancy with standard Gaussian window, the hexagonal lattice yields the smallest upper frame bound among all lattices. We conjecture that the hexagonal lattice should also give the largest lower frame bound in this case. So, we pose the following question: for the second Hermite function, does the Gabor system generated by the hexagonal lattice of density 2 have a positive lower frame bound? The results by Lemvig tempt us to think that this might not be the case, but numerical inspections say that we actually have a Gabor frame with approximate lower frame bound 0.29 . . . .

Our main result, however, concerns odd windows in Feichtinger's algebra, which we denote by $S_0(\mathbb{R}^d)$ (another common notation is $M^1(\mathbb{R}^d)$).

Theorem 1.1 (Main result). *Let $g \in S_0(\mathbb{R}^d)$ be an odd function, that is, $g(t) = -g(-t)$, and let $\Lambda \subset \mathbb{R}^{2d}$ be a symplectic lattice in the time–frequency plane. If $\mathrm{vol}(\Lambda) = 2^{-d}$, then $\mathcal{G}(g, \Lambda)$ cannot be a Gabor frame or, in shorter notation,*

$$\textit{If } g \in S_0(\mathbb{R}^d),\ g(t) = -g(-t) \textit{ and } \mathrm{vol}(\Lambda) = 2^{-d},\ \Lambda \textit{ symplectic} \implies \Lambda \notin \mathfrak{F}_{full}(g).$$

Theorem 1.1 particularly implies that for $d = 1$ no lattice of density 2 can be contained in the frame set of an odd function from Feichtinger's algebra.

This work is structured as follows. In Section 2 we recall the basic properties of Gabor frames for the Hilbert space $L^2(\mathbb{R}^d)$. Then we introduce quadratic representations of a function $f \in L^2(\mathbb{R}^d)$ with respect to a (fixed) window $g \in L^2(\mathbb{R}^d)$, namely the short-time Fourier transform, the ambiguity function and the Wigner distribution. We derive their algebraic relations as well as their relation under the symplectic Fourier transform and introduce the symplectic version of Poisson's summation formula. We will see that Feichtinger's algebra is a convenient setting for our purposes. In Section 3 we show how sharp frame bounds can be calculated, using the results established by Janssen in the 1990s. These results finally lead to the proof of Theorem 1.1.

2. Gabor frames and time–frequency analysis in a nutshell

We consider Gabor frames for the Hilbert space of square integrable functions in d-dimensional Euclidean space $L^2(\mathbb{R}^d)$. Our notation mainly follows the textbook of Gröchenig [9]. A more recent introduction to the topic is the second edition of Christensen's textbook [1].

As our functions will be defined pointwise and at least continuous in the remainder of this work, the following notation for the inner product in $L^2(\mathbb{R}^d)$ is justified:

$$\langle f, g\rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t)}\, dt.$$

For two vectors t and t' in $\mathbb{R}^d$ we denote the Euclidean scalar product by $t \cdot t'$.

The key elements in time–frequency analysis are the translation operator T_x (time shift) and the modulation operator M_ω (frequency shift), which are defined by

$$T_x f(t) = f(t - x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i\omega\cdot t} f(t).$$

For a function in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ we define the Fourier transform by

$$\mathcal{F} f(\omega) = \int_{\mathbb{R}^d} f(t)e^{-2\pi i\omega\cdot t}\, dt,$$

which extends to a unitary operator on $L^2(\mathbb{R}^d)$ by the usual density argument. The Fourier transform has the well-known property of interchanging translation and modulation, that is,

$$\mathcal{F}(T_x f) = M_{-x}\mathcal{F} f \quad \text{and} \quad \mathcal{F}(M_\omega f) = T_\omega \mathcal{F} f.$$

The translation (time shift) and modulation (frequency shift) operators do not commute in general, but they fulfil the following commutation relation:

$$M_\omega T_x = e^{2\pi i\omega\cdot x} T_x M_\omega. \tag{2.1}$$

The combination of the two operators is called a time–frequency shift and usually denoted by

$$\pi(\lambda) = M_\omega T_x, \quad \lambda = (x, \omega) \in \mathbb{R}^{2d},$$

where λ is a point in the time–frequency plane or phase space. The composition of two time–frequency shifts is given by

$$\pi(\lambda)\pi(\lambda') = e^{-2\pi i x\cdot\omega'}\pi(\lambda + \lambda').$$

A Gabor system is a collection of time–frequency shifted copies of a so-called window function $g \in L^2(\mathbb{R}^d)$ with respect to an index set $\Lambda \subset \mathbb{R}^{2d}$ and it is denoted by

$$\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g \mid \lambda \in \Lambda\}.$$

