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ON DERIVATIONS OF LIE ALGEBRAS

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Introduction. A well known result in the theory of Lie algebras, due to H. Zassenhaus, states that if $\mathscr{L}$ is a finite dimensional Lie algebra over the field K such that the killing form of $\mathscr{L}$ is non-degenerate, then the derivations of $\mathscr{L}$ are all inner, [**3**, p. 74]. In particular, this applies to the finite dimensional split simple Lie algebras over fields of characteristic zero. In this paper we extend this result to a class of Lie algebras which generalize the split simple Lie algebras, and which are defined by Cartan matrices (for a definition see § 1). Because of the fact that the algebras we consider are usually infinite dimensional, the method we employ in our investigation is quite different from the standard one used in the finite dimensional case, and makes no reference to any associative bilinear form on the algebras. If $\mathscr{L}$ is one of the Lie algebras under consideration, we let $\mathscr{D}(\mathscr{L})$ denote the derivation algebra of $\mathscr{L}$ and $\mathscr{I}(\mathscr{L})$ the ideal of inner derivations. Our main result states that the dimension of $\mathscr{D}(\mathscr{L})/\mathscr{I}(\mathscr{L})$ equals the nullity of the Cartan matrix which defines $\mathscr{L}$.

In Section 1 we give a brief description of the algebras we consider and in Section 2 we prove our main result. In the final section we present an application of our result to the problem of determining the isomorphism classes of the algebras under consideration. One of the results in this section, Theorem 2, was obtained in joint work with R. Moody, and is of independent interest.

1. Description of the algebras. A Cartan matrix is any $l \times l$ integral matrix (A_{ij}) such that $A_{ii} = 2$, $A_{ij} \leqq 0$ if $i \neq j$, and $A_{ij} = 0$ if and only if $A_{ji} = 0$, for $1 \leqq i, j \leqq l$. We will always assume our Cartan matrix (A_{ij}) is indecomposable, which is the same thing as requiring that the Dynkin diagram associated to it is connected.

Let K be any field of characteristic zero and let $\mathscr{F}\mathscr{L}$ be the free Lie algebras over K generated by the $3l$ elements e_i, h_i, f_i, $1 \leqq i \leqq l$. Let J denote the ideal of $\mathscr{F}\mathscr{L}$ generated by the following elements,

$$[h_i, h_j],$$
$$[e_i, h_j] - A_{ji}e_i,$$
$$[f_i, h_j] + A_{ji}f_i,$$
$$[e_i, f_j] - \delta_{ij}h_i, \text{ for } 1 \leqq i, j \leqq l,$$

and let $\mathscr{L}_U$ denote the factor algebra. $\mathscr{L}_U$ is called the universal heffalump algebra over K attached to (A_{ij}). We let θ denote the ideal of $\mathscr{L}_U$ generated

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by the elements $e_i(\text{ad } e_j)^{-A_{ji}+1}$, $f_i(\text{ad } f_j)^{-A_{ji}+1}$ for $1 \leqq i \neq j \leqq l$, and let $\mathscr{L}_S$ denote the corresponding factor algebras. $\mathscr{L}_S$ is called the standard heffalump algebra. Finally, we let $\mathscr{R}$ denote the radical of $\mathscr{L}_U$ (for a description of $\mathscr{R}$ see [**1**] or [**5**]) and let $\mathscr{L}_R$ be the factor algebra. $\mathscr{L}_R$ is called the reduced heffalump algebra.

When working with any heffalump algebra we let e_i, h_i, f_i again denote their images in the algebra and we let $\mathscr{H}$ be the linear span of the elements h_i, $1 \leqq i \leqq l$, so that $\mathscr{H}$ is abelian. We also assume, from now on, that our Cartan matrix is not of Euclidean type, (see [**2**] for an enumeration of these). Then it is known [**1**], that $\mathscr{R}$ is the only maximal ideal of $\mathscr{L}_U$ so that $\mathscr{L}_R$ is simple.

