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CARLSON–GRIFFITHS THEORY FOR COMPLETE KÄHLER MANIFOLDS

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Abstract We investigate Carlson–Griffiths' equidistribution theory of meormorphic mappings from a complete Kähler manifold into a complex projective algebraic manifold. By using a technique of Brownian motions developed by Atsuji, we obtain a second main theorem in Nevanlinna theory provided that the source manifold is of nonpositive sectional curvature. In particular, a defect relation follows if some growth condition is imposed.

Keywords and phrases: Nevanlinna theory; value distribution; second main theorem; logarithmic derivative lemma; defect relation.

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1. Introduction

Early in the 1970s, Carlson and Griffiths [7, 13] made significant progress in the study of Nevanlinna theory, which devised the equi-distribution theory for holomorphic mappings from $\mathbb{C}^m$ into complex projective algebraic manifolds intersecting divisors. Later, Griffiths and King [14, 13] further extended this theory from $\mathbb{C}^m$ to algebraic manifolds. More generalisations were done by Sakai [24] in terms of Kodaira dimension, and the singular divisor was considered by Shiffman [25]. To begin with, let us review Carlson–Griffiths' work briefly.

Let V be a complex projective algebraic manifold. Given two holomorphic line bundles L_1, L_2 over V, we set

$$\overline{\left[\frac{c_1(L_2)}{c_1(L_1)}\right]} = \inf\left\{t\in\mathbb{R}:\ \omega_2 < t\omega_1;\ {}^\exists\omega_1\in c_1(L_1),\ {}^\exists\omega_2\in c_1(L_2)\right\},$$
$$\underline{\left[\frac{c_1(L_2)}{c_1(L_1)}\right]} = \sup\left\{t\in\mathbb{R}:\ \omega_2 > t\omega_1;\ {}^\exists\omega_1\in c_1(L_1),\ {}^\exists\omega_2\in c_1(L_2)\right\}.$$

Let $f:\mathbb{C}^m\to V$ be a holomorphic mapping. The defect $\delta_f(D)$ of f with respect to D is defined by

$$\delta_f(D)=1-\limsup_{r\to\infty}\frac{N_f(r,D)}{T_f(r,L)},$$

where $N_f(r,D)$,$T_f(r,L)$ are respectively the counting function and the characteristic function of f (see definition in Remark 3.3). Carlson–Griffiths proved the following:

Theorem A. *Let $f:\mathbb{C}^m\to V$ be a differentiably nondegenerate holomorphic mapping with $\dim_{\mathbb{C}}V=m$. Let $D\in|L|$ be a divisor of simple normal crossing type, where L is a positive line bundle over V. Then*

$$\delta_f(D)\le\overline{\left[\frac{c_1(K_V^*)}{c_1(L)}\right]}.$$

The purpose of this article is to generalize Theorem A to complete Kähler manifolds. The method is to combine the logarithmic derivative lemma (LDL) with a stochastic technique developed by Carne and Atsuji. So, the first task here is to establish the LDL for meromorphic functions on complete Kähler manifolds (see Theorem 1.1), which may be of its own interest. Recall that the first probabilistic proof of Nevanlinna's second main theorem of meromorphic functions on $\mathbb{C}$ is due to Carne [8], who reformulated Nevanlinna's functions in terms of Brownian motions. Later, Atsuji wrote a series of papers to study the second main theorem of meromorphic functions on complete Kähler manifolds; see [1, 2, 3, 4]. Recently, Dong–He–Ru [11] re-visited this technique and gave a probabilistic proof of H. Cartan's theory of holomorphic curves.

Let M be a complete Kähler manifold. In what follows, we state the main results of the article, and some notations will be provided later. For technical reasons, we assume that M is connected and noncompact in this article.

We first establish the following LDL.

Theorem 1.1. *Let ψ be a nonconstant meromorphic function on M. Then for any $\delta>0$, there exist a function $C(o,r,\delta)>0$ (independent of ψ) and a set $E_\delta\subset(1,\infty)$ of finite Lebesgue measure such that*

$$m\Big(r,\frac{\|\nabla_M\psi\|}{|\psi|}\Big)\le\Big(1+\frac{(1+\delta)^2}{2}\Big)\log T(r,\psi)+\log C(o,r,\delta)$$

holds for $r>1$ outside E_δ, where o is a fixed reference point in M.

The estimate of term $C(o,r,\delta)$ will be provided when M is nonpositively curved (see (19)). Let Ric_M and $\mathscr{R}_M$ be the Ricci curvature tensor and Ricci curvature form of M, respectively. Set

$$\kappa(t)=\frac{1}{2\dim_{\mathbb{C}}M-1}\min_{x\in\overline{B_o(t)}}R_M(x),\tag{1}$$

where $R_M(x)$ is the pointwise lower bound of the Ricci curvature defined by

$$R_M(x) = \inf_{\xi \in T_x M} \frac{\mathrm{Ric}_M(\xi,\bar{\xi})}{\|\xi\|^2}.$$

Based on the LDL, we obtain a second main theorem as follows:

Theorem 1.2. *Let $f : M \to V$ be a differentiably nondegenerate meromorphic mapping with $\dim_{\mathbb{C}} M \geq \dim_{\mathbb{C}} V$. Let $D \in |L|$ be a divisor of simple normal crossing type, where L is a holomorphic line bundle over V. Then for any $\delta > 0$, there exist a function $C(o,r,\delta) > 0$ (independent of ψ) and a set $E_\delta \subset (1,\infty)$ of finite Lebesgue measure such that*

$$\begin{aligned} &T_f(r,L) + T_f(r,K_V) + T(r,\mathscr{R}_M) \\ &\leq \overline{N}_f(r,D) + O\big(\log T_f(r,\omega) + \log C(o,r,\delta)\big) \end{aligned}$$

holds for $r > 1$ outside E_δ.

If M is nonpositively curved, then we prove the following:

Theorem 1.3. *Let $f : M \to V$ be a differentiably nondegenerate meromorphic mapping with $\dim_{\mathbb{C}} M \geq \dim_{\mathbb{C}} V$. Let $D \in |L|$ be a divisor of simple normal crossing type, where L is a holomorphic line bundle over V. Fix a Hermitian metric ω on V. Then for any $\delta > 0$,*

$$\begin{aligned} &T_f(r,L) + T_f(r,K_V) \\ &\leq \overline{N}_f(r,D) + O\big(\log T_f(r,\omega) - \kappa(r) r^2 + \delta \log r\big) \end{aligned}$$

holds for $r > 1$ outside a set $E_\delta \subset (1,\infty)$ of finite Lebesgue measure.

Let $\Theta_f(D)$ be the *simple defect* of f with respect to D defined by

$$\Theta_f(D) = 1 - \limsup_{r\to\infty} \frac{\overline{N}_f(r,D)}{T_f(r,L)},$$

where $\overline{N}_f(r,D)$ is the *simple counting function* of f with respect to D.

Corollary 1.4 (Defect relation). *Assume the same conditions as in Theorem 1.3. If f satisfies the growth condition*

$$\liminf_{r\to\infty} \frac{r^2 \kappa(r)}{T_f(r,\omega)} = 0,$$

then

$$\Theta_f(D) \underline{\left[\frac{c_1(L)}{\omega}\right]} \leq \overline{\left[\frac{c_1(K_V^*)}{\omega}\right]}.$$

In particular, if $M = \mathbb{C}^m$ with standard Euclidean metric, then $\kappa(r) \equiv 0$. Hence, Corollary 1.4 implies Theorem A. More generally, we further consider the second main theorem for singular divisors.

Theorem 1.5. *Let $f: M \to V$ be a differentiably nondegenerate meromorphic mapping with $\dim_{\mathbb{C}} M \geq \dim_{\mathbb{C}} V$. Let D be a hypersurface of V. Then for any $\delta > 0$,*

$$\begin{aligned} &T_f(r, L_D) + T_f(r, K_V) - \overline{N}_f(r, D) \\ &\leq m_f\big(r, \mathrm{Sing}(D)\big) + O\big(\log T_f(r, \omega) - \kappa(r) r^2 + \delta \log r\big) \end{aligned}$$

holds for $r > 1$ outside a set $E_\delta \subset (1, \infty)$ of finite Lebesgue measure.

2. Preliminaries

We introduce some basics concerning the Poincaré–Lelong formula, Brownian motion and Ricci curvature. We refer the reader to [5, 6, 9, 14, 16, 17, 18, 22].

2.1. Poincaré–Lelong formula

Let M be an m-dimensional complex manifold. A divisor D on M is said to be of *normal crossings* if D is locally defined by an equation $z_1 \cdots z_k = 0$ for a local holomorphic coordinate system $z_1, \cdots, z_m$. Additionally, if every irreducible component of D is smooth, one says that D is of *simple normal crossings*. A holomorphic line bundle $L \to M$ is said to be *Hermitian* if L is equipped with a Hermitian metric $h = (\{h_\alpha\}, \{U_\alpha\})$, where

$$h_\alpha : U_\alpha \to \mathbb{R}^+$$

are positive smooth functions such that $h_\beta = |g_{\alpha\beta}|^2 h_\alpha$ on $U_\alpha \cap U_\beta$, and $\{g_{\alpha\beta}\}$ is a transition function system of L. Let $\{e_\alpha\}$ be a local holomorphic frame of L; then we have $\|e_\alpha\|_h^2 = h_\alpha$. A Hermitian metric h of L defines a global, closed and smooth (1,1)-form $-dd^c \log h$ on M, where

$$d = \partial + \bar{\partial}, \quad d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial), \quad dd^c = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}.$$

We call $-dd^c \log h$ the Chern form denoted by $c_1(L, h)$ associated with metric h, which determines a Chern class $c_1(L) \in H^2_{\mathrm{DR}}(M, \mathbb{R})$; $c_1(L, h)$ is also called the curvature form of L. If $c_1(L) > 0$, namely, there exists a Hermitian metric h such that $-dd^c \log h > 0$, then we say that L is positive, written as $L > 0$.

Let TM denote the holomorphic tangent bundle of M. The *canonical line bundle* of M is defined by

$$K_M = \bigwedge^m T^* M$$

with transition functions $g_{\alpha\beta} = \det(\partial z_j^\beta / \partial z_i^\alpha)$ on $U_\alpha \cap U_\beta$. Given a Hermitian metric h on K_M, it well defines a global, positive and smooth (m,m)-form

$$\Omega = \frac{1}{h} \bigwedge_{j=1}^m \frac{\sqrt{-1}}{2\pi} dz_j \wedge d\bar{z}_j$$

on M, which is therefore a volume form of M. The Ricci form of Ω is defined by $\mathrm{Ric}\Omega = dd^c \log h$. Clearly, $c_1(K_M, h) = -\mathrm{Ric}\Omega$. Conversely, if we let Ω be a volume form

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on M which is compact, there exists a unique Hermitian metric h on K_M such that $dd^c \log h = \mathrm{Ric}\Omega$.