Throughout this work, Λ will be a lattice, that is, a discrete subgroup of $\mathbb{R}^{2d}$. A lattice can be represented by an invertible matrix $M \in \mathrm{GL}(2d, \mathbb{R})$ and is then given by $\Lambda = M\mathbb{Z}^{2d}$. The matrix M is not unique since we can choose from countably many possible bases for $\mathbb{Z}^{2d}$. For example, if $d = 1$, then any matrix $\mathcal{B}$ with integer entries and determinant 1, that is, $\mathcal{B} \in \mathrm{SL}(2, \mathbb{Z})$, satisfies $\mathcal{B}\mathbb{Z}^2 = \mathbb{Z}^2$. Although the representing matrix is not unique, its determinant is. We define the volume of a lattice $\Lambda = M\mathbb{Z}^{2d}$ by

$$\mathrm{vol}(\Lambda) = |\det(M)|.$$

The density of a lattice, $\delta(\Lambda)$, is given by the reciprocal of the volume, that is, $\delta(\Lambda) = \mathrm{vol}(\Lambda)^{-1}$. Usually, a lattice is called separable if it can be written as $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ with $\alpha, \beta \in \mathbb{R}_+$. Alternative definitions of a separable lattice are that the generating matrix is a diagonal matrix or, in the most general case, that the lattice separates as $M_1\mathbb{Z}^d \times M_2\mathbb{Z}^d$ with $M_1, M_2 \in \mathrm{GL}(d, \mathbb{R})$. For $d = 1$ all definitions coincide.

A Gabor system $\mathcal{G}(g, \Lambda)$ is called a Gabor frame if and only if the frame inequality is fulfilled, that is,

$$A\|f\|^2 \le \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \le B\|f\|^2 \quad \text{for all } f \in L^2(\mathbb{R}^d), \tag{2.2}$$

with positive constants $0 < A \le B < \infty$ called frame bounds. In general, a Gabor frame is a redundant system and the redundancy of a Gabor system is given by the density of the underlying lattice. If all elements of the Gabor system $\mathcal{G}(g, \Lambda)$ have unit norm, the redundancy also reflects itself in the frame bounds. We note that in the case of an orthonormal basis we have $A = B = 1$.

2.1. Symmetric time–frequency shifts. It will be advantageous to consider symmetric time–frequency shifts instead of the usual time–frequency shifts. The symmetric time–frequency shift is given by

$$\rho(\lambda) = M_{\omega/2}T_xM_{\omega/2} = T_{x/2}M_\omega T_{x/2} = e^{-\pi i x\cdot\omega}\,\pi(\lambda). \tag{2.3}$$

We note that

$$\rho(\lambda)\rho(\lambda') = e^{-\pi i(x\cdot\omega' - x'\cdot\omega)}\rho(\lambda + \lambda').$$

The Gabor system under consideration is then

$$\widetilde{\mathcal{G}}(g, \Lambda) = \{\rho(\lambda)g \mid \lambda \in \Lambda\}.$$

https://doi.org/10.1017/S0004972718000746 Published online by Cambridge University Press

This system is a frame if and only if there exist positive constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{\lambda\in\Lambda} |\langle f, \rho(\lambda)g\rangle|^2 \le B\|f\|^2 \quad \text{for all } f \in L^2(\mathbb{R}^d). \tag{2.4}$$

It follows from (2.3) that the optimal constants A, B in equations (2.2) and (2.4) are the same. In particular, $\mathcal{G}(g, \Lambda)$ is a frame if and only if $\widetilde{\mathcal{G}}(g, \Lambda)$ is a frame. We will work with the Gabor system $\widetilde{\mathcal{G}}(g, \Lambda)$ as the phase factors are easier to handle in this case.

2.2. Phase-space methods. The short-time Fourier transform (STFT) and the ambiguity function are often used to measure time–frequency concentration. They are defined in similar ways and, in fact, they only differ by a phase factor, that is, a complex exponential of modulus 1. We will now introduce the necessary tools to prove Theorem 1.1. For more details, we refer to the textbooks of Folland [5], de Gosson [6, 7] and Gröchenig [9].

Definition 2.1 (STFT). For $f \in L^2(\mathbb{R}^d)$, the short-time Fourier transform with respect to the window $g \in L^2(\mathbb{R}^d)$ is defined by

$$\mathcal{V}_g f(x, \omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i\omega\cdot t}\,dt = \langle f, \pi(\lambda)g\rangle, \quad \lambda = (x, \omega) \in \mathbb{R}^{2d}.$$

Before we continue, we introduce the function space which will be most suitable for our intentions, namely Feichtinger's algebra $S_0(\mathbb{R}^d)$, introduced by Feichtinger in the early 1980s [4]. There are several equivalent definitions of $S_0(\mathbb{R}^d)$ and we prefer to use the following definition.

Definition 2.2 (Feichtinger's algebra). Feichtinger's algebra $S_0(\mathbb{R}^d)$ consists of all elements $g \in L^2(\mathbb{R}^d)$ such that

$$\|\mathcal{V}_g g\|_{L^1(\mathbb{R}^{2d})} = \iint_{\mathbb{R}^{2d}} \big|\mathcal{V}_g g(\lambda)\big|\,d\lambda < \infty, \quad \lambda = (x, \omega) \in \mathbb{R}^{2d}.$$

We note the following properties of $S_0(\mathbb{R}^d)$. It is a Banach space, invariant under the Fourier transform and time–frequency shifts. It contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and it is dense in $L^p(\mathbb{R}^d)$, $p \in [1, \infty)$. It is for these properties that it is a quite popular function space in time–frequency analysis and the literature on the subject is huge. For more details on S_0, we refer to the survey by Jakobsen [11] and the references therein.

We turn to another time–frequency representation, which is defined similarly to the STFT.