Let $\mathscr{L}$ be any of our heffalump algebras. The following facts are well known and can be found in [**1**; **4**; **6**]. Let $V = \mathbf{Z}\alpha_1 \oplus \ldots \oplus \mathbf{Z}\alpha_l$ be the free $\mathbf{Z}$-module with generators $\alpha_1, \ldots, \alpha_l$ and let V act on $\mathscr{H}$ via $\alpha_i(h_j) = A_{ji}$. Then there is a subset Δ of V such that

$$\mathscr{L} = \mathscr{H} + \sum_{\alpha \in \Delta} \mathscr{L}_\alpha$$

(all sums are direct), where $\mathscr{L}_\alpha$ is a subspace of $\mathscr{L}$, and $[\mathscr{L}_\alpha, \mathscr{L}_\beta] \subseteq \mathscr{L}_{\alpha+\beta}$. If $x \in \mathscr{L}_\alpha$, $h \in \mathscr{H}$ then $[x, h] = \alpha(h)x$. Also, $\alpha_i \in \Delta$ and $\mathscr{L}_{\alpha_i} = Ke_i$, $\mathscr{L}_{-\alpha_i} = Kf_i$ for $1 \leqq i \leqq l$. If

$$\alpha = \sum_{i=1}^{l} d_i\alpha_i \in \Delta$$

then either $d_i \geqq 0$ or $d_i \leqq 0$ for all i from 1 to l, and $0 \notin \Delta$. The elements of Δ are called roots of $\mathscr{L}$ and we can speak of positive and negative roots. If Δ^+ (respectively Δ^-) denotes the collection of positive (respectively negative) roots then $\Delta = \Delta^+ \cup \Delta^-$ and $-\Delta^+ = \Delta^-$. $\mathscr{L} = \mathscr{L}^- \oplus \mathscr{H} \oplus \mathscr{L}^+$ where $\mathscr{L}^+ = \sum_{\alpha \in \Delta^+} \mathscr{L}_\alpha$ and $\mathscr{L}^- = \sum_{\alpha\in\Delta^-} \mathscr{L}_\alpha$. $\mathscr{L}$ has has an automorphism of period two, which is denoted η, and $(e_i)\eta = f_i$, $(h_i)\eta = -h_i$ for $1 \leqq i \leqq l$ so that $(\mathscr{L}^+)\eta = \mathscr{L}^-$. Since we assume that (A_{ij}) is not Euclidean and the characteristic of K is zero, we have that if $0 \neq \alpha = \sum_{i=1}^{l} d_i\alpha_i \in V$ and $d_i \geqq 0$ for $1 \leqq i \leqq l$, then $\alpha(h) \neq 0$ for some $h \in \mathscr{H}$. However, it may be that for $\alpha, \beta \in \Delta$, $\alpha \neq \beta$ but $\alpha(h) = \beta(h)$ for all $h \in \mathscr{H}$.

For $\alpha \in \Delta$ we let $\mathscr{L}_{\tilde{\alpha}}$ denote the sum of the spaces $\mathscr{L}_\beta$ for which $\beta(h) = \alpha(h)$ for all $h \in \mathscr{H}$. It is important to note that $\mathscr{L}_{\alpha_i} = \mathscr{L}_{\tilde{\alpha}_i}$ is one dimensional as is $\mathscr{L}_{-\tilde{\alpha}_i} = \mathscr{L}_{-\alpha_i}$. This follows because if $\beta(h) = \alpha_i(h)$ for some $\beta \in \Delta$ and all $h \in \mathscr{H}$, then the coefficient of α_i in β must be non-zero because $\alpha_i(h_i) = 2$, and hence if $\beta \neq \alpha_i$, $\beta - \alpha_i$ would give rise to a null root, which is impossible since (A_{ij}) is not Euclidean (see [**2**]).

