ANZIAM J. **62** (2020), 302–317
doi:10.1017/S1446181120000243

POINTWISE RESIDUAL METHOD FOR SOLVING PRIMAL AND DUAL ILL-POSED LINEAR PROGRAMMING PROBLEMS WITH APPROXIMATE DATA

A. Y. IVANITSKIY[✉1], V. V. EJOV[2,3] and F. P. VASILYEV[3]

(Received 10 November, 2019; accepted 15 September, 2020; first published online 12 January, 2021)

Abstract

We propose a variation of the pointwise residual method for solving primal and dual ill-posed linear programming with approximate data, sensitive to small perturbations. The method leads to an auxiliary problem, which is also a linear programming problem. Theorems of existence and convergence of approximate solutions are established and optimal estimates of approximation of initial problem solutions are achieved.

2020 *Mathematics subject classification*: 90C05.

Keywords and phrases: ill-posed linear programming problems, approximate data, pointwise residual method, primal and dual linear programming problems.

1. Introduction

The traditional approach to solving linear programming (LP) problems is the following: there is a preset scheme of computation that fixes a solution method, for example, the simplex method, proposed by von Neumann [23], Dantzig [4] and Kantorovich [8] in the early to mid 20th century, or polynomial methods [9, 10, 13, 21], and then the influence of errors that occur in the process of calculation or inscribed in the initial data on the final result is subsequently estimated [14–17, 19]. Such approach is well justified for LP problems with a well-conditioned constraint matrix.

Some articles contain a discussion of linear programming problems with a linear perturbation introduced through a parameter $\varepsilon > 0$. An interesting approach is given by Avrachenkov et al. [3]. Here the authors have constructed a linear programming

[1]Faculty of Applied Mathematics, Physics and Information Technology, Chuvash State University, Cheboksary, Russia; e-mail: ivanitskiy@hotmail.com.
[2]Mathematical Sciences Laboratory at Flinders University of South Australia, College of Science and Engineering, 1284 South Road, Tonsley, SA 5042, Australia; e-mail: vladimir.ejov@flinders.edu.au.
[3]Department of Mathematical Analysis, Faculty of Computational Mathematics and Cybernetics, Moscow State University, Leninskie Gory, 1, Moscow 119234, Russia; e-mail: vasiliev.fp@gmail.ru.

problem, which is independent of ε, and such that its optimal solutions are the so-called *limiting optimal* solutions for the perturbed linear program.

The expansion of computational practice has led to the need to find solutions that are resistant to small changes (perturbations) in the coefficients of the objective function, the matrix and the right-hand side of the constraints for ill-posed LP problems. For such problems, traditional methods are ineffective and lead to solutions that are difficult to interpret [7, 12] (for example, model problems in this article).

The ability to solve unstable LP problems appeared thanks to regularization methods invented in the 1960s. The most common method of these was Tikhonov's regularization method [15–17]. Its main idea is to stabilize the problem by introducing a regularization parameter based on the residual principle, which is to balance the residual value of the regularized solution with the errors in the matrix and the right-hand side of the LP constraints. Application of this regularization method in computational practice to solve unstable LP problems requires, as a rule, iterative solutions of nonlinear optimization problems in order to select the optimal parameter [11, 17]. Tikhonov's regularization method was further developed in [18, 21, 22]; it offered variations that lead to a regularized problem, which is also an auxiliary LP problem.

In this paper, we propose a simple idea for a method (which we call the *pointwise residual method*) to solve unstable LP problems with approximate data, taking into account the pointwise setting of initial data. Such approach was also considered in the literature [6, 7, 21, 22]. This method, first of all, unlike traditional methods of regularization, leads to just a single iteration of solving a regularized problem (also LP) that allows us to use standard computational resources. Secondly, it allows us to obtain approximate solutions that approximate the exact solution of the problem with accurate data, with the same order of accuracy as the accuracy of the initial data, that, in turn, verifies the optimality of our method in terms of the order of the error.

2. Pointwise residual method

Let us consider a fundamental LP problem

$$\varphi(\mathbf{u}) = \langle \mathbf{c}, \mathbf{u} \rangle \to \inf, \quad \mathbf{u} \in U = \{\mathbf{u} \in R^n_+ \mid \mathbf{B}\mathbf{u} \le \mathbf{d}\}, \tag{2.1}$$

where $\langle \mathbf{c}, \mathbf{u} \rangle = \sum_{j=1}^n c_j u_j$ is an objective function, $\mathbf{c} = [c_1, c_2, \ldots, c_n]^T$, $R^n_+ = \{\mathbf{u} = [u_1, u_2, \ldots, u_n]^T \mid u_j \ge 0, j = 1, \ldots, n\}$ is a set of nonnegative vectors,

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b_{p1} & b_{p2} & \ldots & b_{pn} \end{bmatrix} \in R^{p\times n}, \quad \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \ldots \\ d_p \end{bmatrix} \in R^p$$

 https://doi.org/10.1017/S1446181120000243

and $R^{p\times n}$ and R^p are the space of $p \times n$ matrices and the space of p-dimensional vectors, respectively. We assume that

$$U \neq \varnothing, \quad \varphi_* = \inf_{\mathbf{u}\in U} \varphi(\mathbf{u}) > -\infty. \tag{2.2}$$

Then the solution set of the LP problem (2.1) is not empty [19, 21], that is,

$$U_* = \{\mathbf{u} \in U \mid \varphi(\mathbf{u}) = \varphi_*\} \neq \varnothing.$$

Alongside (2.1), consider its dual problem

$$\psi(\mathbf{v}) = -\langle \mathbf{d}, \mathbf{v}\rangle \to \sup, \quad \mathbf{v} \in V = \{\mathbf{v} \in R^p_+ \mid \mathbf{c} + \mathbf{B}^T\mathbf{v} \geq 0\}, \tag{2.3}$$

where $\mathbf{B}^T \in R^{n\times p}$ is a transposed matrix $\mathbf{B}$. Under the conditions (2.2), the solution set of (2.3) is

$$V^* = \{\mathbf{v} \in V \mid \psi(\mathbf{v}) = \psi^*\} \neq \varnothing,$$

where $\psi^* = \sup_{\mathbf{v}\in V} \psi(\mathbf{v}) < \infty$ [15, 23]. LP duality theory yields that $\mathbf{u}_* \in U_*$, $\mathbf{v}^* \in V^*$ if and only if

$$\mathbf{B}\mathbf{u}_* - \mathbf{d} \leq 0, \quad -\mathbf{B}^T\mathbf{v}^* - \mathbf{c} \leq 0, \quad \langle \mathbf{c}, \mathbf{u}_*\rangle + \langle \mathbf{d}, \mathbf{v}^*\rangle \leq 0.$$