Definition 2.3 (Ambiguity function). For $f, g \in L^2(\mathbb{R}^d)$, the (cross) ambiguity function is defined by

$$\begin{aligned}\mathcal{A}_g f(x, \omega) &= \int_{\mathbb{R}^d} f\Big(t + \frac{1}{2}x\Big)\overline{g\Big(t - \frac{1}{2}x\Big)}e^{-2\pi i\omega\cdot t}\,dt \\ &= \Big\langle \pi\Big(-\frac{1}{2}\lambda\Big)f, \pi\Big(\frac{1}{2}\lambda\Big)g\Big\rangle = \langle f, \rho(\lambda)g\rangle, \quad \lambda = (x, \omega) \in \mathbb{R}^{2d}.\end{aligned}$$

https://doi.org/10.1017/S0004972718000746 Published online by Cambridge University Press

Both $\mathcal{V}_g f$ and $\mathcal{A}_g f$ are uniformly continuous on $\mathbb{R}^{2d}$. From relation (2.3), which is a consequence of the commutation relation (2.1),

$$\mathcal{A}_g f(x,\omega) = e^{\pi i \omega \cdot x} \mathcal{V}_g f(x,\omega).$$

In particular, this means that $|\mathcal{V}_g f| \equiv |\mathcal{A}_g f|$. We will now introduce a quadratic representation of a function $f \in L^2(\mathbb{R}^d)$ which is usually used in quantum mechanics, the Wigner distribution.

Definition 2.4 (Wigner distribution). For $f, g \in L^2(\mathbb{R}^d)$, the (cross) Wigner distribution is defined by

$$\mathcal{W}_g f(x,\omega) = \int_{\mathbb{R}} f\Big(x + \frac{1}{2}t\Big)\overline{g\Big(x - \frac{1}{2}t\Big)} e^{-2\pi i \omega \cdot t}\, dt, \qquad x, \omega \in \mathbb{R}^d.$$

For the rest of this work, we will drop the index in all of the above definitions if $f = g$. The Wigner distribution is related to the ambiguity function (and, hence, in a similar way to the STFT) by the symplectic Fourier transform. In order to define the symplectic Fourier transform, we first equip our phase space with a symplectic structure. In what follows the vectors $\lambda = (x,\omega)$ and $\lambda' = (x',\omega')$ in $\mathbb{R}^{2d}$ are always seen as column vectors and the scalar product of two vectors in the phase space is again denoted by $\lambda \cdot \lambda'$. We define the symplectic form

$$\sigma(\lambda,\lambda') = x \cdot \omega' - \omega \cdot x' = \lambda \cdot J\lambda' = \lambda^T J \lambda,$$

where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ is the standard symplectic matrix and I the $d \times d$ identity matrix. A matrix S is called symplectic if and only if it preserves the symplectic form, that is,

$$\sigma(S\lambda, S\lambda') = \sigma(\lambda,\lambda')$$

or, equivalently,

$$S^T J S = J.$$

As mentioned, it will turn out to be convenient to use a slightly different version of the Fourier transform in phase space, the symplectic Fourier transform.

Definition 2.5 (Symplectic Fourier transform). For $F \in \mathcal{S}(\mathbb{R}^{2d})$, the symplectic Fourier transform is given by

$$\mathcal{F}_\sigma F(x,\omega) = \iint_{\mathbb{R}^{2d}} F(\lambda') e^{-2\pi i \sigma(\lambda,\lambda')}\, d\lambda', \qquad \lambda = (x,\omega),\ \lambda' = (x',\omega') \in \mathbb{R}^{2d}.$$

Of course, the symplectic Fourier transform extends to all of $L^2(\mathbb{R}^{2d})$ by the usual density argument (just like the Fourier transform). A tool which is heavily exploited in time–frequency analysis is the Poisson summation formula, which we will use for $2d$-dimensional lattices. The technical details for the Poisson summation formula to hold pointwise have been worked out by Gröchenig in [8]. Since our functions under consideration are in $S_0(\mathbb{R}^d)$, their Wigner distributions as well as their ambiguity functions will be elements of Feichtinger's algebra in phase space, that is, elements of $S_0(\mathbb{R}^{2d})$ (see [11, Ch. 5]). This assumption is sufficient for Poisson's summation formula to hold pointwise.

https://doi.org/10.1017/S0004972718000746 Published online by Cambridge University Press

Proposition 2.6 (Poisson summation formula). *For $F \in S_0(\mathbb{R}^{2d})$ and a lattice $\Lambda = M\mathbb{Z}^{2d}$ with dual lattice $\Lambda^\perp = M^{-T}\mathbb{Z}^{2d}$,*

$$\sum_{\lambda\in\Lambda} F(\lambda+z) = \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\perp\in\Lambda^\perp} \mathcal{F}F(\lambda^\perp)e^{2\pi i\lambda^\perp\cdot z}, \quad \lambda,\lambda^\perp,z\in\mathbb{R}^{2d}.$$

Instead of using the $2d$-dimensional Fourier transform, we can adjust this result by using the symplectic Fourier transform and the adjoint lattice instead of the dual lattice. The adjoint of a lattice $\Lambda = M\mathbb{Z}^{2d}$ is given by $\Lambda^\circ = JM^{-T}\mathbb{Z}^{2d}$. Under the assumptions of Poisson's summation formula,

$$\sum_{\lambda\in\Lambda} F(\lambda+z) = \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{F}_\sigma F(\lambda^\circ)e^{2\pi i\,\sigma(\lambda^\circ,z)}, \quad \lambda,\lambda^\circ,z\in\mathbb{R}^{2d}.$$