Let $\mathscr{H}_0 = \{h \in \mathscr{H} | \alpha(h) = 0 \text{ for all } \alpha \in \Delta\}$ and note that $\mathscr{H}_0 = (0)$ if $\mathscr{L} = \mathscr{L}_R$. We always have, for any of our algebras, that the dimension of $\text{ad}_{\mathscr{L}}\mathscr{H} = \{\text{ad } h | h \in \mathscr{H}\}$ equals the rank of our Cartan matrix (A_{ij}). Finally, we let $V_K = K \otimes_{\mathbf{Z}} V$, and define a non-degenerate symmetric bilinear form on V_K, $\langle .,. \rangle$, by taking the basis $\alpha_1, \ldots, \alpha_l$ to be orthonormal. Thus, if $\alpha =$

$\sum_{i=1}^{l} C_i\alpha_i$, $\beta = \sum_{i=1}^{l} d_i\alpha_i$ are in V_K then $\langle\alpha,\beta\rangle = \sum_{i=1}^{l} C_i d_i$. In Section 3 we will recall some more facts about $\mathscr{L}_R$ which we will use there.

2. Derivations of the algebras. Let $\mathscr{L} = \mathscr{H} + \sum_{\alpha\in\Delta}\mathscr{L}_\alpha$ be any one of our three heffalump algebras associated to the non-Euclidean Cartan matrix (A_{ij}) over the field K of characteristic zero. We define

$$\mathscr{D}_0 = \{D \in \mathscr{D}(\mathscr{L}) | \mathscr{H}D = (0)\}.$$

LEMMA 1. *Let $D \in \mathscr{D}(\mathscr{L})$ and assume that $\mathscr{H}D \subseteq \mathscr{H}$. Then $D \in \mathscr{D}_0$ and there exist scalars $\gamma_i \in K$ such that $e_iD = \gamma_i e_i$, $f_iD = -\gamma_i f_i$ for $1 \leqq i \leqq l$.*

Proof. Let D be a derivation of $\mathscr{L}$ which preserves $\mathscr{H}$ and let $\alpha \in \Delta$, $0 \neq \chi_\alpha \in \mathscr{L}_\alpha$. Say $\chi_\alpha D = h' + \sum_{\beta\in\Delta} e_\beta$ where $h' \in \mathscr{H}$ and $e_\beta \in \mathscr{L}_\beta$ for all $\beta \in \Delta$. Then for any $h \in \mathscr{H}$ we have

$$[\chi_\alpha, h]D = [\chi_\alpha D, h] + [\chi_\alpha, hD]$$

which implies that

$$\alpha(h)h' + \sum_{\beta\in\Delta} \alpha(h)e_\beta = \sum_{\beta\in\Delta} \beta(h)e_\beta + \alpha(hD)\chi_\alpha.$$

From this it follows that $h' = 0$ and that $e_\beta = 0$ unless $\mathscr{L}_\beta \subseteq \mathscr{L}_{\tilde{\alpha}}$. Thus, $\alpha(hD)\chi_\alpha = 0$ for all $h \in \mathscr{H}$ so that $hD \in \mathscr{H}_0$. Moreover, we have $\mathscr{L}_\alpha D \subseteq \mathscr{L}_{\tilde{\alpha}}$ for any $\alpha \in \Delta$, so in particular, $\mathscr{L}_{\alpha_i}D \subseteq \mathscr{L}_{\alpha_i}$ and $\mathscr{L}_{-\alpha_i}D \subseteq \mathscr{L}_{-\alpha_i}$ for $1 \leqq i \leqq l$. Thus, there are scalars $\gamma_i^+, \gamma_i^- \in K$ for which $e_iD = \gamma_i^+e_i$, $f_iD = \gamma_i^-f_i$ for $1 \leqq i \leqq l$. Now $h_iD = [e_i, f_i]D = (\gamma_i^+ + \gamma_i^-)h_i \in \mathscr{H}_0$, but $h_i \notin \mathscr{H}_0$ for $1 \leqq i \leqq l$. It follows that $-\gamma_i^+ = \gamma_i^-$ and $D \in \mathscr{D}_0$.