Let

$$W = \left\{\mathbf{w} = \begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix} \in R^{n+p}_+ \mid \mathbf{B}\mathbf{u} - \mathbf{d} \leq 0,\ -\mathbf{B}^T\mathbf{v} - \mathbf{c} \leq 0,\ \langle \mathbf{c}, \mathbf{u}\rangle + \langle \mathbf{d}, \mathbf{v}\rangle \leq 0\right\}.$$

We consider the normal solution of a system of inequalities that determine W,

$$f(\mathbf{w}) = \|\mathbf{w}\|_1 \to \inf, \quad \mathbf{w} = \begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix} \in W, \tag{2.4}$$

where $\|\mathbf{w}\|_1 = |u_1| + |u_2| + \cdots + |u_n| + |v_1| + |v_2| + \cdots + |v_p|$ is the octahedral norm of $\mathbf{w}$. In this case $\mathbf{u} \in R^n_+$, $\mathbf{v} \in R^p_+$; therefore, we have $\|\mathbf{w}\|_1 = u_1 + u_2 + \cdots + u_n + v_1 + v_2 + \cdots + v_p$. As has been mentioned above, under the conditions (2.2) the solution set U_* of the problem (2.1) and the solution set V^* of the problem (2.3) are not empty. Therefore, $W \neq \varnothing$ and the function $f(\mathbf{w}) \geq 0$, which is bounded below, attains its infimum on a nonempty set W. Let $f_* = \inf_{\mathbf{w}\in W} f(\mathbf{w})$ and $\{\mathbf{w}^k\}$ be a sequence that minimizes $f(\mathbf{w})$:

$$\lim_{k\to\infty} f(\mathbf{w}^k) = f_*.$$

Then, due to Weierstrass's theorem [20], the following theorem holds.

THEOREM 2.1. *The solution set for the problem (2.1) is not empty, that is,*

$$W_* = \{\mathbf{w} \in W \mid f(\mathbf{w}) = f_*\} \neq \varnothing$$

and the sequence $\{\mathbf{w}^k \in W\}$ converges to some point $\mathbf{w}_* \in W_*$.

Let the data set $\{\mathbf{B}, \mathbf{d}, \mathbf{c}\}$ in (2.1) be replaced by approximate values $\{\widetilde{\mathbf{B}}, \widetilde{\mathbf{d}}, \widetilde{\mathbf{c}}\}$, where

$$\widetilde{\mathbf{B}} = \begin{bmatrix} \widetilde{b}_{11} & \widetilde{b}_{12} & \dots & \widetilde{b}_{1n} \\ \widetilde{b}_{21} & \widetilde{b}_{22} & \dots & \widetilde{b}_{2n} \\ \dots & \dots & \dots & \dots \\ \widetilde{b}_{p1} & \widetilde{b}_{p2} & \dots & \widetilde{b}_{pn} \end{bmatrix} \in R^{p\times n}, \quad \widetilde{\mathbf{d}} = \begin{bmatrix} \widetilde{d}_1 \\ \widetilde{d}_2 \\ \dots \\ \widetilde{d}_p \end{bmatrix} \in R^p, \quad \widetilde{\mathbf{c}} = \begin{bmatrix} \widetilde{c}_1 \\ \widetilde{c}_2 \\ \dots \\ \widetilde{c}_p \end{bmatrix} \in R^n$$

satisfy the conditions

$$|\widetilde{b}_{sj} - b_{sj}| \le \Delta_{sj}, \quad |\widetilde{d}_s - d_s| \le \delta_s, \quad |\widetilde{c}_j - c_j| \le \xi_j, \tag{2.5}$$

where $\Delta_{sj} \ge 0$, $\delta_s \ge 0$, $\xi_j \ge 0$, $s = \overline{1,p}$, $j = 1, \dots, n$, are estimation levels in assigning the data $\{\mathbf{B}, \mathbf{d}, \mathbf{c}\}$. In general, the problem (2.4) with approximate data

$$f(\mathbf{w}) = \|\mathbf{w}\|_1 \to \inf, \quad \mathbf{w} \in \widetilde{W},$$

where

$$\widetilde{W} = \{\mathbf{w} = [\mathbf{u}, \mathbf{v}]^T \in R_+^{n+p} \mid \widetilde{\mathbf{B}}\mathbf{u} - \widetilde{\mathbf{d}} \le 0, -\widetilde{\mathbf{B}}^T\mathbf{v} - \widetilde{\mathbf{c}} \le 0, \langle \widetilde{\mathbf{c}}, \mathbf{u}\rangle + \langle \widetilde{\mathbf{d}}, \mathbf{v}\rangle \le 0\},$$

can be insoluble. Also, in case it is soluble, its solution may be unstable (see example in Section 4). Therefore, one may apply methods for solving such problems [6, 7, 12, 17, 20–22]. One of such regularization methods is the so-called "pointwise residual method".

Let

$$\Delta = \begin{bmatrix} \Delta_{11} & \Delta_{12} & \dots & \Delta_{1n} \\ \Delta_{21} & \Delta_{22} & \dots & \Delta_{2n} \\ \dots & \dots & \dots & \dots \\ \Delta_{p1} & \Delta_{p2} & \dots & \Delta_{pn} \end{bmatrix}, \quad \delta = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \dots \\ \delta_p \end{bmatrix}, \quad \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \dots \\ \xi_n \end{bmatrix};$$

$\sigma = \{\Delta, \delta, \xi\}$ are error margins in assigning initial data $\{\widetilde{\mathbf{B}}, \widetilde{\mathbf{d}}$ in (2.5). Consider a set

$$W(\sigma) = \{\mathbf{w} = [\mathbf{u}, \mathbf{v}]^T \in R_+^{n+p} \mid \widetilde{\mathbf{B}}\mathbf{u} - \widetilde{\mathbf{d}} \le \Delta\mathbf{u} + \delta, -\widetilde{\mathbf{B}}^T\mathbf{v} - \widetilde{\mathbf{c}} \le \Delta^T\mathbf{v} + \xi,$$
$$\langle \widetilde{\mathbf{c}}, \mathbf{u}\rangle + \langle \widetilde{\mathbf{d}}, \mathbf{v}\rangle \le \langle \xi, \mathbf{u}\rangle + \langle \delta, \mathbf{v}\rangle\}$$

or

$$W(\sigma) = \{\mathbf{w} = [\mathbf{u}, \mathbf{v}]^T \in R_+^{n+p} \mid (\widetilde{\mathbf{B}} - \Delta)\mathbf{u} \le \widetilde{\mathbf{d}} + \delta, -(\widetilde{\mathbf{B}} + \Delta)^T\mathbf{v} \le \widetilde{\mathbf{c}} + \xi,$$
$$\langle \widetilde{\mathbf{c}} - \xi, \mathbf{u}\rangle + \langle \widetilde{\mathbf{d}} - \delta, \mathbf{v}\rangle \le 0\}.$$

The inequalities of the set $W(\sigma)$ are linking residual vectors $(\widetilde{\mathbf{B}}\mathbf{u} - \widetilde{\mathbf{d}})$ and $(-\widetilde{\mathbf{B}}^T\mathbf{v} - \widetilde{\mathbf{c}})$ with the errors Δ, δ, ξ. It is easy to verify that $W \subseteq W(\sigma)$; therefore, the set $W(\sigma)$ is a special extension of the set W. We should note that $W(\sigma) \ne \varnothing$ because $W \ne \varnothing$.