We say that a lattice is symplectic if its generating matrix is a multiple of a symplectic matrix, that is, $\Lambda = c\,S\mathbb{Z}^{2d}$ with $c > 0$ and $S \in \mathrm{Sp}(d)$, with $\mathrm{Sp}(d)$ being the set of all symplectic $2d\times 2d$ matrices. We note that symplectic matrices actually form a group under matrix multiplication and that any symplectic matrix has determinant 1 and, hence, $\mathrm{Sp}(d) \subset \mathrm{SL}(2d,\mathbb{R})$. In general, $\mathrm{Sp}(d)$ is a proper subgroup of the special linear group $\mathrm{SL}(2d,\mathbb{R})$, except that for $d = 1$ we have that $\mathrm{Sp}(1) = \mathrm{SL}(2,\mathbb{R})$. In particular, it follows that any two-dimensional lattice is symplectic. In general, it follows from the definition of a symplectic matrix that

$$\Lambda^\circ = \mathrm{vol}(\Lambda)^{-1/d}\Lambda, \quad \Lambda \text{ symplectic},$$

because, by definition, $S \in \mathrm{Sp}(d) \Leftrightarrow S = JS^{-T}J^{-1}$ and, for $\Lambda = c\,S\mathbb{Z}^2$ with $c > 0$, we have $\Lambda^\circ = c^{-1}JS^{-T}J^{-1}\mathbb{Z}^{2d} = c^{-1}JS^{-T}\mathbb{Z}^{2d}$, as J^{-1} is just another choice of basis for $\mathbb{Z}^{2d}$. Hence, for Λ symplectic the adjoint lattice is only a scaled version of the original lattice.

As a last point in this section, we take a closer look at the relation between the ambiguity function (and hence the STFT) and the Wigner distribution. We start with their relation given by the symplectic Fourier transform.

Proposition 2.7. *For $f,g \in L^2(\mathbb{R}^d)$, the ambiguity function and the Wigner distribution are symplectic Fourier transforms of each other, that is,*

$$\mathcal{F}_\sigma(\mathcal{A}_g f)(x,\omega) = \mathcal{W}_g f(x,\omega) \quad \textit{and} \quad \mathcal{F}_\sigma(\mathcal{W}_g f)(x,\omega) = \mathcal{A}_g f(x,\omega).$$

Also, we have the following algebraic relation between the ambiguity function and the Wigner distribution.

Proposition 2.8. *For $f,g \in L^2(\mathbb{R}^d)$, the ambiguity function and the Wigner distribution satisfy*

$$\mathcal{W}_g f(x,\omega) = 2^d\mathcal{A}_{g^\vee} f(2x,2\omega) \quad \textit{and} \quad \mathcal{A}_g f(x,\omega) = 2^{-d}\mathcal{W}_{g^\vee} f(\tfrac{1}{2}x,\tfrac{1}{2}\omega),$$

where $g^\vee(t) = g(-t)$ denotes the reflection of g.

https://doi.org/10.1017/S0004972718000746 Published online by Cambridge University Press

We proceed with some more results regarding the ambiguity function and the Wigner distribution which we will need to prove our main result. But, first, we introduce some notation. For a function F in phase space, the isotropic dilation is

$$D_\alpha F(x,\omega) = F(\alpha x, \alpha\omega), \quad \alpha \in \mathbb{R}_+.$$

The behaviour of this operator under the symplectic Fourier transform is given by

$$\mathcal{F}_\sigma(D_\alpha F)(x,\omega) = \alpha^{-2d} D_{1/\alpha}\mathcal{F}_\sigma F(x,\omega). \tag{2.5}$$

Lemma 2.9. *For $f, g \in L^2(\mathbb{R}^d)$ with $g^\vee = g$,*

$$\mathcal{F}_\sigma(D_{\sqrt{2}}\mathcal{A}_g f)(x,\omega) = D_{\sqrt{2}}\mathcal{A}_g f(x,\omega),$$
$$\mathcal{F}_\sigma(D_{1/\sqrt{2}}\mathcal{W}_g f)(x,\omega) = D_{1/\sqrt{2}}\mathcal{W}_g f(x,\omega).$$

If $-g^\vee = g$,

$$\mathcal{F}_\sigma(D_{\sqrt{2}}\mathcal{A}_g f)(x,\omega) = -D_{\sqrt{2}}\mathcal{A}_g f(x,\omega),$$
$$\mathcal{F}_\sigma(D_{1/\sqrt{2}}\mathcal{W}_g f)(x,\omega) = -D_{1/\sqrt{2}}\mathcal{W}_g f(x,\omega).$$

Proof. This is an immediate consequence of Propositions 2.7 and 2.8. From Proposition 2.7 and (2.5),

$$\mathcal{F}_\sigma(D_{\sqrt{2}}\mathcal{A}_g f)(x,\omega) = 2^{-d} D_{1/\sqrt{2}}\mathcal{W}_g f(x,\omega) = D_{\sqrt{2}}(2^{-d}\mathcal{W}_g f(\tfrac{1}{2}x, \tfrac{1}{2}\omega)).$$