Let $H^0(M,L)$ denote the vector space of holomorphic global sections of L over M. For any $s \in H^0(M,L)$, the divisor D_s is well defined by $D_s \cap U_\alpha = (s)|_{U_\alpha}$. Denote by $|L|$ the *complete linear system* of all effective divisors D_s with $s \in H^0(M,L)$. Let D be a divisor on M; then D defines a holomorphic line bundle L_D over M in such manner: let $(\{g_\alpha\},\{U_\alpha\})$ be the local defining function system of D; then the transition system is given by $\{g_{\alpha\beta} = g_\alpha/g_\beta\}$. Note that $\{g_\alpha\}$ defines a global meromorphic section on M written as s_D of L_D over M, called the *canonical section* associated with D.

Lemma 2.1 (Poincaré–Lelong formula, [7]). *Let $L \to M$ be a holomorphic line bundle equipped with a Hermitian metric h, and let s be a holomorphic section of L over M with zero divisor D_s. Then $\log\|s\|_h$ is locally integrable on M and defines a current satisfying*

$$dd^c\left[\log\|s\|_h^2\right] = D_s - c_1(L,h).$$

2.2. Brownian motions

Let (M,g) be a Riemannian manifold with the Laplace–Beltrami operator Δ_M associated with metric g. A *Brownian motion* X_t in M is a heat diffusion process generated by $\Delta_M/2$ with *transition density function* $p(t,x,y)$ being the minimal positive fundamental solution of the heat equation

$$\frac{\partial}{\partial t}u(t,x) - \frac{1}{2}\Delta_M u(t,x) = 0.$$

In particular, when $M = \mathbb{R}^m$,

$$p(t,x,y) = \frac{1}{(2\pi t)^{\frac{m}{2}}} e^{-\|x-y\|^2/2t}.$$

Let X_t be the Brownian motion in M with generator $\Delta_M/2$. We denote by $\mathbb{P}_x$ the law of X_t starting from $x \in M$ and denote by $\mathbb{E}_x$ the expectation with respect to $\mathbb{P}_x$.

A. Co-area formula

Let D be a bounded domain with the smooth boundary ∂D in M. Denote by $d\pi_x^{\partial D}(y)$ the harmonic measure on ∂D with respect to x and by $g_D(x,y)$ the Green function of $\Delta_M/2$ for D with Dirichlet boundary condition and a pole at x; that is,

$$-\frac{1}{2}\Delta_M g_D(x,y) = \delta_x(y), \quad y \in D; \quad g_D(x,y) = 0, \quad y \in \partial D.$$

For each $\phi \in \mathscr{C}_\flat(D)$ (space of bounded and continuous functions on D), the *co-area formula* [5] says that

$$\mathbb{E}_x\left[\int_0^{\tau_D} \phi(X_t)dt\right] = \int_D g_D(x,y)\phi(y)dV(y), \tag{2}$$

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where dV is the Riemannian volume element on M. From Proposition 2.8 in [5], we have the relation of harmonic measures and hitting times as follows:

$$\mathbb{E}_x[\psi(X_{\tau_D})] = \int_{\partial D} \psi(y) d\pi_x^{\partial D}(y) \tag{3}$$

for $\psi \in \mathscr{C}(\overline{D})$. The co-area formulas (3) and (2) still work when ϕ,ψ are of a pluripolar set of singularities.

B. Itô formula

The following identity is called the *Itô formula* (see [1, 17, 18]):

$$u(X_t) - u(x) = B\left(\int_0^t \|\nabla_M u\|^2(X_s)ds\right) + \frac{1}{2}\int_0^t \Delta_M u(X_s)dt, \ \ \mathbb{P}_x - a.s.$$

for $u \in \mathscr{C}_\flat^2(M)$ (space of bounded $\mathscr{C}^2$-class functions on M), where B_t is the standard Brownian motion in $\mathbb{R}$ and ∇_M is the gradient operator on M. It follows the *Dynkin formula*

$$\mathbb{E}_x[u(X_T)] - u(x) = \frac{1}{2}\mathbb{E}_x\left[\int_0^T \Delta_M u(X_t)dt\right]$$

for a stopping time T such that each term makes sense. The Dynkin formula still works if u is of a pluripolar set of singularities.

2.3. Ricci curvatures

Let (M,g) be a Kähler manifold of complex dimension m. Write the Ricci curvature of M in the form $\mathrm{Ric}_M = \sum_{i,j} R_{i\bar{j}} dz_i \otimes d\bar{z}_j$, where

$$R_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\det(g_{s\bar{t}}). \tag{4}$$

A well-known theorem by S. S. Chern asserts that the Ricci form of M

$$\mathscr{R}_M := -dd^c \log\det(g_{s\bar{t}}) = \frac{\sqrt{-1}}{2\pi}\sum_{i,j=1}^m R_{i\bar{j}} dz_i \wedge d\bar{z}_j$$

is a real and closed (1,1)-form which represents a cohomology class of the de Rham cohomology group $H^2_{\mathrm{DR}}(M,\mathbb{R})$. Let s_M be the scalar curvature of M defined by

$$s_M = \sum_{i,j=1}^m g^{i\bar{j}} R_{i\bar{j}},$$

where $(g^{i\bar{j}})$ is the inverse of $(g_{i\bar{j}})$. Since M is Kählerian, then by

$$\Delta_M = 2\sum_{i,j=1}^m g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}$$

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acting on a function, which yields from (4) that

$$s_M = -\frac{1}{2}\Delta_M \log\det(g_{s\bar{t}}).$$

Lemma 2.2. *Let R_M be the pointwise lower bound of Ricci curvature of M. Then*

$$s_M \geq mR_M.$$

Proof. Fix a point $x \in M$; we take local holomorphic coordinates $z_1,\cdots,z_m$ near x such that $g_{i\bar{j}}(x) = \delta^i_j$. Then we obtain

$$s_M(x) = \sum_{j=1}^{m} R_{j\bar{j}}(x) = \sum_{j=1}^{m} \mathrm{Ric}_M\Big(\frac{\partial}{\partial z_j},\frac{\partial}{\partial \bar{z}_j}\Big)_x \geq mR_M(x),$$

which proves the lemma. □

3. First main theorem

We first extend the notion of Nevanlinna's functions to the general Kähler manifolds and then give the first main theorem of meromorphic mappings on Kähler manifolds. Let (M,g) be a Kähler manifold of complex dimension m, the associated Kähler form is defined by

$$\alpha = \frac{\sqrt{-1}}{\pi}\sum_{i,j=1}^{m} g_{i\bar{j}}dz_i \wedge d\bar{z}_j.$$

Fix $o \in M$ as a reference point. Denote by $B_o(r)$ the geodesic ball centred at o with radius r and by $S_o(r)$ the geodesic sphere centred at o with radius r. By Sard's theorem, $S_o(r)$ is a submanifold of M for almost all $r > 0$. Also, one denotes by $g_r(o,x)$ the Green function of $\Delta_M/2$ for $B_o(r)$ with Dirichlet boundary condition and a pole at o and by $d\pi^r_o(x)$ the harmonic measure on $S_o(r)$ with respect to o.

3.1. Nevanlinna's functions

Let

$$f : M \to N$$

be a meromorphic mapping to a compact complex manifold N, which means that f is defined by such a holomorphic mapping $f_0 : M \setminus I \to N$, where I is some analytic subset of M with $\dim_{\mathbb{C}} I \leq m-2$, called the *indeterminacy set* of f such that the closure $\overline{G(f_0)}$ of the graph of f_0 is an analytic subset of $M \times N$ and the natural projection $\overline{G(f_0)} \to M$ is proper. Let η be a (1,1)-form on M, we use the following convenient notation:

$$e_\eta(x) = 2m\frac{\eta \wedge \alpha^{m-1}}{\alpha^m}.$$

Given a smooth (1,1)-form ω on N, since I is an indeterminacy set of f, one could confirm the local integrability of $g_r(o,x)e_{f^*\omega}(x)$ on M with respect to measure α^m by using the

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arguments in Noguchi–Ochiai [[20], Subsection 5.2]. We define the *characteristic function* of f with respect to ω by

$$\begin{aligned} T_f(r,\omega) &= \frac{1}{2}\int_{B_o(r)} g_r(o,x)e_{f^*\omega}(x)dV(x) \\ &= \frac{\pi^m}{(m-1)!}\int_{B_o(r)} g_r(o,x)f^*\omega\wedge\alpha^{m-1}, \end{aligned}$$

where $dV = \pi^m\alpha^m/m!$ is the Riemannian volume element on M. Let (L,h) be a Hermitian line bundle over N. By the compactness of N, we well define

$$T_f(r,L) := T_f\big(r,c_1(L,h)\big)$$

up to a bounded term. We further remark that the indeterminacy set I does not affect the local integrability of integrands in those quantities treated and hence the definitions of the following introduced proximity function $m_f(r,D)$ and counting function $N_f(r,D)$ (including Nevanlinna's functions in Section 5) make sense. We refer the reader to Noguchi–Ochiai [[20], Subsection 5.2].

In what follows, we define the *proximity function* and *counting function.*

Lemma 3.1. *$\Delta_M\log(h\circ f)$ is well defined on $M\setminus I$ satisfying*

$$\Delta_M\log(h\circ f) = -4m\frac{f^*c_1(L,h)\wedge\alpha^{m-1}}{\alpha^m}.$$

Hence, we have

$$e_{f^*c_1(L,h)} = -\frac{1}{2}\Delta_M\log(h\circ f).$$

Proof. Let $(\{U_\alpha\},\{e_\alpha\})$ be a local trivialisation covering of (L,h) with transition function system $\{g_{\alpha\beta}\}$ of local holomorphic frames $\{e_\alpha\}$. On $U_\alpha\cap U_\beta$,

$$e_\beta = g_{\alpha\beta}e_\alpha, \quad h_\alpha = h|_{U_\alpha} = \|e_\alpha\|^2, \quad h_\beta = h|_{U_\beta} = \|e_\beta\|^2.$$

We get

$$\Delta_M\log(h_\beta\circ f) = \Delta_M\log(h_\alpha\circ f) + \Delta_M\log|g_{\alpha\beta}\circ f|^2$$

on $f^{-1}(U_\alpha\cap U_\beta)\setminus I$. Notice that $g_{\alpha\beta}$ is holomorphic and nowhere vanishing on $U_\alpha\cap U_\beta$; we see that $\log|g_{\alpha\beta}\circ f|^2$ is harmonic on $f^{-1}(U_\alpha\cap U_\beta)\setminus I$. So, $\Delta_M\log(h_\beta\circ f) = \Delta_M\log(h_\alpha\circ f)$ on $f^{-1}(U_\alpha\cap U_\beta)\setminus I$. Thus, $\Delta_M\log(h\circ f)$ is well defined on $M\setminus I$. Fix $x\in M$; then we choose a normal holomorphic coordinate system z near x in the sense that $g_{i\bar{j}}(x) = \delta^i_j$ and all of the first-order derivatives of $g_{i\bar{j}}$ vanish at x. Then at x, we have

$$\Delta_M = 2\sum_{j=1}^m \frac{\partial^2}{\partial z_j\partial\bar{z}_j}, \quad \alpha^m = m!\bigwedge_{j=1}^m \frac{\sqrt{-1}}{\pi}dz_j\wedge d\bar{z}_j \tag{5}$$

as well as

$$f^*c_1(L,h)\wedge\alpha^{m-1} = -\frac{(m-1)!}{2}\mathrm{tr}\left(\frac{\partial^2\log(h\circ f)}{\partial z_i\partial\bar{z}_j}\right)\bigwedge_{j=1}^m\frac{\sqrt{-1}}{\pi}dz_j\wedge d\bar{z}_j,$$

where 'tr' means the trace of a square matrix. Indeed, by (5),

$$\Delta_M \log(h\circ f) = 2\mathrm{tr}\left(\frac{\partial^2 \log(h\circ f)}{\partial z_i \partial \bar{z}_j}\right)$$

at x. This proves the lemma. □

Take $0 \neq s \in H^0(N,L)$. Locally, we can write $s = \tilde{s}e$, where e is a local holomorphic frame of L. Then

$$\Delta_M \log\|s\circ f\|^2 = \Delta_M \log(h\circ f) + \Delta_M \log|\tilde{s}\circ f|^2.$$