Let us consider the problem

$$f(\mathbf{w}) = \|\mathbf{w}\|_1 \to \inf, \quad \mathbf{w} \in W(\sigma) \tag{2.6}$$

and denote $f_*(\sigma) = \inf_{\mathbf{w}\in W(\sigma)} f(\mathbf{w})$. Analogously to Theorem 2.1, the following theorem is valid.

THEOREM 2.2. *The solution set for the problem* (2.6)

$$W_*(\sigma) = \{\mathbf{w} \in W(\sigma) : f(\mathbf{w}) = f_*(\sigma)\} \neq \emptyset$$

is nonempty and every minimizing sequence $\{\mathbf{w}^k\} \in W(\sigma)$ *for the function* $f(\mathbf{w})$ *converges to some point* $\mathbf{w}_*(\sigma) \in W_*(\sigma)$ *for* $k \to \infty$.

For the purpose of finding numerical solutions, as it is not necessary to solve the problem exactly, it is sufficient to determine the vector $\mathbf{w}(\sigma, \varepsilon) \in W(\sigma)$ through the conditions

$$f(\mathbf{w}) \le f_*(\sigma) + \varepsilon, \quad \varepsilon \ge 0. \tag{2.7}$$

We denote vectors satisfying the property (2.7) by $W_*(\sigma, \varepsilon) \subseteq W(\sigma)$. Now we show that the vectors in $W_*(\sigma, \varepsilon)$ can be taken as approximate solutions of (2.4).

3. Estimation of the speed of convergence

Let

$$\beta(W_*(\sigma, \varepsilon), W_*) = \sup_{\mathbf{w}(\sigma)\in W_*(\sigma,\varepsilon)} \inf_{\mathbf{w}\in W_*} \|\mathbf{w}(\sigma) - \mathbf{w}\|$$

be the β-distance between nonempty sets $W_*(\sigma, \varepsilon)$ and W_*.

THEOREM 3.1. *Under the constraints* (2.2) *and* (2.5)*, we have* $\beta(W_*(\sigma, \varepsilon), W_*) \to 0$ *if* $\sigma = \{\Delta, \delta, \xi\} \to 0$ *and* $\varepsilon \to 0$.

PROOF. Take arbitrary sequences

$$\{\sigma^k\} = \{\Delta^k, \delta^k, \xi^k\} \to 0, \quad \varepsilon^k \to 0, \quad k \to \infty.$$

By the definition of exact upper bound, there exists a sequence $\{\mathbf{w}^k\} \in W_*(\sigma^k, \varepsilon^k)$ such that

$$\inf_{\mathbf{w}\in W_*} \|\mathbf{w}^k - \mathbf{w}\| \ge \beta(W_*(\sigma^k, \varepsilon^k), W_*) - \frac{1}{k}, \quad k = 1, 2, \ldots. \tag{3.1}$$

As $W \subseteq W(\sigma)$, then $f_*(\sigma) \le f_*$ and, for all $\{\mathbf{w}^k\} \in W_*(\sigma^k, \varepsilon^k) \subseteq W(\sigma)$ from (2.7),

$$f(\mathbf{w}^k) = \|\mathbf{w}^k\|_1 \le f_*(\sigma) + \varepsilon^k \le f_* + \varepsilon^k. \tag{3.2}$$

Therefore, the sequence $\{\mathbf{w}^k\}$ is bounded and so one can extract a convergent subsequence out of it. Without loss of generality, we assume that the sequence is itself convergent: $\{\mathbf{w}^k\} \to \overline{\mathbf{w}}_* = [\overline{\mathbf{u}}_*, \overline{\mathbf{v}}^*]^T$ for $k \to \infty$. Considering that $\mathbf{u}^k \in R^n_+$ for all $\mathbf{w}^k = [\mathbf{u}^k, \mathbf{v}^k]^T \in W_*(\sigma^k, \varepsilon^k)$, $\mathbf{u}^k \in R^n_+$, $\mathbf{v}^k \in R^p_+$,

$$\mathbf{B}\mathbf{u}^k - \mathbf{d} \le 2(\Delta^k \mathbf{u}^k + \delta^k). \tag{3.3}$$

Component-wise,

$$(\mathbf{B}\mathbf{u}^k - \mathbf{d})_s \le 2((\Delta^k \mathbf{u}^k)_s + \delta_s^k) \le 2[\Delta_s^k(u_1^k + u_2^k + \cdots + u_n^k) + \delta_s^k]$$
$$\le 2(\Delta_s^k \|\mathbf{u}^k\|_1 + \delta_s^k) \le 2(\Delta_s^k \|\mathbf{w}^k\|_1 + \delta_s^k), \quad \Delta_s^k = \max_{1\le j\le n} \Delta_{sj}^k.$$

From (3.2),

$$(\mathbf{B}\mathbf{u}^k - \mathbf{d})_s \le 2[\widehat{\Delta}^k(f_* + \varepsilon^k) + \widehat{\delta}^k], \tag{3.4}$$
$$s = \overline{1,p}, \quad \widehat{\Delta}^k = \max_{1\le s\le p} \Delta_s^k, \quad \widehat{\delta}^k = \max_{1\le s\le p} \delta_s^k.$$

Analogously,

$$(-\mathbf{B}^T\mathbf{v}^k - \mathbf{c})_j \le 2[\widehat{\Delta}^k(f_* + \varepsilon^k) + \widehat{\xi}^k], \quad j = 1, \ldots, n, \tag{3.5}$$

$$\langle \mathbf{c}, \mathbf{u}^k\rangle + \langle \mathbf{d}, \mathbf{v}^k\rangle \le 2(\widehat{\xi}^k \|\mathbf{u}^k\|_1 + \widehat{\delta}^k \|\mathbf{v}^k\|_1) \le 2M(\widehat{\delta}^k, \widehat{\xi}^k)(\|\mathbf{u}^k\|_1 + \|\mathbf{v}^k\|_1)$$
$$\le 2M(\widehat{\delta}^k, \widehat{\xi}^k)\|\mathbf{w}\|_1 \le 2M(\widehat{\delta}^k, \widehat{\xi}^k)(f_* + \varepsilon^k), \tag{3.6}$$

where $\widehat{\xi}^k = \max_{1\le j\le n} \xi_j^k$, $M(\widehat{\delta}^k, \widehat{\xi}^k) = \max\{\widehat{\delta}^k, \widehat{\xi}^k\}$. Taking the limit in (3.4), (3.5) and (3.6) for $k \to \infty$ or $\{\widehat{\sigma}^k\} = \{\widehat{\Delta}^k, \widehat{\delta}^k, \widehat{\xi}^k\} \to 0$, $\varepsilon^k \to 0$,