Now, by the algebraic property from Proposition 2.8, we conclude that

$$\mathcal{F}_\sigma(D_{\sqrt{2}}\mathcal{A}_g f)(x,\omega) = D_{\sqrt{2}}\mathcal{A}_{g^\vee} f(x,\omega).$$

In a similar manner we derive the analogous statement for $\mathcal{W}_g f$. The results follow from the definitions of $\mathcal{A}_g f$ and $\mathcal{W}_g f$ and the assumptions that $\pm g^\vee = g$. □

In [17], it was shown that the (suitably scaled) cross Wigner distributions of two Hermite functions as well as tensor products of Hermite functions are eigenfunctions of the planar (two-dimensional) Fourier transform with eigenvalues ±1, depending on the pairing. In [14], another example of a 'nonstandard' eigenfunction of the planar Fourier transform was given, namely the function $F(x,\omega) = \sqrt{x^2+\omega^2}/x\omega$ (integrals have to be understood as Cauchy principal values in this case). All these examples are invariant under rotation (also the presented set of eigenfunctions is countable). Lemma 2.9 gives us an uncountable set of examples of eigenfunctions of the symplectic Fourier transform which do not necessarily possess any rotational symmetries.

For the next result, we recall that $\mathcal{W}_g f \in L^1(\mathbb{R}^{2d})$ if and only if $f, g \in S_0(\mathbb{R}^d)$ (see [7, Ch. 7] or [11]). Also, if $f, g \in S_0(\mathbb{R}^d)$, then the Wigner distribution $\mathcal{W}_g f$ is in $S_0(\mathbb{R}^{2d})$ (see [11]), which means that

$$\mathcal{W}_g f \in L^1(\mathbb{R}^{2d}) \iff \mathcal{W}_g f \in S_0(\mathbb{R}^{2d}).$$

This statement holds, of course, for the ambiguity function $\mathcal{A}_g f$ and for the STFT $\mathcal{V}_g f$. Also, the assumptions for Poisson's summation formula to hold pointwise are met and we derive the following result.

https://doi.org/10.1017/S0004972718000746 Published online by Cambridge University Press

LEMMA 2.10. *Let* $f, g \in S_0(\mathbb{R}^d)$ *and let* g *be an odd function and* Λ *a symplectic lattice with* $\mathrm{vol}(\Lambda)^{-1} = 2^d$. *Then*

$$\sum_{\lambda\in\Lambda} \mathcal{W}_g f(\lambda) = -\sum_{\lambda\in\Lambda} \mathcal{W}_g f(\lambda) = 0,$$
$$\sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{A}_g f(\lambda^\circ) = -\sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{A}_g f(\lambda^\circ) = 0.$$

PROOF. By the symplectic version of Poisson's summation formula,

$$\sum_{\lambda\in\Lambda} \mathcal{W}_g f(\lambda) = \underbrace{\mathrm{vol}(\Lambda)^{-1}}_{=2^d} \sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{F}_\sigma(\mathcal{W}_g f)(\lambda^\circ).$$

By Proposition 2.7,

$$2^d \sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{F}_\sigma(\mathcal{W}_g f)(\lambda^\circ) = 2^d \sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{A}_g f(\lambda^\circ)$$

and by the algebraic relation in Proposition 2.8,

$$2^d \sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{A}_g f(\lambda^\circ) = 2^d \sum_{\lambda^\circ\in\Lambda^\circ} 2^{-d}\mathcal{W}_{g^\vee} f(2^{-1}\lambda^\circ) = -\sum_{\lambda\in\Lambda} \mathcal{W}_g f(\lambda),$$

since $g^\vee = -g$, $\mathrm{vol}(\Lambda)^{-1} = 2^d$ and, hence, $2^{-1}\Lambda^\circ = \Lambda$ as Λ is symplectic. Therefore, the statement about the Wigner distribution follows. The statement for the ambiguity function follows analogously. □

An alternative (but equivalent) proof can be established by using Lemma 2.9. Let $f, g \in S_0(\mathbb{R}^d)$, $-g^\vee = g$ and $\mathrm{vol}(\Lambda)^{-1} = 1$ (note that in this case $\Lambda = \Lambda^\circ$); then

$$\sum_{\lambda\in\Lambda} D_{1/\sqrt{2}}\mathcal{W}_g f(\lambda) = \sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{F}_\sigma(D_{1/\sqrt{2}}\mathcal{W}_g f)(\lambda) = \sum_{\lambda\in\Lambda} -D_{1/\sqrt{2}}\mathcal{W}_g f(\lambda).$$

The analogous statement for $\mathcal{A}_g f$ obviously holds as well. Now, note that dilating the lattice and dilating the Wigner distribution are two equivalent ways to establish the result.