Using similar arguments as in the proof of Lemma 3.1, we get

$$\Delta_M \log|\tilde{s}\circ f|^2 = 4m\frac{dd^c\log|\tilde{s}\circ f|^2\wedge\alpha^{m-1}}{\alpha^m}.$$

Lemma 3.2. *Let $s\in H^0(N,L)$ with zero divisor D. If $(L,h)\geq 0$, then*

(i) $\log\|s\circ f\|^2$ is locally the difference of two plurisubharmonic functions, and hence $\log\|s\circ f\|^2 \in \mathscr{L}_{loc}(M)$.

(ii) $dd^c[\log\|s\circ f\|^2] = f^*D - f^*c_1(L,h)$ in the sense of currents.

Proof. Locally, we can write $s = \tilde{s}e$, where e is a local holomorphic frame of L with $h = \|e\|^2$. Then

$$\log\|s\circ f\|^2 = \log|\tilde{s}\circ f|^2 + \log(h\circ f).$$

Since $c_1(L,h)\geq 0$, one obtains $-dd^c\log(h\circ f)\geq 0$. Indeed, $\tilde{s}$ is holomorphic; hence, $dd^c\log|\tilde{s}\circ f|^2 \geq 0$. This follows (i). The Poincaré–Lelong formula implies that $dd^c[\log|\tilde{s}\circ f|^2] = f^*D$ in the sense of currents; hence, (ii) holds. □

Let $D\in|L|$, where (L,h) is a Hermitian positive line bundle over N. We define the *proximity function* of f with respect to D by

$$m_f(r,D) = \int_{S_o(r)} \log\frac{1}{\|s_D\circ f(x)\|}d\pi_o^r(x).$$

Write

$$\log\|s_D\circ f\|^{-2} = \log(h\circ f)^{-1} - \log|\tilde{s}_D\circ f|^2$$

as the difference of two pluri-subharmonic functions. It defines a Riesz charge $d\mu = d\mu_1 - d\mu_2$, where $d\mu_2$ is a Riesz measure for f^*D. The *counting function* of f with respect to D is defined by

$$N_f(r,D) = \frac{1}{4}\int_{B_o(r)} g_r(o,x)d\mu_2(x) = \frac{\pi^m}{(m-1)!}\int_{f^*D\cap B_o(r)} g_r(o,x)\alpha^{m-1}.$$

Similarly, we can define $\overline{N}_f(r,D) := N(r,\mathrm{Supp}f^*D)$.

3.2. Probabilistic expressions of Nevanlinna's functions

Let us formulate Nevanlinna's functions in terms of Brownian motion X_t. Since I is a thin analytic subset contained in some pluripolar subset of M, X_t hits I in probability 0,

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I will not affect the expectation of those quantities involving f treated with respect to probability measure $d\mathbb{P}_o$. We define the stopping time

$$\tau_r = \inf\left\{t > 0 : X_t \notin B_o(r)\right\}.$$

Set $\omega := -dd^c \log h$. By the co-area formula, we have

$$T_f(r,L) = \frac{1}{2}\mathbb{E}_o\left[\int_0^{\tau_r} e_{f^*\omega}(X_t)dt\right].$$

By the relation between harmonic measures and hitting times, it gives that

$$m_f(r,D) = \mathbb{E}_o\left[\log\frac{1}{\|s_D\circ f(X_{\tau_r})\|}\right].$$

For the counting function $N_f(r,D)$, we use an alternative probabilistic expression (see [1, 4, 8, 12]) as follows:

$$N_f(r,D) = \lim_{\lambda\to\infty}\lambda\mathbb{P}_o\left(\sup_{0\le t\le\tau_r}\log\frac{1}{\|s_D\circ f(X_t)\|} > \lambda\right). \tag{6}$$

Remark 3.3. The definitions of Nevanlinna's functions in the above are natural extensions of the classical ones. To see that, we recall the $\mathbb{C}^m$-case:

$$\begin{aligned} T_f(r,L) &= \int_0^r \frac{dt}{t^{2m-1}}\int_{B_o(t)} f^*c_1(L,h)\wedge\alpha^{m-1}, \\ m_f(r,D) &= \int_{S_o(r)}\log\frac{1}{\|s_D\circ f\|}\gamma, \\ N_f(r,D) &= \int_0^r \frac{dt}{t^{2m-1}}\int_{f^*D\cap B_o(t)}\alpha^{m-1}, \end{aligned}$$

where o is taken as the coordinate origin of $\mathbb{C}^m$, and

$$\alpha = dd^c\|z\|^2, \quad \gamma = d^c\log\|z\|^2\wedge\left(dd^c\log\|z\|^2\right)^{m-1}.$$

Notice the following facts:

$$\gamma = d\pi_o^r(z), \quad g_r(o,z) = \begin{cases} \frac{\|z\|^{2-2m}-r^{2-2m}}{(m-1)\omega_{2m-1}}, & m\ge 2; \\ \frac{1}{\pi}\log\frac{r}{|z|}, & m=1. \end{cases},$$

where ω_{2m-1} is the volume of unit sphere in $\mathbb{R}^{2m}$. By integration by part, it is not difficult to see that they are a match.

3.3. First main theorem

Let N be a complex projective algebraic manifold. There is a very ample holomorphic line bundle L' over V. Equip L' with a Hermitian metric h' such that $\omega' := -dd^c\log h' > 0$. For an arbitrary holomorphic line bundle $L \to N$ equipped with a Hermitian metric h, whose Chern form says $\omega := -dd^c\log h$, we can pick $k\in\mathbb{N}$ large enough so that $\omega + k\omega' > 0$. Take the natural product Hermitian metric $\|\cdot\|$ on $L\otimes L'^{\otimes k}$; then the Chern form is $\omega + k\omega'$. Choose $\sigma\in H^0(M,L')$ such that $f(M)\not\subset \mathrm{Supp}(\sigma)$. Due to $\omega + k\omega' > 0$ and $\omega' > 0$, we see

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that $\log\|(s_D\otimes\sigma^k)\circ f\|^2$ and $\log\|\sigma\circ f\|^2$ are locally the difference of two pluri-subharmonic functions, where $D\in|L|$. Thus,

$$\log\|s_D\circ f\|^2=\log\|(s_D\otimes\sigma^k)\circ f\|^2-k\log\|\sigma\circ f\|^2$$

is locally the difference of two pluri-subharmonic functions. Hence, $m_f(r,D)$ can be defined.

We have the first main theorem (FMT).

Theorem 3.4 (FMT). *Assume that $f(o)\notin D$. Then*

$$T_f(r,L)=m_f(r,D)+N_f(r,D)+O(1).$$

Proof. Since I is an indeterminacy set and X_t meets I in probability 0, we may ignore I. Set

$$T_\lambda=\inf\left\{t>0:\sup_{s\in[0,t]}\log\frac{1}{\|s_D\circ f(X_s)\|}>\lambda\right\}.$$

Due to the definition of T_λ, X_t does not hit $\mathrm{Supp} f^*D$ when $0\le t\le\tau_r\wedge T_\lambda$. By Dynkin's formula, it follows that

$$\begin{aligned}&\mathbb{E}_o\left[\log\frac{1}{\|s_D\circ f(X_{\tau_r\wedge T_\lambda})\|}\right]\\&=\frac{1}{2}\mathbb{E}_o\left[\int_0^{\tau_r\wedge T_\lambda}\Delta_M\log\frac{1}{\|s_D\circ f(X_t)\|}dt\right]+\log\frac{1}{\|s_D\circ f(o)\|},\end{aligned}\tag{7}$$

where $\tau_r\wedge T_\lambda=\min\{\tau_r,T_\lambda\}$. Note that $\Delta_M\log|\tilde{s}_D\circ f|=0$ outside f^*D. We see that

$$\Delta_M\log\frac{1}{\|s_D\circ f(X_t)\|}=-\frac{1}{2}\Delta_M\log h\circ f(X_t)$$

for $t\in[0,T_\lambda]$. Thus, (7) becomes

$$\begin{aligned}&\mathbb{E}_o\left[\log\frac{1}{\|s_D\circ f(X_{\tau_r\wedge T_\lambda})\|}\right]\\&=-\frac{1}{4}\mathbb{E}_o\left[\int_0^{\tau_r\wedge T_\lambda}\Delta_M\log h\circ f(X_t)dt\right]+O(1).\end{aligned}$$

The monotone convergence theorem leads to

$$\frac{1}{4}\mathbb{E}_o\left[\int_0^{\tau_r\wedge T_\lambda}\Delta_M\log h\circ f(X_t)dt\right]\to\frac{1}{2}\mathbb{E}_o\left[\int_0^{\tau_r}e_{f^*\omega}(X_t)dt\right]=T_f(r,L)$$

as $\lambda\to\infty$. We handle the first term in (7) and write it as two parts:

$$\mathrm{I}+\mathrm{II}=\mathbb{E}_o\left[\log\frac{1}{\|s_D\circ f(X_{\tau_r})\|}:\tau_r<T_\lambda\right]+\mathbb{E}_o\left[\log\frac{1}{\|s_D\circ f(X_{T_\lambda})\|}:T_\lambda\le\tau_r\right].$$

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Using the monotone convergence theorem again,

$$\mathrm{I} \to \mathbb{E}_o\left[\log\frac{1}{\|s_D\circ f(X_{\tau_r})\|}\right] = m_f(r,D)$$

as $\lambda\to\infty$. Finally, we deal with II. By the definition of T_λ, we see that

$$\mathrm{II} = \lambda\mathbb{P}_o\left(\sup_{t\in[0,\tau_r]}\log\frac{1}{\|s_D\circ f(X_t)\|} > \lambda\right) \to N_f(r,D)$$

as $\lambda\to\infty$. Putting the above together, we show the theorem. □

4. Logarithmic derivative lemma

The LDL is an important tool in derivation of the second main theorem. The goal of this section is to prove the LDL for Kähler manifolds (i.e., Theorem 1.1).

