This department welcomes short notes and problems believed to be new. Contributors should include solutions where known, or background material in case the problem is unsolved. Send all communications concerning this department to I. G. Connell, Department of Mathematics, McGill University, Montreal, P. Q.

AN APPLICATION OF RAMSAY'S THEOREM TO A PROBLEM OF ERDÖS AND HAJNAL

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A family $\mathcal{F}$ of sets is said to possess property $\mathcal{B}$ if there exists a set $B \subset \cup \mathcal{F}$ such that $B \cap F \neq \Phi$ and $F \not\subset B$ for each $F \in \mathcal{F}$. In [1], P. Erdős and A. Hajnal ask the following question: Does there exist for every positive integer k a finite family $\mathcal{F}_k$ of finite sets satisfying

(i) $|F| = k$ for each $F \in \mathcal{F}_k$

(ii) $|F \cap G| \leq 1$ for $F, G \in \mathcal{F}_k$, $F \neq G$

(iii) $\mathcal{F}_k$ does not possess property $\mathcal{B}$?

They observed that such families exist for $k = 1, 2, 3$. For $k = 1$, there is no problem. For $k = 2$, one can take $\mathcal{F}_2 = \{(1,2), (1,3), (2,3)\}$ and for $k = 3$ one can take $\mathcal{F}_3 = \{(1,2,3), (1,4,5), (1,6,7), (2,4,6), (2,5,7), (3,4,7), (3,5,6)\}$. It is not difficult to verify that $\mathcal{F}_2$ and $\mathcal{F}_3$ satisfy (i), (ii) and (iii).

The object of this note is to prove that such families exist for every positive integer k. In fact, we shall construct such families explicitly. We make use of a well known theorem of F. P. Ramsay [4] which can be formulated as follows:

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RAMSAY'S THEOREM. To each pair of positive integers k and r with $k \geq r$ there corresponds a positive integer $N(k, r)$, which we take to be minimal, such that if $\ell \geq N(k, r)$ and L is a set of ℓ elements, then the following is true. If $P_r(L)$ (i.e. the set of all subsets of L with r elements) is partitioned in an arbitrary manner into two classes $\mathcal{L}_1$ and $\mathcal{L}_2$, then there exists a subset K of L with k elements such that either $P_r(K) \subset \mathcal{L}_1$ or $P_r(K) \subset \mathcal{L}_2$.

Now we prove

THEOREM 1. Let $\ell \geq N(k, r)$ and let L be a set of ℓ elements. Let K be a subset of L with k elements and let F be the set whose elements are the $\binom{k}{r}$ subsets of K with r elements. Let $\mathcal{F}_{k,r}$ be the family of all possible sets constructed in this way. Then $\mathcal{F}_{k,r}$ does not possess property $\mathcal{B}$.

Proof. Assume that $\mathcal{F}_{k,r}$ possesses property $\mathcal{B}$. Then there exists a set $B \subset \cup \mathcal{F}_{k,r}$ such that $B \cap F \neq \Phi$ and $F \not\subset B$ for each $F \in \mathcal{F}_{k,r}$. Partition $P_r(L)$ into two classes $\mathcal{L}_1$ and $\mathcal{L}_2$ by placing $R \in P_r(L)$ in $\mathcal{L}_1$ if $R \in B$ and in $\mathcal{L}_2$ if $R \notin B$. Then it is not difficult to see that Ramsay's Theorem is contradicted. Thus $\mathcal{F}_{k,r}$ does not possess property $\mathcal{B}$.

The question of Erdős and Hajnal can now be settled by observing that the family $\mathcal{F}_{k,k-1}$ satisfies conditions (i), (ii) and (iii).

If we choose $\ell = N(k, r)$ in Theorem 1, the total number of sets in the family $\mathcal{F}_{k,r}$ is $\binom{N(k,r)}{k}$ and each set has $\binom{k}{r}$ elements. In [2], it is proved that if $\{A_1, A_2, \ldots, A_t\}$ is a family of sets which does not possess property $\mathcal{B}$ and if $|A_i| = n$ for $i = 1, 2, \ldots, t$, then $t > 2^{n-1}$. We must therefore have

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$$\binom{N(k,r)}{k} > 2^{\binom{k}{r} - 1} .$$

This result was obtained by Erdős [3] using a different argument.

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