3. Sharp frame bounds

In this section, we take a closer look at the frame operator and its spectrum. We will mainly follow Janssen's articles [12, 13]. The main differences are that we formulate the results for symplectic lattices in $2d$-dimensional phase space rather than for separable lattices in two-dimensional phase space. Also, we use symmetric time–frequency shifts which only change the phase factors. Nonseparable lattices are easier to handle with this approach. Building on the results of the previous section, we will show finally that for odd windows in $S_0(\mathbb{R}^d)$ and a lattice $\Lambda \subset \mathbb{R}^{2d}$ in phase space with $\mathrm{vol}(\Lambda)^{-1} = 2^d$, the lower frame bound of the Gabor system $\widetilde{\mathcal{G}}(g, \Lambda)$ vanishes. By the comments in Section 2.1, this is equivalent to the fact that the Gabor system $\mathcal{G}(g, \Lambda)$ does not generate a frame, which is our main result.

https://doi.org/10.1017/S0004972718000746 Published online by Cambridge University Press

The frame operator associated to the Gabor system $\widetilde{\mathcal{G}}(g,\Lambda)$ is denoted by $\widetilde{S}_{g,\Lambda}$ and given by

$$\widetilde{S}_{g,\Lambda} f = \sum_{\lambda\in\Lambda} \langle f, \rho(\lambda) g\rangle \rho(\lambda) g, \quad f \in L^2(\mathbb{R}^d).$$

Another useful representation is due to Janssen [12] and usually called Janssen's representation of the frame operator:

$$\widetilde{S}_{g,\Lambda} = \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} \langle g, \rho(\lambda^\circ) g\rangle \rho(\lambda^\circ).$$

The frame operator is the composition of the analysis and the synthesis operators, which are adjoint to each other. The analysis operator maps a function from $L^2(\mathbb{R}^d)$ to $\ell^2(\Lambda)$, $\Lambda \subset \mathbb{R}^{2d}$, and is given by

$$\widetilde{G}_{g,\Lambda} f = (\langle f, \rho(\lambda) g\rangle)_{\lambda\in\Lambda}.$$

Its adjoint is the synthesis operator, mapping sequences $c = (c_\lambda)_{\lambda\in\Lambda} \in \ell^2(\Lambda)$ to $L^2(\mathbb{R}^d)$, and is given by

$$\widetilde{G}^*_{g,\Lambda} c = \sum_{\lambda\in\Lambda} c_\lambda \rho(\lambda) g.$$

The frame operator can be written as

$$\widetilde{S}_{g,\Lambda} = \widetilde{G}^*_{g,\Lambda} \widetilde{G}_{g,\Lambda}.$$

The following result is a straightforward generalisation of the main result in [12], where Janssen showed it for $d = 1$ and Λ separable.

Proposition 3.1. *The following statements are equivalent:*

(i) $\widetilde{\mathcal{G}}(g,\Lambda)$ *is a frame with bounds A and B;*
(ii) $A\, I_{L^2(\mathbb{R}^d)} \leq \widetilde{S}_{g,\Lambda} \leq B\, I_{L^2(\mathbb{R}^d)}$;
(iii) $A\, I_{\ell^2(\Lambda^\circ)} \leq \mathrm{vol}(\Lambda)^{-1} \widetilde{G}_{g,\Lambda^\circ} \widetilde{G}^*_{g,\Lambda^\circ} \leq B\, I_{\ell^2(\Lambda^\circ)}$.

The most interesting part for this work is that we can compute the frame bounds via the eigenvalues of the bi-infinite matrix, indexed by the adjoint lattice:

$$\widetilde{G}_{g,\Lambda^\circ} \widetilde{G}^*_{g,\Lambda^\circ} = (\langle \rho(\lambda^\circ) g, \rho(\lambda^{\circ\prime}) g\rangle)_{\lambda^\circ, \lambda^{\circ\prime} \in \Lambda^\circ}.$$

We proceed by calculating the values of the above matrix by means of

$$\langle \rho(\lambda^\circ) g, \rho(\lambda^{\circ\prime}) g\rangle = \langle g, \rho(-\lambda^\circ)\rho(\lambda^{\circ\prime}) g\rangle = e^{\pi i \sigma(\lambda^\circ, \lambda^{\circ\prime})} \langle g, \rho(\lambda^{\circ\prime} - \lambda^\circ) g\rangle, \quad \lambda^\circ, \lambda^{\circ\prime} \in \Lambda^\circ.$$

Assume that $\mathrm{vol}(\Lambda)^{-1/d} \in \mathbb{N}$. Then the entries in $\mathrm{vol}(\Lambda)^{-1} \widetilde{G}_{g,\Lambda^\circ} \widetilde{G}^*_{g,\Lambda^\circ}$ are constant along diagonals, that is, $\mathrm{vol}(\Lambda)^{-1} \widetilde{G}_{g,\Lambda^\circ} \widetilde{G}^*_{g,\Lambda^\circ}$ has a Laurent structure. For the time–frequency shifts $\rho(\lambda^{\circ\prime} - \lambda^\circ)$, the argument is obvious. The only justification we have to make is that the phase factor $e^{\pi i \sigma(\lambda^\circ, \lambda^{\circ\prime})}$ is constant along diagonals. We show that $\sigma(\lambda^\circ, \lambda^{\circ\prime})$ is an integer multiple of $\mathrm{vol}(\Lambda)^{-1/d}$. If $\Lambda = \alpha^{1/2d} S\mathbb{Z}^{2d}$, then $\mathrm{vol}(\Lambda) = \alpha$ and

https://doi.org/10.1017/S0004972718000746 Published online by Cambridge University Press