4.1. Logarithmic derivative lemma

Let (M,g) be an m-dimensional complete Kähler manifold and ∇_M be the gradient operator on M associated with g. Let X_t be the Brownian motion in M with generator $\Delta_M/2$.

Lemma 4.1 (Calculus lemma, [1]). *Let $k\geq 0$ be a locally integrable function on M such that it is locally bounded at $o\in M$. Then for any $\delta>0$, there exist a function $C(o,r,\delta)>0$ (independent of k) and a set $E_\delta\subset[0,\infty)$ of finite Lebesgue measure such that*

$$\mathbb{E}_o\left[k(X_{\tau_r})\right] \leq C(o,r,\delta)\left(\mathbb{E}_o\left[\int_0^{\tau_r}k(X_t)dt\right]\right)^{(1+\delta)^2} \tag{8}$$

holds for $r>1$ outside E_δ.

Let ψ be a meromorphic function on M. The norm of the gradient of ψ is defined by

$$\|\nabla_M\psi\|^2 = 2\sum_{i,j=1}^m g^{i\bar{j}}\frac{\partial\psi}{\partial z_i}\overline{\frac{\partial\psi}{\partial z_j}},$$

where $(g^{i\bar{j}})$ is the inverse of $(g_{i\bar{j}})$. Locally, we write $\psi=\psi_1/\psi_0$, where ψ_0,ψ_1 are holomorphic functions so that $\operatorname{codim}_{\mathbb{C}}(\psi_0=\psi_1=0)\geq 2$ if $\dim_{\mathbb{C}}M\geq 2$. Identify ψ with a meromorphic mapping into $\mathbb{P}^1(\mathbb{C})$ by $x\mapsto[\psi_0(x):\psi_1(x)]$. The characteristic function of ψ with respect to the Fubini–Study form ω_{FS} on $\mathbb{P}^1(\mathbb{C})$ is defined by

$$T_\psi(r,\omega_{FS}) = \frac{1}{4}\int_{B_o(r)}g_r(o,x)\Delta_M\log(|\psi_0(x)|^2+|\psi_1(x)|^2)dV(x).$$

Let $i:\mathbb{C}\hookrightarrow\mathbb{P}^1(\mathbb{C})$ be an inclusion defined by $z\mapsto[1:z]$. Via the pullback by i, we have a (1,1)-form $i^*\omega_{FS}=dd^c\log(1+|\zeta|^2)$ on $\mathbb{C}$, where $\zeta:=w_1/w_0$ and $[w_0:w_1]$ is the homogeneous coordinate system of $\mathbb{P}^1(\mathbb{C})$. The characteristic function of ψ with respect to $i^*\omega_{FS}$ is defined by

$$\hat{T}_\psi(r,\omega_{FS}) = \frac{1}{4}\int_{B_o(r)} g_r(o,x)\Delta_M \log(1+|\psi(x)|^2)dV(x).$$

Clearly,

$$\hat{T}_\psi(r,\omega_{FS}) \le T_\psi(r,\omega_{FS}).$$

We adopt the spherical distance $\|\cdot,\cdot\|$ on $\mathbb{P}^1(\mathbb{C})$; then the proximity function of ψ with respect to $a \in \mathbb{P}^1(\mathbb{C}) = \mathbb{C}\cup\{\infty\}$ is defined by

$$\hat{m}_\psi(r,a) = \int_{S_o(r)} \log\frac{1}{\|\psi(x),a\|} d\pi_o^r(x).$$

Again, set

$$\hat{N}_\psi(r,a) = \frac{\pi^m}{(m-1)!}\int_{\psi^{-1}(a)\cap B_o(r)} g_r(o,x)\alpha^{m-1}.$$

Using similar arguments as in the proof of Theorem 3.4, we easily show that $\hat{T}_\psi(r,\omega_{FS}) = \hat{m}_\psi(r,a) + \hat{N}_\psi(r,a) + O(1)$. We also define Nevanlinna's characteristic function

$$T(r,\psi) := m(r,\psi) + N(r,\psi),$$

where

$$m(r,\psi) = \int_{S_o(r)} \log^+|\psi(x)| d\pi_o^r(x),$$
$$N(r,\psi) = \frac{\pi^m}{(m-1)!}\int_{\psi^{-1}(\infty)\cap B_o(r)} g_r(o,x)\alpha^{m-1}.$$

We have

$$T(r,\psi) = \hat{T}_\psi(r,\omega_{FS}) + O(1), \quad T\Big(r,\frac{1}{\psi-a}\Big) = T(r,\psi) + O(1). \tag{9}$$

On $\mathbb{P}^1(\mathbb{C})$, we take a singular metric

$$\Phi = \frac{1}{|\zeta|^2(1+\log^2|\zeta|)}\frac{\sqrt{-1}}{4\pi^2}d\zeta\wedge d\bar{\zeta}.$$

A direct computation shows that

$$\int_{\mathbb{P}^1(\mathbb{C})}\Phi = 1, \quad 4m\pi\frac{\psi^*\Phi\wedge\alpha^{m-1}}{\alpha^m} = \frac{\|\nabla_M\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}. \tag{10}$$

Set

$$T_\psi(r,\Phi) = \frac{1}{2}\int_{B_o(r)} g_r(o,x)e_{\psi^*\Phi}(x)dV(x).$$

Invoking (10), we obtain

$$T_\psi(r,\Phi)=\frac{1}{4\pi}\int_{B_o(r)}g_r(o,x)\frac{\|\nabla_M\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}(x)dV(x). \tag{11}$$

Lemma 4.2. *We have*

$$T_\psi(r,\Phi)\le T(r,\psi)+O(1).$$

Proof. Using Fubini's theorem,

$$\begin{aligned}T_\psi(r,\Phi)&=m\int_{B_o(r)}g_r(o,x)\frac{\psi^*\Phi\wedge\alpha^{m-1}}{\alpha^m}dV(x)\\&=\frac{\pi^m}{(m-1)!}\int_{\mathbb{P}^1(\mathbb{C})}\Phi(\zeta)\int_{\psi^{-1}(\zeta)\cap B_o(r)}g_r(o,x)\alpha^{m-1}\\&=\int_{\zeta\in\mathbb{P}^1(\mathbb{C})}N_\psi(r,\zeta)\Phi(\zeta)\\&\le\int_{\zeta\in\mathbb{P}^1(\mathbb{C})}\big(T(r,\psi)+O(1)\big)\Phi(\zeta)\\&=T(r,\psi)+O(1).\end{aligned}$$

The proof is completed. □

Lemma 4.3. *Assume that $\psi(x)\not\equiv 0$. For any $\delta>0$, there are $C(o,r,\delta)>0$ independent of ψ and $E_\delta\subset(1,\infty)$ of finite Lebesgue measure such that*

$$\mathbb{E}_o\left[\log^+\frac{\|\nabla_M\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}(X_{\tau_r})\right]\le(1+\delta)^2\log T(r,\psi)+\log C(o,r,\delta)$$

holds for $r>1$ outside E_δ.

Proof. By Jensen's inequality, it is clear that

$$\begin{aligned}\mathbb{E}_o\left[\log^+\frac{\|\nabla_M\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}(X_{\tau_r})\right]&\le\mathbb{E}_o\left[\log\left(1+\frac{\|\nabla_M\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}(X_{\tau_r})\right)\right]\\&\le\log^+\mathbb{E}_o\left[\frac{\|\nabla_M\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}(X_{\tau_r})\right]+O(1).\end{aligned}$$

By Lemma 4.1 and the co-area formula, there is $C(o,r,\delta)>0$ such that

$$\begin{aligned}&\log^+\mathbb{E}_o\left[\frac{\|\nabla_M\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}(X_{\tau_r})\right]\\&\le(1+\delta)^2\log^+\mathbb{E}_o\left[\int_0^{\tau_r}\frac{\|\nabla_M\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}(X_t)dt\right]+\log C(o,r,\delta)\\&\le(1+\delta)^2\log T(r,\psi)+\log C(o,r,\delta)+O(1),\end{aligned}$$

where Lemma 4.2 and (11) are applied. Modify $C(o,r,\delta)$ such that the term $O(1)$ is removed; then we get the desired inequality. □

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Define

$$m\left(r,\frac{\|\nabla_M\psi\|}{|\psi|}\right)=\int_{S_o(r)}\log^+\frac{\|\nabla_M\psi\|}{|\psi|}(x)d\pi_o^r(x).$$

Now we prove Theorem 1.1.

Proof. On the one hand,

$$\begin{aligned}
m\left(r,\frac{\|\nabla_M\psi\|}{|\psi|}\right)&\leq\frac{1}{2}\int_{S_o(r)}\log^+\frac{\|\nabla_M\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}(x)d\pi_o^r(x)\\
&\quad+\frac{1}{2}\int_{S_o(r)}\log^+\left(1+\log^2|\psi(x)|\right)d\pi_o^r(x)\\
&=\frac{1}{2}\mathbb{E}_o\left[\log^+\frac{\|\nabla_M\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}(X_{\tau_r})\right]\\
&\quad+\frac{1}{2}\int_{S_o(r)}\log\left(1+\log^2|\psi(x)|\right)d\pi_o^r(x)\\
&\leq\frac{1}{2}\mathbb{E}_o\left[\log^+\frac{\|\nabla_M\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}(X_{\tau_r})\right]\\
&\quad+\frac{1}{2}\int_{S_o(r)}\log\left(1+\left(\log^+|\psi(x)|+\log^+\frac{1}{|\psi(x)|}\right)^2\right)d\pi_o^r(x)\\
&\leq\frac{1}{2}\mathbb{E}_o\left[\log^+\frac{\|\nabla_M\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}(X_{\tau_r})\right]\\
&\quad+\int_{S_o(r)}\log\left(1+\log^+|\psi(x)|+\log^+\frac{1}{|\psi(x)|}\right)d\pi_o^r(x).
\end{aligned}$$

Lemma 4.3 implies that

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}_o\left[\log^+\frac{\|\nabla_M\psi\|^2}{|\psi|^2(1+\log^2|\psi|)}(X_{\tau_r})\right]\\
&\leq\frac{(1+\delta)^2}{2}\log T(r,\psi)+\frac{1}{2}\log C(o,r,\delta)+O(1).
\end{aligned}$$

On the other hand, by Jensen's inequality and (9),

$$\begin{aligned}
&\int_{S_o(r)}\log\left(1+\log^+|\psi(x)|+\log^+\frac{1}{|\psi(x)|}\right)d\pi_o^r(x)\\
&\leq\log\int_{S_o(r)}\left(1+\log^+|\psi(x)|+\log^+\frac{1}{|\psi(x)|}\right)d\pi_o^r(x)\\
&\leq\log\left(m(r,\psi)+m(r,1/\psi)\right)+O(1)\\
&\leq\log T(r,\psi)+O(1).
\end{aligned}$$

Replacing $C(o,r,\delta)$ by $C^2(o,r,\delta)$ and combining the above, the theorem is proved. □

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4.2. Estimate of $C(o,r,\delta)$