$$(\mathbf{B}\overline{\mathbf{u}}_* - \mathbf{d})_s \le 0, \quad (-\mathbf{B}^T\overline{\mathbf{v}}_* - \mathbf{c})_j \le 0, \quad \langle \mathbf{c}, \overline{\mathbf{u}}_*\rangle + \langle \mathbf{d}, \overline{\mathbf{v}}_*\rangle \le 0,$$

$$s = \overline{1,p}, \quad j = 1, \ldots, n,$$

or

$$\mathbf{B}\overline{\mathbf{u}}_* - \mathbf{d} \le 0, \quad -\mathbf{B}^T\mathbf{v}^* - \mathbf{c} \le 0, \quad \langle \mathbf{c}, \overline{\mathbf{u}}_*\rangle + \langle \mathbf{d}, \overline{\mathbf{v}}_*\rangle \le 0,$$

that is, $\overline{\mathbf{w}}_* = [\overline{\mathbf{u}}_*, \overline{\mathbf{v}}_*]^T \in W$; thus, (3.2) yields $\|\overline{\mathbf{w}}_*\| \le f_*$. Therefore, $\overline{\mathbf{w}}_* = [\overline{\mathbf{u}}_*, \overline{\mathbf{v}}_*]^T$ is the solution of (2.4), that is, $\overline{\mathbf{w}}_* \in W_*$. Taking the limit in (3.1) for $k \to \infty$ implies that $\beta(W_*(\sigma^k, \varepsilon^k), W_*) \to 0$. □

Let us now estimate the approximation order of vectors $\mathbf{w}_* \in W_*$ by vectors $\mathbf{u}(\sigma, \varepsilon) \in W_*(\sigma, \varepsilon)$. Denote

$$\widehat{\Delta} = \max_{1\le s\le p} \max_{1\le j\le n} \Delta_{sj}, \quad \widehat{\delta} = \max_{1\le s\le p} \delta_s = \|\delta\|_\infty, \quad \widehat{\xi} = \max_{1\le j\le n} \xi_j = \|\xi\|_\infty$$

and $\rho(\mathbf{x}, Y) = \inf_{\mathbf{y}\in Y} \|\mathbf{x} - \mathbf{y}\|$ the distance between the vector $\mathbf{x} \in \mathbf{R}^n$ and the set $Y \subseteq \mathbf{R}^n$, where

$$\|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{j=1}^n |x_i - y_i|^2}.$$

THEOREM 3.2. *Let $U \ne \emptyset$, $\varphi_* > -\infty$ and the conditions (2.5) be satisfied. Then, for sufficiently small $\widehat{\Delta}$, $\widehat{\delta}$, $\widehat{\xi}$ and ε, the following estimate holds:*

$$\sup_{\mathbf{w}\in W_*(\sigma,\varepsilon)} \rho(\mathbf{w}, W_*) = O(\widehat{\Delta}, \widehat{\delta}, \widehat{\xi}, \varepsilon).$$

PROOF. The solution set W_* of (2.4) can be interpreted as a polyhedron

$$W_* = \{\mathbf{w} = [\mathbf{u}, \mathbf{v}]^T \geq 0 \mid \|\mathbf{w}\|_1 \leq f_*, \mathbf{B}\mathbf{u} \leq \mathbf{d}, -\mathbf{B}^T\mathbf{v} \leq \mathbf{c}, \langle \mathbf{c}, \mathbf{u}\rangle + \langle \mathbf{d}, \mathbf{v}\rangle \leq 0\}.$$

Then, according to Vasilyev and Ivanitskiy [21, Theorem 2.5.1] (see also the article by Hoffman [5]), there exists a constant $M > 0$, depending just on the matrix $\mathbf{B}$ and vectors $\mathbf{d}$, $\mathbf{c}$, such that for all $\mathbf{w} = [\mathbf{u}, \mathbf{v}]^T \in R^{n+p}$ and the inequality

$$\rho(\mathbf{w}, W_*) \leq M \max\{(\|\mathbf{w}\|_1 - f_*)^+, \|(\mathbf{B}\mathbf{u} - \mathbf{d})^+\|_\infty\}, \|(-\mathbf{B}^T\mathbf{v} - \mathbf{c})^+\|_\infty, (\langle \mathbf{c}, \mathbf{u}\rangle + \langle \mathbf{d}, \mathbf{v}\rangle)^+\}, \tag{3.7}$$

$$(\|\mathbf{w}\|_1 - f_*)^+ = \max\{0; \|\mathbf{w}\|_1 - f_*\},$$

$$\|(\mathbf{B}\mathbf{u} - \mathbf{d})^+\|_\infty = \max_{1\leq s\leq p}(\max\{0; (\mathbf{B}\mathbf{u} - \mathbf{d})_s\}),$$

$$\|(-\mathbf{B}^T\mathbf{v} - \mathbf{c})^+\|_\infty = \max_{1\leq j\leq n}(\max\{0; (-\mathbf{B}^T\mathbf{v} - \mathbf{c})_j\}),$$

$$(\langle \mathbf{c}, \mathbf{u}\rangle + \langle \mathbf{d}, \mathbf{v}\rangle)^+ = \max\{0; \langle \mathbf{c}, \mathbf{u}\rangle + \langle \mathbf{d}, \mathbf{v}\rangle\}.$$

In particular, this is true for $\mathbf{w}(\sigma, \varepsilon) \in W_*(\sigma, \varepsilon) \subseteq R_+^{n+p}$. Consider the inequality (3.7) for $\mathbf{w} = \mathbf{w}(\sigma, \varepsilon) = [\mathbf{u}(\sigma, \varepsilon), \mathbf{v}(\sigma, \varepsilon)]^T \in W_*(\sigma, \varepsilon)$. As $W \subseteq W(\sigma)$, then $\|\mathbf{w}(\sigma, \varepsilon)\|_1 \leq f_*(\sigma) + \varepsilon \leq f_* + \varepsilon$ and $\|\mathbf{w}(\sigma, \varepsilon)\|_1 - f_* \leq \varepsilon$. Using the same arguments as in the proof of (3.4),

$$(\mathbf{B}\mathbf{u}(\sigma, \varepsilon) - \mathbf{d})_s \leq 2(\widehat{\Delta}(f_* + \varepsilon) + \widehat{\delta}), \quad s = \overline{1, p}.$$