Definition. For any $\gamma \in V_K$ we define a map $D_\gamma : \mathscr{L} \to \mathscr{L}$ as follows: $h_iD_\gamma = 0$ for $1 \leqq i \leqq l$, and if $\alpha \in \Delta$, $\chi_\alpha \in \mathscr{L}_\alpha$ we let $\chi_\alpha D_\gamma = \langle\alpha, \gamma\rangle\chi_\alpha$. We extend D_γ by linearity to all of $\mathscr{L}$ and note the fact that $[\mathscr{L}_\alpha, \mathscr{L}_\beta] \subseteq \mathscr{L}_{\alpha+\beta}$ implies that $D_\gamma \in \mathscr{D}_0$ for all $\gamma \in V_K$.

LEMMA 2. $\mathscr{D}_0 \cap \mathscr{I}(\mathscr{L}) = \mathrm{ad}_{\mathscr{L}}\mathscr{H}$ *and the dimension of $\mathscr{D}_0$ is l.*

Proof. If ad $\chi \in \mathscr{D}_0 \cap \mathscr{I}(\mathscr{L})$ then $[h, \chi] = 0$ for all $h \in \mathscr{H}$. This clearly implies that $\chi \in \mathscr{H}$. Also, by Lemma 1, we have that $\mathscr{D}_0 = \{D_\gamma | \gamma \in V_K\}$ and hence is of dimension l.

THEOREM 1. *$\mathscr{D}(\mathscr{L}) = \mathscr{D}_0 + \mathscr{I}(\mathscr{L})$ and the dimension of $\mathscr{D}(\mathscr{L})/\mathscr{I}(\mathscr{L})$ equals the nullity of (A_{ij}).*

Proof. Let $D \in \mathscr{D}(\mathscr{L})$ and let $\{e_{\beta_i}\}_{i=1}^{\infty}$ be a basis of $\mathscr{L}^+$ such that $e_{\beta_i} \in \mathscr{L}_{\beta_i}$ for all $i \geqq 1$ and $e_{\beta_j} = e_j$ for $1 \leqq j \leqq l$. Let $e_{-\beta_j} = (e_{\beta_j})\eta$ so that $\{e_{-\beta_j}\}_{j=1}^{\infty}$ is a basis of $\mathscr{L}^-$ and $e_{-\beta_j} = f_j$ for $1 \leqq j \leqq l$. For each $i \geqq 1$ we choose $h_i \in \mathscr{H}$ such that $\beta_i(h_i) = 2$ for all $i \geqq 1$ and $h_j = [e_j, f_j]$ for $1 \leqq j \leqq l$. This choice of an infinite collection of h_i's is possible since (A_{ij}) is not Euclidean.

For each $i \geqq 1$ let

$$h_i D = \sum_{j=1}^{\infty} a_{ij} e_{-\beta_j} + \hat{h}_i + \sum_{j=1}^{\infty} b_{ij} e_{\beta_j}$$

where $\hat{h}_i \in \mathscr{H}$ and the coefficients a_{ij}, b_{ij} are in K and almost all of them are zero. Since $\mathscr{H}$ is abelian we have that $[h_i D, h_k] = [h_k D, h_i]$, so upon comparing coefficients we obtain that $a_{ij}\beta_j(h_k) = a_{kj}\beta_j(h_i)$ and $b_{ij}\beta_j(h_k) = b_{kj}\beta_j(h_i)$ for all $i, j, k \geqq 1$. Taking $i = j$ we get that $a_{ii}\beta_i(h_k) = 2a_{ki}$, $b_{ii}\beta_i(h_k) = 2b_{ki}$ for all $i, k \geqq 1$.
Thus,

$$2(h_i D - \hat{h}_i) = \sum_{j=1}^{\infty} a_{jj}\beta_j(h_i)e_{-\beta_j} + \sum_{j=1}^{\infty} b_{jj}\beta_j(h_i)e_{\beta_j} = h_i \operatorname{ad} \chi,$$