Let M be a complete Kähler manifold of nonpositive sectional curvature. Indeed, we let κ be defined by (1). Clearly, κ is a nonpositive, nonincreasing and continuous function on $[0,\infty)$. Treat the ordinary differential equation

$$G''(t)+\kappa(t)G(t)=0; \quad G(0)=0, \ \ G'(0)=1 \tag{12}$$

on $[0,\infty)$. Now compare (12) with $y''(t)+\kappa(0)y(t)=0$ under the same initial conditions; we see that G can be estimated simply as

$$G(t)=t \ \text{ for } \kappa\equiv 0; \quad G(t)\geq t \ \text{ for } \kappa\not\equiv 0.$$

This follows that

$$G(r)\geq r \ \text{ for } r\geq 0; \quad \int_1^r \frac{dt}{G(t)}\leq \log r \ \text{ for } r\geq 1. \tag{13}$$

On the other hand, we rewrite (12) as the form

$$\log' G(t)\cdot \log' G'(t)=-\kappa(t).$$

Since $G(t)\geq t$ is increasing, the decrease and nonpositivity of κ imply that for each fixed t,G must satisfy one of the following two inequalities:

$$\log' G(t)\leq \sqrt{-\kappa(t)} \ \text{ for } t>0; \quad \log' G'(t)\leq \sqrt{-\kappa(t)} \ \text{ for } t\geq 0.$$

By virtue of $G(t)\to 0$ as $t\to 0$, by integration, G is bounded from above by

$$G(r)\leq r\exp\left(r\sqrt{-\kappa(r)}\right) \ \text{ for } r\geq 0. \tag{14}$$

In what follows, one assumes that M is simply connected. The purpose of this section is to show the following LDL by estimating $C(o,r,\delta)$.

Theorem 4.4 (LDL). *Let ψ be a nonconstant meromorphic function on M. Then*

$$m\left(r,\frac{\|\nabla_M\psi\|}{|\psi|}\right)\leq \left(1+\frac{(1+\delta)^2}{2}\right)\log T(r,\psi)+O\left(r\sqrt{-\kappa(r)}+\delta\log r\right) \ \|,$$

where κ is defined by (1).

We need some lemmas.

Lemma 4.5 ([4]). *Let $\eta>0$ be a number. Then there is a constant $C>0$ such that*

$$g_r(o,x)\int_\eta^r G^{1-2m}(t)dt\geq C\int_{r(x)}^r G^{1-2m}(t)dt$$

holds for $r>\eta$ and $x\in B_o(r)\setminus\overline{B_o(\eta)}$, where G is defined by (12).

Lemma 4.6 ([10, 16]). *Let M be a simply connected, nonpositively curved and complete Hermitian manifold of complex dimension m. Then*

$$(i)\ \ g_r(o,x) \leq \begin{cases} \frac{1}{\pi}\log\frac{r}{r(x)}, & m=1 \\ \frac{1}{(m-1)\omega_{2m-1}}\left(r^{2-2m}(x)-r^{2-2m}\right), & m\geq 2 \end{cases};$$

$$(ii)\quad d\pi_o^r(x) \leq \frac{1}{\omega_{2m-1}r^{2m-1}}d\sigma_r(x),$$

where $g_r(o,x)$ denotes the Green function of $\Delta_M/2$ for $B_o(r)$ with Dirichlet boundary condition and a pole at o, $d\pi_o^r(x)$ is the harmonic measure on $S_o(r)$ with respect to o, ω_{2m-1} is the Euclidean volume of a unit sphere in $\mathbb{R}^{2m}$ and $d\sigma_r(x)$ is the induced Riemannian volume element on $S_o(r)$.

Lemma 4.7 (Borel lemma, [23]). *Let T be a strictly positive nondecreasing function of $\mathscr{C}^1$-class on $(0,\infty)$. Let $\gamma>0$ be a number such that $T(\gamma)\geq e$ and ϕ be a strictly positive nondecreasing function such that*

$$c_\phi=\int_e^\infty \frac{1}{t\phi(t)}dt<\infty.$$

Then, the inequality

$$T'(r)\leq T(r)\phi(T(r))$$

holds for $r\geq\gamma$ outside a set of Lebesgue measure not exceeding c_ϕ. Particularly, take $\phi(T)=T^\delta$ for a number $\delta>0$; then we have $T'(r)\leq T^{1+\delta}(r)$ holds for $r>0$ outside a set $E_\delta\subset(0,\infty)$ of finite Lebesgue measure.

Now we prove the following so-called calculus lemma (see also [4]) which gives an estimate of $C(o,r,\delta)$.

Lemma 4.8 (Calculus lemma). *Let $k\geq 0$ be a locally integrable function on M such that it is locally bounded at $o\in M$. Then for any $\delta>0$, there is a constant $C>0$ independent of k,δ and a set $E_\delta\subset(1,\infty)$ of finite Lebesgue measure such that*

$$\mathbb{E}_o[k(X_{\tau_r})]\leq \frac{C^{(1+\delta)^2}\log^{(1+\delta)^2} r}{r^{(1-2m)\delta}e^{(1-2m)(1+\delta)r\sqrt{-\kappa(r)}}}\left(\mathbb{E}_o\left[\int_0^{\tau_r}k(X_t)dt\right]\right)^{(1+\delta)^2}$$

holds for $r>1$ outside E_δ, where κ is defined by (1).

Proof. By Lemma 4.5 and Lemma 4.6 with (13), we get

$$\begin{aligned}\mathbb{E}_o\left[\int_0^{\tau_r}k(X_t)dt\right] &= \int_{B_o(r)} g_r(o,x)k(x)dV(x) \\ &= \int_0^r dt\int_{S_o(t)} g_r(o,x)k(x)d\sigma_t(x) \\ &\geq C_0\int_0^r \frac{\int_t^r G^{1-2m}(s)ds}{\int_1^r G^{1-2m}(s)ds}dt\int_{S_o(t)}k(x)d\sigma_t(x) \\ &= \frac{C_0}{\log r}\int_0^r dt\int_t^r G^{1-2m}(s)ds\int_{S_o(t)}k(x)d\sigma_t(x)\end{aligned}$$

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and

$$\mathbb{E}_o\big[k(X_{\tau_r})\big] = \int_{S_o(r)} k(x)d\pi_o^r(x) \le \frac{1}{\omega_{2m-1}r^{2m-1}}\int_{S_o(r)} k(x)d\sigma_r(x),$$

where ω_{2m-1} denotes the Euclidean volume of a unit sphere in $\mathbb{R}^{2m}$ and $d\sigma_r$ is the induced volume measure on $S_o(r)$. Hence,

$$\mathbb{E}_o\left[\int_0^{\tau_r} k(X_t)dt\right] \ge \frac{C_0}{\log r}\int_0^r dt \int_t^r G^{1-2m}(s)ds \int_{S_o(t)} k(x)d\sigma_t(x)$$

and

$$\mathbb{E}_o\big[k(X_{\tau_r})\big] \le \frac{1}{\omega_{2m-1}r^{2m-1}}\int_{S_o(r)} k(x)d\sigma_r(x). \tag{15}$$

Put

$$\Gamma(r) = \int_0^r dt \int_t^r G^{1-2m}(s)ds \int_{S_o(t)} k(x)d\sigma_t(x).$$

Then

$$\Gamma(r) \le \frac{\log r}{C_0}\mathbb{E}_o\left[\int_0^{\tau_r} k(X_t)dt\right]. \tag{16}$$

A simple computation shows that

$$\Gamma'(r) = G^{1-2m}(r)\int_0^r dt \int_{S_o(t)} k(x)d\sigma_t(x).$$

By this with (15),

$$\mathbb{E}_o\big[k(X_{\tau_r})\big] \le \frac{1}{\omega_{2m-1}r^{2m-1}}\frac{d}{dr}\left(\frac{\Gamma'(r)}{G^{1-2m}(r)}\right). \tag{17}$$

Using Lemma 4.7 twice, for any $\delta > 0$ we have

$$\frac{d}{dr}\left(\frac{\Gamma'(r)}{G^{1-2m}(r)}\right) \le \frac{\Gamma^{(1+\delta)^2}(r)}{G^{(1-2m)(1+\delta)}(r)} \tag{18}$$

holds outside a set $E_\delta \subset (1,\infty)$ of finite Lebesgue measure. Using (16)–(18) and (14), it is not hard to conclude that

$$\mathbb{E}_o\big[k(X_{\tau_r})\big] \le \frac{C^{(1+\delta)^2}\log^{(1+\delta)^2} r}{r^{(1-2m)\delta}e^{(1-2m)(1+\delta)r\sqrt{-\kappa(r)}}}\left(\mathbb{E}_o\left[\int_0^{\tau_r} k(X_t)dt\right]\right)^{(1+\delta)^2}$$

with $C = 1/C_0 > 0$ being a constant independent of k,δ. □

Lemma 4.8 implies an estimate

$$C(o,r,\delta) \le \frac{C^{(1+\delta)^2}\log^{(1+\delta)^2} r}{r^{(1-2m)\delta}e^{(1-2m)(1+\delta)r\sqrt{-\kappa(r)}}}.$$

Thus, we get

$$\log C(o,r,\delta) \le O\Big(r\sqrt{-\kappa(r)} + \delta\log r\Big). \tag{19}$$

We prove Theorem 4.4.

Proof. Combining Theorem 1.1 with (19), we show the theorem. □

5. Second main theorem

5.1. Meromorphic mappings into $\mathbb{P}^n(\mathbb{C})$

Let $\psi: M \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping from Kähler manifold M into $\mathbb{P}^n(\mathbb{C})$; that is, there is an open covering $\{U_\alpha\}$ of M such that ψ has a local representation $[\psi_0^\alpha : \cdots : \psi_n^\alpha]$ on each U_α, where $\psi_0^\alpha, \cdots, \psi_n^\alpha$ are holomorphic functions on U_α satisfying

$$\mathrm{codim}_{\mathbb{C}}(\psi_0^\alpha = \cdots = \psi_n^\alpha = 0) \geq 2.$$

Let $[w_0 : \cdots : w_n]$ denote the homogeneous coordinate of $\mathbb{P}^n(\mathbb{C})$. Assume that $w_0 \circ \psi \not\equiv 0$. Let $i: \mathbb{C}^n \hookrightarrow \mathbb{P}^n(\mathbb{C})$ be an inclusion given by $(z_1, \cdots, z_n) \mapsto [1 : z_1 : \cdots : z_n]$. Clearly, ω_{FS} induces a (1,1)-form $i^*\omega_{FS} = dd^c \log(|\zeta_0|^2 + |\zeta_1|^2 + \cdots + |\zeta_n|^2)$ on $\mathbb{C}^n$, where $\zeta_j := w_j/w_0$ for $0 \leq j \leq n$. The characteristic function of ψ with respect to $i^*\omega_{FS}$ is well defined by

$$\hat{T}_\psi(r, \omega_{FS}) = \frac{1}{4}\int_{B_o(r)} g_r(o,x)\Delta_M \log\Big(\sum_{j=0}^n |\zeta_j \circ \psi(x)|^2\Big) dV(x).$$