From these inequalities,

$$(\mathbf{B}\mathbf{u}(\delta, \varepsilon) - \mathbf{d}_s)_s^+ = \max\{0; (\mathbf{B}\mathbf{u}(\delta, \varepsilon) - \mathbf{d})_s\} \leq 2(\widehat{\Delta}(f_* + \varepsilon) + \widehat{\delta}), \quad s = \overline{1, p},$$

and, therefore,

$$\begin{aligned}\|(\mathbf{B}\mathbf{u}(\sigma, \varepsilon) - \mathbf{d})^+\|_\infty &= \max_{1\leq s\leq p}(\mathbf{B}\mathbf{u}(\sigma, \varepsilon) - \mathbf{d})_s^+ \\ &\leq 2(\widehat{\Delta}\|\mathbf{u}(\sigma, \varepsilon)\|_1 + \|\delta\|_\infty) \leq 2[\widehat{\Delta}(f_* + \varepsilon) + \widehat{\delta}].\end{aligned}$$

Analogously,

$$\|(-\mathbf{B}^T\mathbf{v}(\sigma, \varepsilon) - \mathbf{c})^+\|_\infty \leq 2(\widehat{\Delta}(f_* + \varepsilon) + \widehat{\xi})$$

and

$$\langle \mathbf{c}, \mathbf{u}(\sigma, \varepsilon)\rangle + \langle \mathbf{d}, \mathbf{v}(\sigma, \varepsilon)\rangle \leq 2M(\widehat{\delta}, \widehat{\xi})(f_* + \varepsilon),$$

where $M(\widehat{\delta}, \widehat{\xi}) = \max\{\widehat{\delta}, \widehat{\xi}\}$. Then

$$\begin{aligned}\rho(\mathbf{w}, W_*) &\leq M \max\{\varepsilon; 2(\widehat{\Delta}(f_* + \varepsilon) + \widehat{\delta}); 2(\widehat{\Delta}(f_* + \varepsilon) + \widehat{\xi}); 2M(\delta, \xi)(f_* + \varepsilon)\} \\ &\leq M\{\varepsilon + 2(\widehat{\Delta}(f_* + \varepsilon) + \widehat{\delta}) + 2(\widehat{\Delta}(f_* + \varepsilon) + \widehat{\xi}) + 2M(\delta, \xi)(f_* + \varepsilon)\} \\ &\leq M\{(4\widehat{\Delta} + 2M(\delta, \xi))(f_* + \varepsilon) + 2(\widehat{\delta} + \widehat{\xi}) + \varepsilon\} \quad \text{for all } \mathbf{w} \in W_*(\sigma, \varepsilon).\end{aligned}$$

□

As follows from the last estimate, one can determine solutions of (2.4) with approximate data with the same order of approximation as the order of approximation of the initial given data set. Auxiliary problem (2.6) that determines the pointwise residual method is also an LP problem as in the original problem (2.4), for which quite effective methods have been developed [21].

4. Model problem 1

Consider the fundamental LP problem: determine a vector $\mathbf{u} = [\mathbf{u}_1, \mathbf{u}_2]^T$ by the conditions

$$\varphi(u) = u_1 + u_2 \to \inf, \quad \mathbf{u} \in \mathbf{U}, \tag{4.1}$$

where $\mathbf{U}$ is determined by the inequalities

$$\begin{cases} u_1 + 2u_2 \le 6, \\ -\sqrt{5}u_1 - \sqrt{20}u_2 \le -\sqrt{180}, \\ u_1 \ge 0, \ u_2 \ge 0. \end{cases} \tag{4.2}$$

The second inequality after division by $-\sqrt{5}$ takes the form $u_1 + 2u_2 \ge 6$. Thus, the problem can be reformulated as follows: find the vector $\mathbf{u} = [u_1, u_2]^T$ from the conditions

$$u_1 + u_2 \to \inf,$$

$$\begin{cases} u_1 + 2u_2 = 6, \\ u_1 \ge 0, \ u_2 \ge 0. \end{cases}$$

The solution of this problem easily follows from geometric arguments (see Figure 1):

$$\mathbf{u}_* = [0; 3]^T, \quad \varphi_* = \inf_{\mathbf{u} \in U} \varphi(\mathbf{u}) = 3.$$

Let k be the number of meaningful digits in the fractional part of $\sqrt{5} = 2.23606797\ldots$, $\sqrt{20} = 4.47213595\ldots$, $\sqrt{180} = 13.41640786\ldots$. The rounding error does not exceed $\Delta = 0.5 \cdot 10^{-k}$. Using $\sqrt{5}$, $\sqrt{20}$, $\sqrt{180}$ helps to make imitation relations (2.5). Table 1 contains approximate solutions of (4.1) by the simplex method in Maple 2015 with rounding k digits in the fractional-part precision.

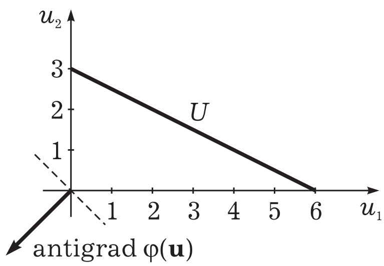

FIGURE 1. $u_* = [0; 3]^T$, $\varphi_* = 3$.

TABLE 1. Solution of problem (4.1) by simplex method for various values of k.

$u(k)$	Exact solution	k: 0	1	2	3	4	5	...
$u_1(k)$	0	-	0.0	2.00	0.000	2.0000	0.00000	...
$u_2(k)$	3	-	3.0	2.00	3.000	2.0000	3.00000	...

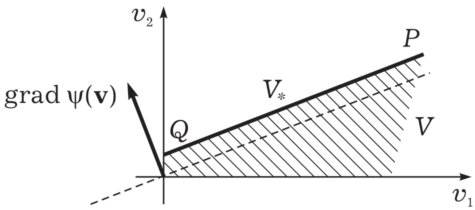

FIGURE 2. $V^* = QP$, $\psi^* = 3$.

Rounding of irrational numbers $\sqrt{5}$, $\sqrt{20}$, $\sqrt{180}$ up to k significant digits in the fractional part one can interpret as a perturbation in the construction (4.1). Slightly different perturbations for various k lead to LP problems that either do not have solutions for $k = 0$ (also for $k = 9$, 18, 19 if you continue to calculate) or for $k = 1, 2, 3, 4, 5$ those problems may be unstable, as evident from Table 1.

The dual to the problem (4.1) has the form: find vectors $\mathbf{v} = [v_1, v_2]^T$ such that

$$\psi(\mathbf{v}) = -6v_1 + \sqrt{180}v_2 \to \sup, \quad \mathbf{v} \in V, \tag{4.3}$$

where V is determined by the inequalities

$$\begin{cases} v_1 - \sqrt{5}v_2 \geq -1, \\ 2v_1 - \sqrt{20}v_2 \geq -1, \\ v_1 \geq 0, \ v_2 \geq 0. \end{cases}$$

Figure 2 shows that the positive axis $V^* = QP = \{\mathbf{v} \mid \mathbf{v} = [-0.5 + \sqrt{5}\alpha, \alpha]^T, \ \alpha \in [1/\sqrt{20}; \infty)\}$ is a solution set of the dual problem (4.3).