where

$$\chi = \sum_{j=1}^{\infty} a_{jj} e_{-\beta_j} - \sum_{j=1}^{\infty} b_{jj} e_{\beta_j},$$

and the finite dimensionality of $\mathscr{H}$ insures that almost all coefficients in χ are zero. We now have that $h_i(D - 1/2 \operatorname{ad} \chi) = \hat{h}_i \in \mathscr{H}$ for all $i \geqq 1$ and hence, since $\{h_i\}_{i=1}^{\infty}$ spans $\mathscr{H}$, that $D - 1/2 \operatorname{ad} \chi \in \mathscr{D}_0$, by Lemma 1. Thus $\mathscr{D}(\mathscr{L}) = \mathscr{D}_0 + \mathscr{I}(\mathscr{L})$ and hence $\mathscr{D}(\mathscr{L})/\mathscr{I}(\mathscr{L}) \cong \mathscr{D}_0/\operatorname{ad}_{\mathscr{L}} \mathscr{H}$. From Lemma 2 it follows that the dimension of $\mathscr{D}(\mathscr{L})/\mathscr{I}(\mathscr{L})$ equals the nullity of (A_{ij}).

Remarks. (1). It is perhaps worthwhile to point out that our method can be applied to the Classical Simple Lie Algebras over fields of characteristic p not 2 or 3, which arise from the Cartan matrices of finite type (see [**1**]). In particular, except when the matrix is of type A_l and $p|l + 1$, we see that all derivations of such algebras are inner and hence this covers the case of algebras of type E_8 over fields of characteristic 5. For this case the result that all derivations are inner appears to be new [**8**, p. 112].

(2) If our Cartan matrix is a 1-tiered Euclidean Cartan matrix and $\mathscr{L}$ is the reduced heffalump algebra arising from it over the field K of characteristic zero then $\mathscr{L} \cong \hat{\mathscr{L}} \otimes_K K[\chi, \chi^{-1}]$, where $\hat{\mathscr{L}}$ is a finite dimensional split simple Lie algebra over K [**6**; **7**]. R. Moody has applied our techniques to this situation and concludes that $\mathscr{D}(\mathscr{L})/\mathscr{I}(\mathscr{L})$ is isomorphic to the infinite dimensional abelian Lie algebra $K[\chi, \chi^{-1}]$. Here $K[\chi, \chi^{-1}]$ denotes the ring of finite Laurent series over K.

3. An application to isomorphism classes. Our main result in this section is that if K is a field of characteristic zero then there is a doubly infinite family of simple infinite dimensional Lie algebras of heffalump type over K each of which belongs to a different isomorphism class. More specifically, if $t \geqq 1$ we let I_t denote the $t \times t$ identity matrix and $\chi_t(\rho)$ the $t \times t$ matrix with ρ's on the main diagonal and -1's elsewhere. For $n \geqq 2$ we define two $4n \times 4n$

Cartan matrices (A_{ij}) and (B_{ij}) as follows:

$$(A_{ij}) = \left[\begin{array}{cc|c} \frac{\chi_n(2)}{\chi_n(0)} & \frac{\chi_n(0)}{\chi_n(2)} & -2\,I_{2n} \\ \hline \multicolumn{2}{c|}{-2\,I_{2n}} & 2\,I_{2n} \end{array}\right],$$

$$(B_{ij}) = \begin{bmatrix} \chi_n(2) & \chi_n(-1) & \chi_n(-4) & \chi_n(-1) \\ \chi_n(-2) & \chi_n(2) & \chi_n(-1) & \chi_n(-5) \\ \chi_n(-4) & \chi_n(-1) & \chi_n(2) & \chi_n(-1) \\ \chi_n(-2) & \chi_n(-5) & \chi_n(-1) & \chi_n(2) \end{bmatrix}.$$

We let $\mathscr{L}_n$ (respectively $\hat{\mathscr{L}}_n$) denote the reduced heffalump algebra attached to (A_{ij}) (respectively (B_{ij})) over K. We are going to show that for any $m \geqq 2$ if $\mathscr{L} = \mathscr{L}_m$ or $\mathscr{L} = \hat{\mathscr{L}}_m$ then the only algebra in the collection $\{\mathscr{L}_n, \hat{\mathscr{L}}_n, n \geqq 2\}$ which is isomorphic to $\mathscr{L}$ is $\mathscr{L}$ itself.