Clearly,

$$\hat{T}_\psi(r, \omega_{FS}) \leq \frac{1}{4}\int_{B_o(r)} g_r(o,x)\Delta_M \log\|\psi(x)\|^2 dV(x) = T_\psi(r, \omega_{FS}).$$

The co-area formula leads to

$$\hat{T}_\psi(r, \omega_{FS}) = \frac{1}{4}\mathbb{E}_o\Big[\int_0^{\tau_r} \Delta_M \log\Big(\sum_{j=0}^n |\zeta_j \circ \psi(X_t)|^2\Big) dt\Big].$$

Note that the pole divisor of $\zeta_j \circ \psi$ is pluripolar. By Dynkin's formula,

$$\hat{T}_\psi(r, \omega_{FS}) = \frac{1}{2}\int_{S_o(r)} \log\Big(\sum_{j=0}^n |\zeta_j \circ \psi(x)|^2\Big) d\pi_o^r(x) - \frac{1}{2}\log\Big(\sum_{j=0}^n |\zeta_j \circ \psi(o)|^2\Big),$$

$$\hat{T}_{\zeta_j \circ \psi}(r, \omega_{FS}) = \frac{1}{2}\int_{S_o(r)} \log\big(1 + |\zeta_j \circ \psi(x)|^2\big) d\pi_o^r(x) - \frac{1}{2}\log\big(1 + |\zeta_j \circ \psi(o)|^2\big).$$

Theorem 5.1. *We have*

$$\max_{1 \leq j \leq n} T(r, \zeta_j \circ \psi) + O(1) \leq \hat{T}_\psi(r, \omega_{FS}) \leq \sum_{j=1}^n T(r, \zeta_j \circ \psi) + O(1).$$

Proof. On the one hand,

$$\begin{aligned}\hat{T}_\psi(r, \omega_{FS}) &\leq \frac{1}{2}\sum_{j=1}^n \Big(\int_{S_o(r)} \log\big(1 + |\zeta_j \circ \psi(x)|^2\big) d\pi_o^r(x) - \log\big(1 + |\zeta_j \circ \psi(o)|^2\big)\Big) + O(1) \\ &= \sum_{j=1}^n T(r, \zeta_j \circ \psi) + O(1).\end{aligned}$$

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On the other hand,

$$
\begin{aligned}
T(r,\zeta_j\circ\psi) &= \hat{T}_{\zeta_j\circ\psi}(r,\omega_{FS})+O(1) \\
&\le \frac{1}{4}\int_{B_o(r)} g_r(o,x)\Delta_M \log\Big(\sum_{j=0}^{n}|\zeta_j\circ\psi(x)|^2\Big)dV(x)+O(1) \\
&= \hat{T}_{\psi}(r,\omega_{FS})+O(1).
\end{aligned}
$$

We conclude the proof. □

Corollary 5.2. *We have*

$$\max_{1\le j\le n} T(r,\zeta_j\circ\psi)\le T_{\psi}(r,\omega_{FS})+O(1).$$

Let V be a complex projective algebraic variety and $\mathbb{C}(V)$ be the field of rational functions defined on V over $\mathbb{C}$. Let $V\hookrightarrow\mathbb{P}^N(\mathbb{C})$ be a holomorphic embedding and H_V be the restriction of the hyperplane line bundle H over $\mathbb{P}^N(\mathbb{C})$ to V. Denote by $[w_0:\cdots:w_N]$ the homogeneous coordinate system of $\mathbb{P}^N(\mathbb{C})$ and assume that $w_0\ne 0$ without loss of generality. Notice that the restriction $\{\zeta_j := w_j/w_0\}$ to V gives a transcendental base of $\mathbb{C}(V)$. Hence, any $\phi\in\mathbb{C}(V)$ can be represented by a rational function in $\zeta_1,\cdots,\zeta_N$,

$$\phi = Q(\zeta_1,\cdots,\zeta_N).$$

Theorem 5.3. *Let $f:M\to V$ be an algebraically nondegenerate meromorphic mapping. Then for $\phi\in\mathbb{C}(V)$, there exists a constant $C>0$ such that*

$$T(r,\phi\circ f)\le CT_f(r,H_V)+O(1).$$

Proof. Assume that $w_0\circ f\not\equiv 0$ without loss of generality. Since Q_j is rational, there is constant $C'>0$ such that $T(r,\phi\circ f)\le C'\sum_{j=1}^{N}T(r,\zeta_j\circ f)+O(1)$. By Corollary 5.2, $T(r,\zeta_j\circ f)\le T_f(r,H_V)+O(1)$. This proves the theorem. □

Corollary 5.4. *Let $f:M\to V$ be an algebraically nondegenerate meromorphic mapping. Fix a positive $(1,1)$-form ω on V. Then for any $\phi\in\mathbb{C}(V)$, there is a constant $C>0$ such that*

$$T(r,\phi\circ f)\le CT_f(r,\omega)+O(1).$$

Proof. The compactness of V and Theorem 5.3 deduce the corollary. □

5.2. Estimate of $\mathbb{E}_o[\tau_r]$

Now we assume M is a simply connected complete Kähler manifold of nonpositive sectional curvature, and let X_t be the Brownian motion in M with generator $\Delta_M/2$ started at o. Recall that $\dim_{\mathbb{C}} M=m, \tau_r=\inf\{t>0: X_t\notin B_o(r)\}$.

Lemma 5.5. *We have*

$$\mathbb{E}_o[\tau_r]\le\frac{2r^2}{2m-1}.$$

Proof. The argument follows essentially from Atsuji [4], but here we provide a simpler proof albeit a rougher estimate. We refer the reader to [4] for a better estimate that $\mathbb{E}_o[\tau_r] \le r^2/2m$. Let X_t be the Brownian motion in M started at $o \ne o_1$, where $o_1 \in B_o(r)$. Let $r_1(x)$ be the distance function of x from o_1. Apply Itô's formula to $r_1(x)$,

$$r_1(X_t) - r_1(X_0) = B_t - L_t + \frac{1}{2}\int_0^t \Delta_M r_1(X_s)ds, \tag{20}$$

where B_t is the standard Brownian motion in $\mathbb{R}$ and L_t is a local time on the cut locus of o, an increasing process which increases only at the cut loci of o. Since M is simply connected and nonpositively curved,

$$\Delta_M r_1(x) \ge \frac{2m-1}{r_1(x)}, \quad L_t \equiv 0.$$

By (20), we arrive at

$$r_1(X_t) \ge B_t + \frac{2m-1}{2}\int_0^t \frac{ds}{r_1(X_s)}.$$

Let $t = \tau_r$ and take expectation on both sides of the above inequality; then it yields that

$$\max_{x \in S_o(r)} r_1(x) \ge \frac{(2m-1)\mathbb{E}_o[\tau_r]}{2\max_{x\in S_o(r)} r_1(x)}.$$

Let $o' \to o$, and we are led to the conclusion. □

5.3. Second main theorem

Let M be a complete Kähler manifold of nonpositive sectional curvature. Consider the (analytic) universal covering

$$\pi : \tilde{M} \to M.$$

Via the pullback by $\pi, \tilde{M}$ can be equipped with the induced metric from the metric of M. So, under this metric, $\tilde{M}$ becomes a simply connected complete Kähler manifold of nonpositive sectional curvature. Take a diffusion process $\tilde{X}_t$ in $\tilde{M}$ such that $X_t = \pi(\tilde{X}_t)$, where X_t is the Brownian motion started at $o \in M$. Then $\tilde{X}_t$ is a Brownian motion generated by $\Delta_{\tilde{M}}/2$ induced from the pullback metric. Let $\tilde{X}_t$ start at $\tilde{o} \in \tilde{M}$ with $o = \pi(\tilde{o})$. Then

$$\mathbb{E}_o[\phi(X_t)] = \mathbb{E}_{\tilde{o}}\left[\phi \circ \pi(\tilde{X}_t)\right]$$

for $\phi \in \mathscr{C}_\flat(M)$. Set

$$\tilde{\tau}_r = \inf\left\{t > 0 : \tilde{X}_t \notin B_{\tilde{o}}(r)\right\},$$

where $B_{\tilde{o}}(r)$ is a geodesic ball centred at $\tilde{o}$ with radius r in $\tilde{M}$. If necessary, one can extend the filtration in probability space where $(X_t, \mathbb{P}_o)$ are defined so that $\tilde{\tau}_r$ is a stopping time with respect to a filtration where the stochastic calculus of X_t works. By the above arguments, we may assume M is simply connected by lifting f to the universal covering.

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Let V be a complex projective algebraic manifold with complex dimension $n \leq m = \dim_{\mathbb{C}} M$, and let $L \to V$ be a holomorphic line bundle. Let a divisor $D \in |L|$ be of simple normal crossing type; then one can express $D = \sum_{j=1}^{q} D_j$ as the union of irreducible components. Equip L_{D_j} with a Hermitian metric which then induces a natural Hermitian metric h on $L = \otimes_{j=1}^{q} L_{D_j}$. Fix a Hermitian metric form ω on V, which gives a (smooth) volume form $\Omega := \omega^n$ on V. Pick $s_j \in H^0(V, L_{D_j})$ with $(s_j) = D_j$ and $\|s_j\| < 1$. On V, one defines a singular volume form

$$\Phi = \frac{\Omega}{\prod_{j=1}^{q} \|s_j\|^2}. \tag{21}$$

Set

$$\xi \alpha^m = f^* \Phi \wedge \alpha^{m-n}.$$

Note that

$$\alpha^m = m! \det(g_{i\bar{j}}) \bigwedge_{j=1}^{m} \frac{\sqrt{-1}}{\pi} dz_j \wedge d\bar{z}_j.$$

A direct computation leads to

$$dd^c \big[\log \xi\big] \geq f^* c_1(L, h) - f^* \mathrm{Ric}\Omega + \mathscr{R}_M - \mathrm{Supp} f^* D$$

in the sense of currents, where $\mathscr{R}_M = -dd^c \log \det(g_{i\bar{j}})$. This follows that

$$\begin{aligned} &\frac{1}{4} \int_{B_o(r)} g_r(o,x) \Delta_M \log \xi(x) dV(x) \\ &\geq T_f(r, L) + T_f(r, K_V) + T(r, \mathscr{R}_M) - \overline{N}_f(r, D) + O(1). \end{aligned} \tag{22}$$

We now prove Theorem 1.2.