$\Lambda^\circ = \alpha^{-1/2d} S\,\mathbb{Z}^{2d}$. Since our lattice is symplectic by assumption, the symplectic form σ is independent from the matrix S and

$$e^{\pi i\,\sigma(\lambda^\circ,\lambda^{\circ\prime})} = e^{\pi i\,\sigma\left(\alpha^{-1/2d} S\,(k,l)^T,\,\alpha^{-1/2d} S\,(k',l')^T\right)} = e^{\mathrm{vol}(\Lambda)^{-1/d}\pi i(k\cdot l' - k'\cdot l)}, \quad k, l, k', l' \in \mathbb{Z}^d.$$

In the case that $\mathrm{vol}(\Lambda)^{-1/d}$ is even, the phase factor equals $+1$ and can be neglected. However, if $\mathrm{vol}(\Lambda)^{-1/d}$ is odd, the phase factor takes the role of an alternating sign, which is constant along diagonals, that is, it is either $+1$ or -1 depending on the diagonal built by $\lambda^\circ - \lambda^{\circ\prime}$ being constant. For this reason we focus on the case where $\mathrm{vol}(\Lambda)^{-1/d}$ is even.

It follows from the general theory on Toeplitz (matrices) and Laurent operators that the spectrum of such a (double) bi-infinite matrix can be computed via the essential infimum and supremum of a Fourier series, where the coefficients of the series are derived from the entries in the matrix. Using the above arguments, the following result is a straightforward generalisation of the result derived by Janssen in [13] (see the Appendix for Janssen's result).

Proposition 3.2. *For $g \in L^2(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^{2d}$ with $\mathrm{vol}(\Lambda)^{-1/d} \in \mathbb{N}$, the Gabor system $\widetilde{\mathcal{G}}(g,\Lambda)$ possesses the optimal frame bounds*

$$A = \operatorname*{ess\,inf}_{z\in\mathbb{R}^2}\ \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ - \lambda^{\circ\prime}\in\Lambda^\circ} e^{\pi i\,\sigma(\lambda^\circ,\lambda^{\circ\prime})} \mathcal{A}g(\lambda^{\circ\prime} - \lambda^\circ)\, e^{2\pi i\,\sigma(\lambda^{\circ\prime}-\lambda^\circ,z)},$$
$$B = \operatorname*{ess\,sup}_{z\in\mathbb{R}^2}\ \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ - \lambda^{\circ\prime}\in\Lambda^\circ} e^{\pi i\,\sigma(\lambda^\circ,\lambda^{\circ\prime})} \mathcal{A}g(\lambda^{\circ\prime} - \lambda^\circ)\, e^{2\pi i\,\sigma(\lambda^{\circ\prime}-\lambda^\circ,z)}.$$

The above series is real-valued (since we sum over a lattice the imaginary parts also appear as complex conjugates and cancel out). Note that the above series need not be convergent. In this case the upper bound might not be finite and the Gabor system might not constitute a frame. However, for windows in Feichtinger's algebra the upper bound is always finite.[1] Since the frame operator $\widetilde{S}_{g,\Lambda}$ is self-adjoint and positive semi-definite,

$$0 \le A = \operatorname*{ess\,inf}_{z\in\mathbb{R}^2}\ \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ - \lambda^{\circ\prime}\in\Lambda^\circ} e^{\pi i\,\sigma(\lambda^\circ,\lambda^{\circ\prime})} \mathcal{A}g(\lambda^{\circ\prime} - \lambda^\circ)\, e^{2\pi i\,\sigma(\lambda^{\circ\prime}-\lambda^\circ,z)},$$

by the theory of Laurent operators. We have now all the tools we need to prove Theorem 1.1.

Proof of Theorem 1.1. In order to prove our main result, we will show that the lower frame bound vanishes under the assumptions of Theorem 1.1. For $\mathrm{vol}(\Lambda)^{-1} = 2^d$ and due to the fact that $\lambda^{\circ\prime} - \lambda^\circ \in \Lambda^\circ$, the series in Proposition 3.2 reduces to

$$\phi(z) = \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{A}g(\lambda^\circ)\, e^{2\pi i\,\sigma(\lambda^\circ,z)}.$$

[1] It follows from the results in Tolimieri and Orr [18] that $\mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ\in\Lambda^\circ} |\mathcal{A}g(\lambda^\circ)|$ is always an upper bound, however, usually not the optimal upper bound. For $g \in S_0(\mathbb{R}^d)$, this expression is always finite.

Now, observe that

$$\phi(0) = \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} \mathcal{A}g(\lambda^\circ),$$

which is, up to the factor $\mathrm{vol}(\Lambda)^{-1}$, just the series from Lemma 2.10. Hence, we conclude that for $\mathrm{vol}(\Lambda)^{-1} = 2^d$ and for $g \in S_0(\mathbb{R}^d)$ with $g^\vee = -g$, we have $\phi(0) = 0$. This is equivalent to the statement that the lower frame bound of the system $\widetilde{\mathcal{G}}(g, \Lambda)$ vanishes. The same is true for the lower frame bound of the Gabor system $\mathcal{G}(g, \Lambda)$. This completes the proof of Theorem 1.1. □