It is clear that both (A_{ij}) and (B_{ij}) are indecomposable Cartan matrices. Moreover, for $n \geqq 2$ fixed, and $1 \leqq i \leqq n$ we let v_i denote the column vector

$$[\delta_{1i}, \ldots, \delta_{ni}, -\delta_{1i}, \ldots, -\delta_{ni}, \delta_{1i}, \ldots, \delta_{ni}, -\delta_{1i}, \ldots, -\delta_{ni}]$$

and note that v_i is in the kernel of both of our $4n \times 4n$ Cartan matrices. Also, it is an easy matter to check that the vectors $v_1, \ldots, v_n$ span the kernels of our matrices and hence the nullity of each of the $4n \times 4n$ Cartan matrices is exactly n. Thus, if $\bar{\mathscr{L}}_m$ is one of the reduced algebras $\mathscr{L}_m$ or $\hat{\mathscr{L}}_m$ then Theorem 1 implies that $\bar{\mathscr{L}}_m$ is not isomorphic to $\bar{\mathscr{L}}_n$ if $n \neq m$. Hence, to prove our result, we need only show $\mathscr{L}_n$ and $\hat{\mathscr{L}}_n$ are not isomorphic. This will follow from Theorem 2 below.

At this point we need to recall some more information about reduced heffalump algebras. An $l \times l$ Cartan matrix (C_{ij}) is called symmetrizable if and only if there exist positive rational numbers $\epsilon_1, \ldots, \epsilon_l$ for which $A_{ij}\epsilon_j = A_{ji}\epsilon_i$ for $1 \leqq i, j \leqq l$. Note that our matrix (A_{ij}) is symmetric, hence symmetrizable; but that (B_{ij}) is not symmetrizable. Also, it is known [**6**] that if $\mathscr{L}$ is a reduced heffalump algebra over K attached to a symmetrizable Cartan matrix, then there is a non-degenerate symmetric bilinear form $(.,.) : \mathscr{L} \times \mathscr{L} \to K$ which is associative in the sense that $([x, y], z) = (x, [y, z])$ for all $x, y, z \in \mathscr{L}$.

Let $\mathscr{L}$ denote the reduced heffalump algebras over the field K which is attached to the $l \times l$ indecomposable Cartan matrix (C_{ij}) and assume (C_{ij}) is not Euclidean. The Weyl group, W, of (C_{ij}) is defined to be the subgroup of $GL(V_K)$ generated by the reflections r_i, $1 \leqq i \leqq l$, defined on V_K by

$$\alpha_i r_j = \alpha_i - C_{ji}\alpha_j$$

for $1 \leqq i, j \leqq l$. It is known [**1**; **6**] that if $\omega \in W$ there is an automorphism $\theta(\omega)$ of $\mathscr{L}$ such that $\mathscr{L}_\alpha\theta(\omega) = \mathscr{L}_{\alpha\omega}$ for all $\alpha \in \Delta$ and $\mathscr{H}\theta(\omega) = \mathscr{H}$. In fact, $h_j\theta(r_i) = h_j - C_{ji}h_i$ for $1 \leqq i, j \leqq l$ and each $\theta(\omega)$ is in the subgroup of Aut $(\mathscr{L})$ generated by elements of the form exp (ad z) where $z \in \mathscr{L}$ and ad z is locally nilpotent on $\mathscr{L}$. Since the inverse of exp (ad z) is exp (ad $(-z)$), it is easy to see that if $(.,.) : \mathscr{L} \times \mathscr{L} \to K$ is any non-degenerate symmetric associative bilinear form on $\mathscr{L}$ then $(x\theta(\omega), y\theta(\omega)) = (x, y)$ for all $x, y \in \mathscr{L}$ and $\omega \in W$. We are now in a position to prove the following result which was obtained jointly with R. Moody.