Proof. By Ru–Wong's arguments (see [23], pp. 231–233), the simple normal crossing type of D implies that there exists a finite open covering $\{U_\lambda\}$ of V together with rational functions $w_{\lambda 1}, \cdots, w_{\lambda n}$ on V for λ such that $w_{\lambda 1}, \cdots$ are holomorphic on U_λ as well as

$$\begin{aligned} dw_{\lambda 1} \wedge \cdots \wedge dw_{\lambda n}(y) \neq 0, \ {}^\forall y \in U_\lambda, \\ D \cap U_\lambda = \big\{ w_{\lambda 1} \cdots w_{\lambda h_\lambda} = 0 \big\}, \ {}^\exists h_\lambda \leq n. \end{aligned}$$

In addition, we can require $L_{D_j}|_{U_\lambda} \cong U_\lambda \times \mathbb{C}$ for λ, j. On U_λ, we get

$$\Phi = \frac{e_\lambda}{|w_{\lambda 1}|^2 \cdots |w_{\lambda h_\lambda}|^2} \bigwedge_{k=1}^{n} \frac{\sqrt{-1}}{2\pi} dw_{\lambda k} \wedge d\bar{w}_{\lambda k},$$

where Φ is given by (21) and e_λ is a smooth positive function. Let $\{\phi_\lambda\}$ be a partition of unity subordinate to $\{U_\lambda\}$; then $\phi_\lambda e_\lambda$ is bounded on V. Set

$$\Phi_\lambda = \frac{\phi_\lambda e_\lambda}{|w_{\lambda 1}|^2 \cdots |w_{\lambda h_\lambda}|^2} \bigwedge_{k=1}^{n} \frac{\sqrt{-1}}{2\pi} dw_{\lambda k} \wedge d\bar{w}_{\lambda k}.$$

Put $f_{\lambda k} = w_{\lambda k} \circ f$; then on $f^{-1}(U_\lambda)$ we obtain

$$f^*\Phi_\lambda = \frac{\phi_\lambda \circ f \cdot e_\lambda \circ f}{|f_{\lambda 1}|^2 \cdots |f_{\lambda h_\lambda}|^2} \bigwedge_{k=1}^{n} \frac{\sqrt{-1}}{2\pi} df_{\lambda k} \wedge d\bar{f}_{\lambda k}. \tag{23}$$

Set

$$f^*\Phi \wedge \alpha^{m-n} = \xi \alpha^m, \quad f^*\Phi_\lambda \wedge \alpha^{m-n} = \xi_\lambda \alpha^m$$

which arrives at (22). Clearly, we have $\xi = \sum_\lambda \xi_\lambda$. Again, set

$$f^*\omega \wedge \alpha^{m-1} = \varrho \alpha^m \tag{24}$$

which follows that

$$\varrho = \frac{1}{2m} e_{f^*\omega}. \tag{25}$$

For each λ and any $x \in f^{-1}(U_\lambda)$, take a local holomorphic coordinate system z around x. Since $\phi_\lambda \circ f \cdot e_\lambda \circ f$ is bounded, it is not very hard to see from (23) and (24) that ξ_λ is bounded from above by P_λ, where P_λ is a polynomial in

$$\varrho, \quad g^{i\bar{j}} \frac{\partial f_{\lambda k}}{\partial z_i} \overline{\frac{\partial f_{\lambda k}}{\partial z_j}} \Big/ |f_{\lambda k}|^2, \quad 1 \leq i,j \leq m, \ 1 \leq k \leq n.$$

This yields that

$$\log^+ \xi_\lambda \leq O\Big(\log^+ \varrho + \sum_k \log^+ \frac{\|\nabla_M f_{\lambda k}\|}{|f_{\lambda k}|} \Big) + O(1). \tag{26}$$

Thus, we conclude that

$$\log^+ \xi \leq O\Big(\log^+ \varrho + \sum_{k,\lambda} \log^+ \frac{\|\nabla_M f_{\lambda k}\|}{|f_{\lambda k}|} \Big) + O(1) \tag{27}$$

on M. On the one hand,

$$\frac{1}{4} \int_{B_o(r)} g_r(o,x) \Delta_M \log \xi(x) dV(x) = \frac{1}{2} \mathbb{E}_o\big[\log \xi(X_{\tau_r})\big] + O(1)$$

due to the co-area formula and Dynkin's formula. Hence, by (22) we have

$$\begin{aligned} &\frac{1}{2}\mathbb{E}_o\big[\log \xi(X_{\tau_r})\big] \\ &\geq T_f(r,L) + T_f(r,K_V) + T(r,\mathscr{R}_M) - \overline{N}_f(r,D) + O(1). \end{aligned} \tag{28}$$

On the other hand, since $f_{\lambda k}$ is the pullback of rational function $w_{\lambda k}$ on V by f, Corollary 5.4 implies that

$$T(r, f_{\lambda k}) \leq O(T_f(r,\omega)) + O(1). \tag{29}$$

Using (26) and (29) with Theorem 1.1,

$$\begin{aligned} &\frac{1}{2}\mathbb{E}_o\big[\log \xi(X_{\tau_r})\big] \\ &\leq O\Big(\sum_{k,\lambda} \mathbb{E}_o\Big[\log^+ \frac{\|\nabla_M f_{\lambda k}\|}{|f_{\lambda k}|}(X_{\tau_r}) \Big] \Big) + O\big(\mathbb{E}_o\big[\log^+ \varrho(X_{\tau_r})\big]\big) + O(1) \end{aligned}$$

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$$
\begin{aligned}
&\leq O\Big(\sum_{k,\lambda} m\Big(r, \frac{\|\nabla_M f_{\lambda k}\|}{|f_{\lambda k}|}\Big)\Big) + O\big(\log^+ \mathbb{E}_o[\varrho(X_{\tau_r})]\big) + O(1) \\
&\leq O\Big(\sum_{k,\lambda} \log T(r, f_{\lambda k}) + \log C(o,r,\delta)\Big) + O\big(\log^+ \mathbb{E}_o\big[\varrho(X_{\tau_r})\big]\big) \\
&\leq O\big(\log T_f(r,\omega) + \log C(o,r,\delta)\big) + O\big(\log^+ \mathbb{E}_o\big[\varrho(X_{\tau_r})\big]\big).
\end{aligned}
$$

In the meanwhile, Lemma 4.1 and (25) imply

$$
\begin{aligned}
\log^+ \mathbb{E}_o\big[\varrho(X_{\tau_r})\big] &\leq (1+\delta)^2 \log^+ \mathbb{E}_o\left[\int_0^{\tau_r} \varrho(X_t)dt\right] + \log C(o,r,\delta) \\
&= \frac{(1+\delta)^2}{2m} \log^+ \mathbb{E}_o\left[\int_0^{\tau_r} e_{f^*\omega}(X_t)dt\right] + \log C(o,r,\delta) \\
&\leq \frac{(1+\delta)^2}{m} \log T_f(r,\omega) + \log C(o,r,\delta).
\end{aligned}
$$

By this with (28), we prove the theorem. □

We proceed to prove Theorem 1.3.

Lemma 5.6. *Let κ be defined by* (1). *If M is nonpositively curved, then*

$$
T(r,\mathscr{R}_M) \geq m\kappa(r)r^2.
$$

Proof. Lemma 2.2 implies that $0 \geq s_M \geq mR_M$. By the co-area formula,

$$
\begin{aligned}
T(r,\mathscr{R}_M) &= -\frac{1}{4}\mathbb{E}_o\left[\int_0^{\tau_r} \Delta_M \log\det(g_{i\bar{j}}(X_t))dt\right] \\
&= \frac{1}{2}\mathbb{E}_o\left[\int_0^{\tau_r} s_M(X_t)dt\right] \geq \frac{1}{2}m\mathbb{E}_o\left[\int_0^{\tau_r} R_M(X_t)dt\right] \\
&\geq \frac{m(2m-1)}{2}\kappa(r)\mathbb{E}_o[\tau_r].
\end{aligned}
$$

Since $\mathbb{E}_o[\tau_r] \leq 2r^2/(2m-1)$ by Lemma 5.5, we prove the lemma. □

Proof. With the estimate of $C(o,r,\delta)$ given by (19) and estimate of $T(r,\mathscr{R}_M)$ given by Lemma 5.6, Theorem 1.3 follows from Theorem 1.2. □

Corollary 5.7 (Carlson–Griffiths–King, [7, 14]; Noguchi, [19]). *Let $f: \mathbb{C}^m \to V$ be a differentiably nondegenerate meromorphic mapping with* $\dim_{\mathbb{C}} V \leq m$. *Let D be a divisor of simple normal crossing type, where L is a holomorphic line bundle over V. Fix a Hermitian metric ω on V. Then*

$$
T_f(r,L) + T_f(r,K_V) \leq \overline{N}_f(r,D) + O\big(\log T_f(r,\omega) + \delta \log r\big) \ \|.
$$

6. Second main theorem for singular divisors

We extend the second main theorem for divisors of simply normal crossing type to general divisors. Given a hypersurface D of a complex projective algebraic manifold V, let S

denote the set of the points of D at which D has a nonnormal crossing singularity. By Hironaka's resolution of singularities (see [15]), there exists a proper modification

$$\tau : \tilde{V} \to V$$

such that $\tilde{V} \setminus \tilde{S}$ is biholomorphic to $V \setminus S$ under τ and $\tilde{D}$ is only of normal crossing singularities, where $\tilde{S} = \tau^{-1}(S)$ and $\tilde{D} = \tau^{-1}(D)$. Let $\hat{D} = \overline{\tilde{D} \setminus \tilde{S}}$ be the closure of $\tilde{D} \setminus \tilde{S}$ and $\tilde{S}_j$ be the irreducible components of $\tilde{S}$. Put

$$\tau^* D = \hat{D} + \sum p_j \tilde{S}_j = \tilde{D} + \sum (p_j - 1)\tilde{S}_j, \quad R_\tau = \sum q_j \tilde{S}_j, \tag{30}$$

where R_τ is ramification divisor of τ and $p_j, q_j > 0$ are integers. Again, set

$$S^* = \sum \varsigma_j \tilde{S}_j, \quad \varsigma_j = \max\{p_j - q_j - 1, 0\}. \tag{31}$$

We endow L_{S^*} with a Hermitian metric $\|\cdot\|$ and take a holomorphic section σ of L_{S^*} with $\mathrm{Div}\sigma = (\sigma) = S^*$ and $\|\sigma\| < 1$. Let

$$f : M \to V$$

be a meromorphic mapping from a complete Kähler manifold M into V such that $f(M) \not\subset D$. The *proximity function* of f with respect to the singularities of D is defined by

$$m_f\big(r, \mathrm{Sing}(D)\big) = \int_{S_o(r)} \log \frac{1}{\|\sigma \circ \tau^{-1} \circ f(x)\|} d\pi_o^r(x).$$

Let $\tilde{f} : M \to \tilde{V}$ be the lift of f given by $\tau \circ \tilde{f} = f$. Then $\tilde{f}$ is a holomorphic mapping on $M \setminus \tilde{I}$, where $\tilde{I} = I \cup f^{-1}(S)$ with the indeterminacy set I of f. Here we remark that Nevanlinna's functions of $\tilde{f}$ can be defined similarly as in Section 3.1 by the lift of f via τ. For example, given a smooth (1,1)-form ω on V, we have already noted that $g_r(o,x)e_{f^*\omega}$ is integrable on $B_o(r)$. Since τ is biholomorphic restricted to $V \setminus S$, $g_r(o,x)e_{\tilde{f}^*(\tau^*\omega)}$ is integrable on $B_o(r) \setminus f^{-1}(S)$. And because $f^{-1}(S)$ has measure 0 with respect to α^{m-1}, we see that $g_r(o,x)e_{\tilde{f}^*(\tau^*\omega)}$ is integrable on $B_o(r)$ and $\tilde{I}$ does not affect the definition of $T_{\tilde{f}}(r, \tau^*\omega)$. It is easy to verify that

$$m_f\big(r, \mathrm{Sing}(D)\big) = m_{\tilde{f}}(r, S^*) = \sum \varsigma_j m_{\tilde{f}}(r, \tilde{S}_j). \tag{32}$$

Now we prove Theorem 1.5.