According to Vasilyev and Ivanitskiy [21, Theorem 4.2.3], the direct problem (4.1) does not have a stable solution as the solution set V^* of the dual LP is not bounded. This is confirmed by Table 1.

Note that normal solutions for the problem (4.3)

$$\|\mathbf{v}\|_1 = v_1 + v_2 \to \inf, \quad \mathbf{v} \in V^* = \{\mathbf{v} \mid \|\mathbf{v}\|_1 = \psi^*\}$$

have the form

$$\mathbf{v}^* = \left[0; \frac{1}{\sqrt{20}}\right]^T = [0; 0.2236066797\ldots].$$

TABLE 2. Solution of problem (4.4) by the method of pointwise discrepancy.

k	Δ_{sj}, δ_j	$w_1(k)$	$w_2(k)$	$w_3(k)$	$w_4(k)$
0	0.5	0.000000	2.777778	0.000000	0.230771
1	0.05	0.000000	2.934077	0.000000	0.207243
2	0.005	0.000000	2.997777	0.000000	0.222182
3	0.0005	0.000000	2.999555	0.000000	0.223432
4	0.00005	0.000000	2.999988	0.000000	0.223599
5	0.000005	0.000000	2.999999	0.00000	0.223611

TABLE 3. The order of approximation.

k	0	1	2	3	4	5
$\lvert u_{*1} - w_1(k)\rvert$	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
$\lvert u_{*2} - w_2(k)\rvert$	0.222222	0.065923	0.002223	0.000445	0.000012	0.000001
$\lvert v_1^* - w_3(k)\rvert$	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
$\lvert v_2^* - w_4(k)\rvert$	0.000000	0.016364	0.001425	0.000175	0.000008	0.000005

Instead of the primal problems (4.1) and (4.3) (dual to (4.1)), consider the analogue of the problem (2.4): find vectors $\mathbf{w} = [w_1, w_2, w_3, w_4]^T$ from the conditions

$$f(\mathbf{w}) = \|\mathbf{w}\|_1 = w_1 + w_2 + w_3 + w_4 \to \inf, \quad \mathbf{w} \in W, \tag{4.4}$$

where W is determined by the inequalities

$$\begin{cases} w_1 + 2w_2 \le 6, \\ -\sqrt{5}w_1 - \sqrt{20}w_2 \le -\sqrt{180}, \\ -w_3 + \sqrt{5}w_4 \le 1, \\ -2w_3 + \sqrt{20}w_4 \le 1, \\ w_1 + w_2 + 6w_3 - \sqrt{180}w_4 \le 0, \\ w_i \ge 0, \ i = \overline{1,4}. \end{cases}$$

The exact solution of this problem is

$$\mathbf{w}^* = \left[0; 3; 0; \frac{1}{\sqrt{20}}\right]^T = [0; 3; 0; 0.2236066797\ldots]^T, \quad f_* = 3 + \frac{1}{\sqrt{20}}.$$

Let us now solve it by means of the pointwise residual method (2.6) for various k by Maple 2015. Computational results are recorded in Table 2 with the precision of six digits in the fractional part. The order of approximation of the exact solution $\mathbf{u}_* = [0; 3]^T$ of the primal problem (4.1) and the normal solution $\mathbf{v}^* = [0; 1/\sqrt{20}]^T = [0; 2236066797\ldots]^T$ of the dual problem (4.2)) is listed in Table 3. One can see from Tables 2 and 3 that $w_1(k) \to u_{*1}$, $w_2(k) \to u_{*2}$, $w_3(k) \to v_1^*$, $w_4(k) \to v_2^*$ for increasing values of k.

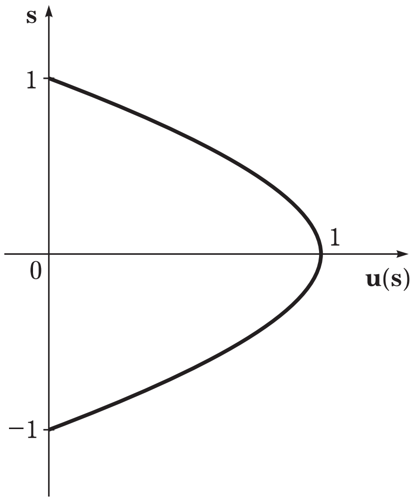

FIGURE 3. $\overline{\mathbf{u}}(\mathbf{s}) = 1 - \mathbf{s}^2$.

On a simple example in model problem 1 we demonstrate the problem of instability and how the pointwise residual method allows us to attain stable approximation. In our example the approximation error for the normal solution of the primal problem (4.1) and the dual problem (4.3) by their approximate solutions has the same order as the order of approximation of the initial data.

Next, we consider model problem 2 that corresponds to the strongly unstable problem [11] in the sense that the condition number of the matrix of a system of equations that approximate an integral equation is high.

5. Model problem 2

Find the solution $u(s)$ of the integral Fredholm equation of the first kind

$$\int_a^b K(x,s)u(s)\,ds = f(x), \qquad c \le x \le d, \tag{5.1}$$

where

$$K(x,s) = [1 + (x-s)^2]^{-1}, \quad [c;d] = [-2;2], \quad [a;b] = [-1;1],$$

$$f(x) = (2 - x^2)[\arctan(1-x) + \arctan(1+x)] - 2 - x\ln\frac{1+(1-x)^2}{1+(1+x)^2}.$$

In Figure 3, the exact solution of (5.1) is $\overline{\mathbf{u}}(\mathbf{s}) = 1 - \mathbf{s}^2$. Approximate (5.1) on uniform lattices $x_{i+1} = x_i + h_i$, $x_1 = -2$, $i = 0, \ldots, 40$, $h_i = 0.1$ and $s_{j+1} = s_j + h_j$, $s_1 = -1$, $j = 0, \ldots, 40$, $h_j = 0.05$, using the Simpson formula [1] for the evaluation of definite integrals. In the end, we obtain a system of linear algebraic equations

$$\widetilde{\mathbf{A}}\mathbf{u} = \widetilde{\mathbf{f}}, \tag{5.2}$$

where $\widetilde{\mathbf{A}} = \mathbf{K}\cdot\mathbf{N}$, $\mathbf{K} = \{k_{ij}\} = \mathbf{K}(x_i, s_j) \in \mathbf{R}^{41\times 41}$, $\mathbf{N} \in \mathbf{R}^{41\times 41}$ is a diagonal matrix corresponding to the integral formula with the diagonal elements N_j or, in detail,

$$\widetilde{\mathbf{A}} = \{\widetilde{a}_{ij}\} \in \mathbf{R}^{41\times 41}, \quad \widetilde{a}_{ij} = \frac{N_j h_i}{1 + (x_i - s_j)^2}, \quad \widetilde{\mathbf{f}} = [f(x_0), f(x_2), \ldots, f(x_{40})]^T, \tag{5.3}$$

where N_j are the integral coefficients of the Simpson formula.