THEOREM 2. *Let (C_{ij}) be an indecomposable Cartan matrix which is not Euclidean and let $\mathscr{L}$ denote the corresponding reduced heffalump algebra over the field K of characteristic zero. Then $\mathscr{L}$ has a non-degenerate symmetric associative bilinear form if and only if (C_{ij}) is symmetrizable.*

Proof. We need only show that if $(.,.) : \mathscr{L} \times \mathscr{L} \to K$ is such a form then (C_{ij}) is symmetrizable. For any $\alpha \in \Delta$, $e_\alpha \in \mathscr{L}_\alpha$ and $h, h' \in \mathscr{H}$ we have that

$$\alpha(h)(e_\alpha, h') = ([e_\alpha, h], h') = (e_\alpha, [h, h']) = 0.$$

Since h and h' are arbitrary we get that $(\mathscr{L}_\alpha, \mathscr{H}) = (0)$ for any $\alpha \in \Delta$. Thus, our form restricted to $\mathscr{H}$ is non-degenerate.

Next, we note that $(h_i, h_i) \neq 0$ for $1 \leqq i \leqq l$. Indeed,

$$(h_i, h_i) = ([e_i, f_i], h_i) = (e_i, [f_i, h_i]) = -2(e_i, f_i).$$

Also, for $h \in \mathscr{H}$, $\beta \in \Delta$, and $e_\beta \in \mathscr{L}_\beta$ we have $([e_i, h], e_\beta) = (e_i, [h, e_\beta])$, so that $\alpha_i(h)(e_i, e_\beta) = -\beta(h)(e_i, e_\beta)$. Thus, since $\mathscr{L}_{\alpha_i} = \mathscr{L}_{\tilde{\alpha}_i}$ is one dimensional we get $(e_i, \mathscr{L}_\beta) = (0)$ unless $\beta = -\alpha_i$. It follows that $(e_i, f_i) \neq 0$, since our form is non-degenerate, and hence that $(h_i, h_i) \neq 0$ for $1 \leqq i \leqq l$.

We now normalize our form, multiplying it by a non-zero scalar if necessary, to assume $(h_1, h_1) = 1$. For $1 \leqq i, j, k \leqq l$ we have $(h_i, h_j) = (h_i\theta(r_k), h_j\theta(r_k))$. It then follows using the formula $h_i\theta(r_k) = h_i - C_{ik}h_k$, that

$$C_{ik}C_{jk}(h_k, h_k) = C_{jk}(h_i, h_k) + C_{ik}(h_k, h_j) \text{ for } 1 \leqq i, j, k \leqq l.$$

Take $i = k$ to get

$$2C_{ji}(h_i, h_i) = C_{ji}(h_i, h_i) + 2(h_i, h_j)$$

and interchange i and j to obtain

$$2C_{ij}(h_j, h_j) = C_{ij}(h_j, h_j) + 2(h_j, h_i).$$

Thus, for $1 \leqq i, j \leqq l$,

$$C_{ji}(h_i, h_i) = 2(h_i, h_j) = 2(h_j, h_i) = C_{ij}(h_j, h_j),$$

so setting $\epsilon_i = (h_i, h_i)$ yields

$$C_{ij}\epsilon_j = C_{ji}\epsilon_i \text{ for } 1 \leqq i, j \leqq l.$$

Also, the fact that (C_{ij}) is indecomposable together with $\epsilon_1 = 1$ implies that each ϵ_i is a positive rational number.

Theorem 2, together with our previous remarks, now implies the following result.

THEOREM 3. *Let K be any field of characteristic zero. Then there is a doubly infinite family of isomorphism classes of simple heffalump algebras over K.*

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