Proof. We first suppose that D is the union of smooth hypersurfaces, namely, no irreducible component of $\tilde{D}$ crosses itself. Let E be the union of generic hyperplane sections of V so that the set $A = \tilde{D} \cup E$ has only normal crossing singularities. By (30) with $K_{\tilde{V}} = \tau^* K_V \otimes L_{R_\tau}$, we have

$$K_{\tilde{V}} \otimes L_{\tilde{D}} = \tau^* K_V \otimes \tau^* L_D \otimes \bigotimes L_{\tilde{S}_j}^{\otimes(1 - p_j + q_j)}. \tag{33}$$

Applying Theorem 1.3 to $\tilde{f}$ for divisor A,

$$\begin{aligned} &T_{\tilde{f}}(r, L_A) + T_{\tilde{f}}(r, K_{\tilde{V}}) \\ &\leq \overline{N}_{\tilde{f}}(r, A) + O\big(\log T_{\tilde{f}}(r, \tau^*\omega) - r^2\kappa(r) + \delta \log r\big). \end{aligned}$$

The first main theorem implies that

$$\begin{aligned}T_{\tilde f}(r,L_A) &= m_{\tilde f}(r,A)+N_{\tilde f}(r,A)+O(1)\\ &= m_{\tilde f}(r,\tilde D)+m_{\tilde f}(r,E)+N_{\tilde f}(r,A)+O(1)\\ &\geq m_{\tilde f}(r,\tilde D)+N_{\tilde f}(r,A)+O(1)\\ &= T_{\tilde f}(r,L_{\tilde D})-N_{\tilde f}(r,\tilde D)+N_{\tilde f}(r,A)+O(1),\end{aligned}$$

which leads to

$$T_{\tilde f}(r,L_A)-\overline{N}_{\tilde f}(r,A)\geq T_{\tilde f}(r,L_{\tilde D})-\overline{N}_{\tilde f}(r,\tilde D)+O(1).$$

Note that $T_{\tilde f}(r,\tau^*\omega)=T_f(r,\omega)$ and $\overline{N}_{\tilde f}(r,\tilde D)=\overline{N}_f(r,D)$. By this together with the above, we obtain

$$\begin{aligned}&T_{\tilde f}(r,L_{\tilde D})+T_{\tilde f}(r,K_{\tilde V})\\ &\leq \overline{N}_{\tilde f}(r,\tilde D)+O\big(\log T_f(r,\omega)-r^2\kappa(r)+\delta\log r\big).\end{aligned}\tag{34}$$

It yields from (33) that

$$\begin{aligned}&T_{\tilde f}(r,L_{\tilde D})+T_{\tilde f}(r,K_{\tilde V})\\ &=T_{\tilde f}(r,\tau^*L_D)+T_{\tilde f}(r,\tau^*K_V)+\sum(1-p_j+q_j)T_{\tilde f}(r,L_{\tilde S_j})\\ &=T_f(r,L_D)+T_f(r,K_V)+\sum(1-p_j+q_j)T_{\tilde f}(r,L_{\tilde S_j}).\end{aligned}\tag{35}$$

Since $N_{\tilde f}(r,\tilde S)=0$, it follows from (31) and (32) that

$$\begin{aligned}\sum(1-p_j+q_j)T_{\tilde f}(r,L_{\tilde S_j}) &= \sum(1-p_j+q_j)m_{\tilde f}(r,\tilde S_j)+O(1)\\ &\leq \sum\varsigma_j m_{\tilde f}(r,\tilde S_j)+O(1)\\ &= m_f\big(r,\mathrm{Sing}(D)\big)+O(1).\end{aligned}\tag{36}$$

Combining (34)–(36), we show the theorem.

To prove the general case, according to the above proved, one only needs to verify this claim for an arbitrary hypersurface D of normal crossing type. Note by the arguments in [[25], p. 175] that there is a proper modification $\tau:\tilde V\to V$ such that $\tilde D=\tau^{-1}(D)$ is only the union of a collection of smooth hypersurfaces of normal crossings. Thus, $m_f(r,\mathrm{Sing}(D))=0$. By the special case of this theorem proved, the claim holds for D by using Theorem 1.3. □

Corollary 6.1 (Shiffman, [25]). *Let $f:\mathbb{C}^m\to V$ be a differentiably nondegenerate meromorphic mapping with $\dim_{\mathbb{C}}V\leq m$. Let $D\subset V$ be an ample hypersurface. Then*

$$\begin{aligned}&T_f(r,L_D)+T_f(r,K_V)\\ &\leq \overline{N}_f(r,D)+m_f\big(r,\mathrm{Sing}(D)\big)+O\big(\log T_f(r,L_D)+\delta\log r\big)\ \big\|.\end{aligned}$$

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Corollary 6.2 (Defect relation). *Assume the same conditions as in Theorem* 1.5. *If f satisfies the growth condition*

$$\liminf_{r\to\infty}\frac{r^2\kappa(r)}{T_f(r,\omega)}=0,$$

where κ is defined by (1), *then*

$$\Theta_f(D)\underline{\left[\frac{c_1(L)}{\omega}\right]}\le\overline{\left[\frac{c_1(K_V^*)}{\omega}\right]}+\limsup_{r\to\infty}\frac{m_f\big(r,\mathrm{Sing}(D)\big)}{T_f(r,\omega)}.$$

For further consideration of defect relations, we introduce some additional notations. Let A be a hypersurface of V such that $A\supset S$, where S is a set of nonnormal crossing singularities of D given before. We write

$$\tau^*A=\hat{A}+\sum t_j\tilde{S}_j,\quad \hat{A}=\overline{\tau^{-1}(A)\setminus\tilde{S}}. \tag{37}$$

Set

$$\gamma_{A,D}=\max\frac{\varsigma_j}{t_j} \tag{38}$$

where ς_j are given by (31). Clearly, $0\le\gamma_{A,D}<1$. Note from (37) that

$$m_f(r,A)=m_{\tilde{f}}(r,\tau^*A)\ge\sum t_jm_{\tilde{f}}(r,\tilde{S}_j)+O(1).$$

By (32), we see that

$$m_f\big(r,\mathrm{Sing}(D)\big)\le\gamma_{A,D}\sum t_jm_{\tilde{f}}(r,\tilde{S}_j)\le\gamma_{A,D}m_f(r,A)+O(1). \tag{39}$$

Theorem 6.3. *Let $f:M\to V$ be a differentiably nondegenerate meromorphic mapping with $\dim_{\mathbb{C}}M\ge\dim_{\mathbb{C}}V$. Let $D_1,\cdots,D_q\in|L|$ be hypersurfaces such that any two among them have no common components, where L is a holomorphic line bundle over V. Let $A\subset V$ be a hypersurface containing the nonnormal crossing singularities of $\sum_{j=1}^qD_j$. If f satisfies the growth condition*

$$\liminf_{r\to\infty}\frac{r^2\kappa(r)}{T_f(r,\omega)}=0,$$

where κ is defined by (1), *then*

$$\sum_{j=1}^q\Theta_f(D_j)\underline{\left[\frac{c_1(L)}{\omega}\right]}\le\frac{1}{q}\overline{\left[\frac{c_1(K_V^*)}{\omega}\right]}+\frac{\gamma_{A,D}}{q}\overline{\left[\frac{c_1(L_A)}{\omega}\right]}.$$

Proof. By (39), we get

$$\sum_{j=1}^q\limsup_{r\to\infty}\frac{m_f\big(r,\mathrm{Sing}(D_j)\big)}{T_f(r,\omega)}\le\gamma_{A,D}\overline{\left[\frac{c_1(L_A)}{\omega}\right]}.$$

Note that $L_{D_1+\cdots+D_q}=L^{\otimes q}$. By Theorem 6.2, we show the theorem. □

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Corollary 6.4 (Shiffman, [25]). *Let $f:\mathbb{C}^m\to V$ be a differentiably nondegenerate meromorphic mapping with $\dim_{\mathbb{C}} V\le m$. Let $D_1,\cdots,D_q\in|L|$ be hypersurfaces such that any two among them have no common components, where L is a positive line bundle over V. Let $A\subset V$ be a hypersurface containing the nonnormal crossing singularities of $\sum_{j=1}^q D_j$. Then*

$$\sum_{j=1}^{q}\Theta_f(D_j)\le\frac{1}{q}\overline{\left[\frac{c_1(K_V^*)}{c_1(L)}\right]}+\frac{\gamma_{A,D}}{q}\overline{\left[\frac{c_1(L_A)}{c_1(L)}\right]}.$$

Proof. Replace ω by $c_1(L,h)$ in Theorem 6.3. □

Corollary 6.5. *Let $D\in|L|$ be a hypersurface, where L is a positive line bundle over V. If there is a hypersurface $A\subset V$ containing the nonnormal crossing singularities of D such that*

$$\overline{\left[\frac{c_1(K_V^*)}{c_1(L)}\right]}+\gamma_{A,D}\overline{\left[\frac{c_1(L_A)}{c_1(L)}\right]}<1,$$

then every meromorphic mapping $f:M\to V\setminus D$ with $\dim_{\mathbb{C}} M\ge\dim_{\mathbb{C}} V$ satisfying

$$\liminf_{r\to\infty}\frac{r^2\kappa(r)}{T_f(r,L)}=0$$

is differentiably degenerate, where κ is defined by (1).

Corollary 6.6. *Let $D\subset\mathbb{P}^n(\mathbb{C})$ be a hypersurface of degree d_D. If there is a hypersurface $A\subset\mathbb{P}^n(\mathbb{C})$ of degree d_A containing the nonnormal crossing singularities of D such that $d_A\gamma_{A,D}+n+1<d_D$, then every meromorphic mapping $f:M\to\mathbb{P}^n(\mathbb{C})\setminus D$ with $\dim_{\mathbb{C}} M\ge n$ satisfying*

$$\liminf_{r\to\infty}\frac{r^2\kappa(r)}{T_f(r,L_D)}=0$$

is differentiably degenerate, where κ is defined by (1).

Proof. The conditions imply that

$$\overline{\left[\frac{c_1(K^*_{\mathbb{P}^n(\mathbb{C})})}{c_1([D])}\right]}+\gamma_{A,D}\overline{\left[\frac{c_1([A])}{c_1([D])}\right]}=\frac{n+1}{d_D}+\gamma_{A,D}\frac{d_A}{d_D}<1.$$

By Corollary 6.5, we see that the corollary holds. □

Competing Interests. None.

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