Let us seek the solution (5.1) in the class of nonnegative ($u(s) \geq 0$), monotonic ($u'(s) \geq 0$, $s \in [-1; 0]$ and $u'(s) \leq 0$, $s \in [0; 1]$) and convex ($u''(s) \leq 0$, $s \in [-1; 1]$) functions. For this, consider the set $\mathbf{D}$ of vertices determined by a discrete analogue of monotonicity

$$u_{j+1} - u_j \geq 0, \quad j = \overline{0, 19}, \quad u_{j+1} - u_j \leq 0, \quad j = \overline{20, 39}, \tag{5.4}$$

and convexity

$$\frac{u_{j+1} - u_j}{h_j} - \frac{u_j - u_{j-1}}{h_{j-1}} \leq 0, \quad j = \overline{1, 39}. \tag{5.5}$$

The problem of finding the normal solution for the system (5.2) is written in the equivalent form

$$\varphi(u) = \|\mathbf{u}\|_1 = u_0 + u_1 + \cdots + u_{40} = \langle \mathbf{c}, \mathbf{u}\rangle \to \inf, \quad \mathbf{u} \in \widetilde{\mathbf{U}}_D,$$

where

$$\mathbf{c} = [1\ 1 \ldots 1]^T \in R^{41}_+, \quad \widetilde{\mathbf{U}}_D = \widetilde{\mathbf{U}}_+ \cap \mathbf{D},$$
$$\widetilde{\mathbf{U}}_+ = \{\mathbf{u} \in \mathbf{R}^{41}_+ \mid \widetilde{\mathbf{A}}\mathbf{u} \leq \widetilde{\mathbf{f}}, -\widetilde{\mathbf{A}}\mathbf{u} \leq -\widetilde{\mathbf{f}}\}$$

or in the form

$$\varphi(u) = \|\mathbf{u}\|_1 = u_0 + u_1 + \cdots + u_{40} \to \inf, \quad \mathbf{u} \in \widetilde{\mathbf{U}}_D, \tag{5.6}$$

where

$$\widetilde{\mathbf{U}}_D = \{\mathbf{u} \in \mathbf{R}^{41}_+ \mid \widetilde{\mathbf{B}}\mathbf{u} \leq \widetilde{\mathbf{d}}\}, \quad \widetilde{\mathbf{B}} = \begin{bmatrix} \widetilde{A} \\ -\widetilde{A} \\ F \\ L \end{bmatrix} \in R^{161\times 41}, \quad \widetilde{\mathbf{d}} = \begin{bmatrix} \widetilde{f} \\ -\widetilde{f} \\ 0 \\ \ldots \\ 0 \end{bmatrix} \in R^{161}, \tag{5.7}$$

$$\mathbf{F} = \begin{bmatrix} 1 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & -1 & 1 \end{bmatrix} \in R^{40\times 41},$$

$$\mathbf{L} = \begin{bmatrix} 1 & -2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & -2 & 1 \end{bmatrix} \in R^{39\times 41}.$$

The matrices $\mathbf{F}$ and $\mathbf{L}$ are matrices of the inequalities of monotonicity and convexity conditions. It is natural to consider them to be known precisely. Thus, the error in assigning elements of the matrices $\mathbf{F}$ and $\mathbf{L}$ equals 0 (zero). Let $O_1 \in R^{40\times 41}$ and $O_2 \in R^{39\times 41}$ be matrices of errors with zero elements in assigning $\mathbf{F}$ and $\mathbf{L}$, respectively. Thus, we reduce our problem to an LP problem (2.1) with approximate matrix $\widetilde{\mathbf{B}}$ and vector $\widetilde{\mathbf{d}}$ that has a special structure.

When we solve the LP problem (5.6) by the simplex method in Maple 2015 we obtain its approximate solution $\widetilde{\mathbf{u}}$ that strongly differs from the exact solution $\bar{\mathbf{u}}(\mathbf{s})$ or we do not have a solution. It depends on the accuracy of calculation $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{f}}$.

Now we describe the pointwise residual method of the type (2.6) for the problem (5.6). Let

$$\overline{\Delta} = \begin{bmatrix} \Delta \\ \Delta^T \\ O_1 \\ O_2 \end{bmatrix} \in R^{161\times 41}, \quad \overline{\delta} = \begin{bmatrix} \delta \\ O_3 \\ O_4 \\ O_5 \end{bmatrix} \in R^{161},$$

where elements Δ_{ij} of the matrix Δ and components δ_i of the vector δ are determined by

$$\Delta_{ij} = |\xi_{ij}| \cdot 10^{-k}, \quad \delta_i = |\eta_i| \cdot 10^{-k},$$

where ξ_{ij} and η_i, $i = \overline{0, 40}$, $j = \overline{0, 40}$, are uniformly distributed random numbers on the segment $[-1; 1]$, obtained by the RandomTools package in Maple 2015, $k = 1, \dots, 6$ and O_3, O_4, O_5 are zero vectors of sizes 41, 40, 39, respectively. Since the matrices $\mathbf{F}$ and $\mathbf{L}$ are known exactly in (5.7), the set $W(\sigma)$ for the problem (5.6) is defined as

$$W(\sigma) = \{\mathbf{w} = [\mathbf{u}, \mathbf{v}] \in \mathbf{R}^{202}_+ \mid \widetilde{\mathbf{B}}\mathbf{u} \le \overline{\Delta}\mathbf{u} + \overline{\delta},\ -\widetilde{\mathbf{B}}^T\mathbf{v} - \mathbf{c} \le \overline{\Delta}^T\mathbf{v},\ \langle \mathbf{c}, \mathbf{u}\rangle + \langle \widetilde{\mathbf{d}}, \mathbf{v}\rangle \le \langle \overline{\delta}, \mathbf{v}\rangle\}.$$

Thus, the pointwise residual method for solving primal and dual LP problems (5.6) leads to the LP problem problem

$$\mathbf{f}(\mathbf{w}) = \|\mathbf{w}\|_1 = w_0 + w_1 + \dots + w_{201} \to \inf, \quad \mathbf{w} \in \mathbf{W}(\sigma). \tag{5.8}$$

Due to Theorem 2.2, the solution set of (5.8) is $W_*(\sigma) \ne \emptyset$. Let

$$w = [w_0, w_1, \dots, w_{40}, w_{41}, \dots, w_{201}]^T = [\widetilde{\mathbf{u}}|\widetilde{\mathbf{v}}] \in W_*(\sigma),$$
$$\widetilde{\mathbf{u}} = [w_0, w_1, \dots, w_{40}]^T \in \mathbf{R}^{41}_+ \quad \text{and}$$
$$\widetilde{\mathbf{v}} = [w_{41}, w_{42}, \dots, w_{201}]^T \in \mathbf{R}^{201}_+.$$

We note that in this particular case we are not concerned with $[w_{41}, w_{42}, \dots, w_{201}]^T$ because we analyse the pointwise residual method for solving primal and dual