Appendix. Janssen's proposition

We state Janssen's proposition from [13], which he used to compute sharp frame bounds for the $L^2(\mathbb{R})$ case. In his formulation, Janssen used the STFT rather than the ambiguity function and separable lattices rather than general lattices. Hence, Janssen's result is a special case of Proposition 3.2, but already carries the general idea in it. For $g \in L^2(\mathbb{R})$ and a lattice $\Lambda_{(\alpha,\beta)} = \alpha\mathbb{Z} \times \beta\mathbb{Z}$, $(\alpha\beta)^{-1} \in \mathbb{N}$, the Gabor system $\mathcal{G}(g, \Lambda_{(\alpha,\beta)})$ possesses the optimal frame bounds

$$A = \operatorname*{ess\,inf}_{(x,\omega)\in\mathbb{R}^2} (\alpha\beta)^{-1} \sum_{k-k',l-l'\in\mathbb{Z}} \mathcal{V}g\Big(\frac{k-k'}{\beta}, \frac{l-l'}{\alpha}\Big) e^{2\pi i((k-k')x+(l-l')\omega)},$$

$$B = \operatorname*{ess\,sup}_{(x,\omega)\in\mathbb{R}^2} (\alpha\beta)^{-1} \sum_{k-k',l-l'\in\mathbb{Z}} \mathcal{V}g\Big(\frac{k-k'}{\beta}, \frac{l-l'}{\alpha}\Big) e^{2\pi i((k-k')x+(l-l')\omega)}.$$

We note that the phase factor implicitly appears in the STFT and that the standard Poisson summation formula (and not its symplectic version) was used.

Acknowledgements

The computational results presented have been achieved (in part) using the Vienna Scientific Cluster (VSC). The author wishes to thank Franz Luef for many fruitful discussions on the topic. The author thanks the anonymous referee for beneficial feedback on the first version of the manuscript.

References

[1] O. Christensen, *An Introduction to Frames and Riesz Bases*, 2nd edn, Applied and Numerical Harmonic Analysis (Birkhäuser, Basel, 2016).
[2] M. Faulhuber, 'Minimal frame operator norms via minimal theta functions', *J. Fourier Anal. Appl.* **24**(2) (2018), 545–559.
[3] M. Faulhuber and S. Steinerberger, 'Optimal Gabor frame bounds for separable lattices and estimates for Jacobi theta functions', *J. Math. Anal. Appl.* **445**(1) (2017), 407–422.
[4] H. G. Feichtinger, 'On a new Segal algebra', *Monatsh. Math.* **92**(4) (1981), 269–289.
[5] G. B. Folland, *Harmonic Analysis in Phase Space*, Annals of Mathematics Studies, 122 (Princeton University Press, Princeton, NJ, 1989).
[6] M. A. de Gosson, *Symplectic Methods in Harmonic Analysis and in Mathematical Physics*, Pseudo-Differential Operators. Theory and Applications, 7 (Birkhäuser/Springer, Basel, 2011).

[7] M. A. de Gosson, *The Wigner Transform* (World Scientific, Singapore, 2017).
[8] K. Gröchenig, 'An uncertainty principle related to the Poisson summation formula', *Studia Math.* **121**(1) (1996), 87–104.
[9] K. Gröchenig, *Foundations of Time–Frequency Analysis*, Applied and Numerical Harmonic Analysis (Birkhäuser, Boston, MA, 2001).
[10] K. Gröchenig, 'The mystery of Gabor frames', *J. Fourier Anal. Appl.* **20**(4) (2014), 865–895.
[11] M. S. Jakobsen, 'On a (no longer) new Segal algebra: a review of the Feichtinger algebra', *J. Fourier Anal. Appl.* (2018), 1–82; published online https://doi.org/10.1007/s00041-018-9596-4.
[12] A. J. E. M. Janssen, 'Duality and biorthogonality for Weyl–Heisenberg frames', *J. Fourier Anal. Appl.* **1**(4) (1995), 403–436.
[13] A. J. E. M. Janssen, 'Some Weyl–Heisenberg frame bound calculations', *Indag. Math. (N.S.)* **7**(2) (1996), 165–183.
[14] F. Lanzara and V. Maz'ya, 'Note on a nonstandard eigenfunction of the planar Fourier transform', *J. Math. Sci.* **224**(5) (2017), 694–698.
[15] J. Lemvig, 'On some Hermite series identities and their applications to Gabor analysis', *Monatsh. Math.* **182**(4) (2017), 889–912.
[16] Y. Lyubarskii and P. G. Nes, 'Gabor frames with rational density', *Appl. Comput. Harmon. Anal.* **34**(3) (2013), 488–494.
[17] S.-C. Pei and C.-L. Liu, 'A general form of 2D Fourier transform eigenfunctions', in: *2012 IEEE Int. Conf. Acoustics, Speech and Signal Processing (ICASSP)* (IEEE, New York, 2012), 3701–3704.
[18] R. Tolimieri and R. S. Orr, 'Poisson summation, the ambiguity function, and the theory of Weyl–Heisenberg frames', *J. Fourier Anal. Appl.* **1**(3) (1995), 233–247.

MARKUS FAULHUBER, Analysis Group,
Department of Mathematical Sciences, NTNU Trondheim, Sentralbygg 2,
Gløshaugen, Trondheim, Norway
e-mail: markus.faulhuber@ntnu.no