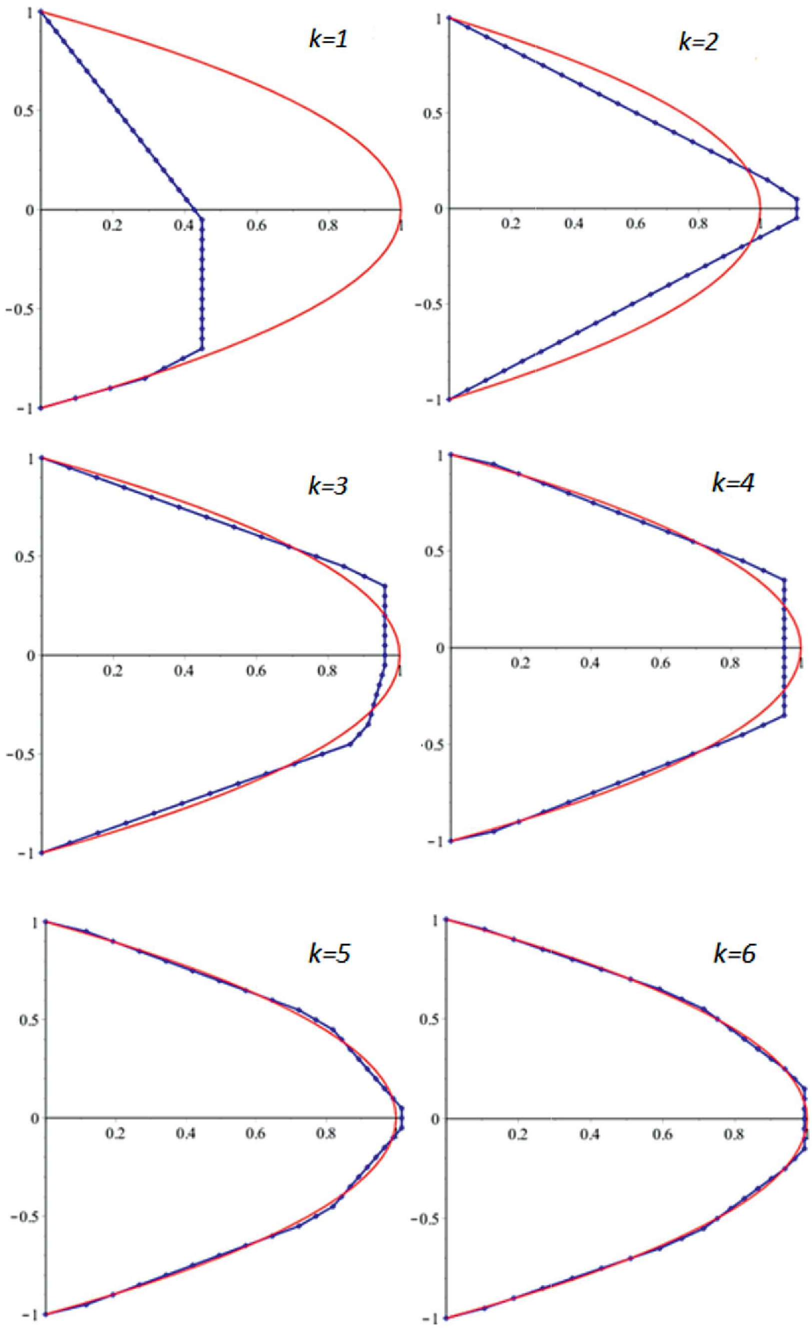

FIGURE 4. Solid red lines – exact solutions, dotted blue lines – approximate solutions, obtained by the pointwise residual method for $k = 1, 2, 3, 4, 5, 6$. (Colour available online.)

TABLE 4. Pointwise residual method for solving integral Fredholm equations in the class of nonnegative, monotonic and convex functions.

	k					
	1	2	3	4	5	6
$\|\|\widetilde{\mathbf{u}} - \overline{\mathbf{u}}\|\|_1$	14.6	4.12	1.356	1.1118	1.07908	0.534178
$\|\|\widetilde{\mathbf{A}}\widetilde{\mathbf{u}} - \widetilde{\mathbf{f}}\|\|_1$	5.7	1.52	0.214	0.0251	0.00223	$4.84 \cdot 10^{-4}$

LP problems for strongly unstable high-dimensional problems and obtain an admissible approximate solution for the function $\overline{\mathbf{u}}(s) = 1 - s^2$ that solves the integral equation (5.1).

Figure 4 displays graphs of exact solutions (solid red lines) of Fredholm type I integral equations [2] and also their approximate solutions (dotted blue lines) in the class of nonnegative, monotonic and convex functions.

Computational results are displayed in Table 4. As we observe from Figure 4 and Table 4, the approximate solutions obtained by the pointwise residual method (2.6) and (2.7) for the problem (5.6) with increasing k converge to the exact solution $u(s) = 1 - s^2$ of the integral equation (5.1).

6. Conclusion

In this paper we have considered the pointwise residual method for solving primal (2.1) and dual (2.3) linear programming problems where instead of exact input data $\mathbf{B}, \mathbf{d}, \mathbf{c}$ we only know their approximations, satisfying the pointwise conditions (2.5). This method leads to the construction of the auxiliary problem (2.6) that is also a linear programming problem. Here we have proved that solutions of the auxiliary problem (2.6) converge (Theorem 3.2) to solutions of the primal (2.1) and dual (2.3) linear programming problems. Also in Theorem 3.2 we obtained an estimate (3.7) for the rate of convergence of the pointwise residual method. In terms of the order of convergence our estimate is optimal and cannot be further improved. However, in the formula (3.7) there is a constant $\mathbf{M}$ that depends only on the coefficients of the matrix $\mathbf{B}$. The aim of further research would be to obtain an accurate estimate for the size of the constant $\mathbf{M}$.

References

[1] M. Abramowitz and I. A. Stegun (eds), *Handbook of mathematical functions with formulas, graphs and mathematical tables*, 9th printing, (Dover, New York, 1972) 886; doi:10.1119/1.15378.

[2] G. B. Arfken, H. J. Weber and F. E. Harris, *Mathematical methods for physicists*, 7th edn, (Academic Press, Oxford, 2012); doi:10.1016/C2009-0-30629-7.

[3] K. Avrachenkov, R. S. Burachik, J. A. Filar and V. Gaitsgory, "Constraint augmentation in pseudo-singularity perturbed linear programs", *Math. Program. Ser. A* **132** (2012) 179–208; doi:10.1007/s10107-010-0388-0.

[4] G. B. Dantzig, "Maximization of a linear function of variables subject to linear inequalities", in: *Activity analysis of production and allocation* (ed. T. C. Koopmans), (Wiley, New York, 1951) 339–347.

[5] A. Hoffman, "On approximate solutions of system of linear inequalities", *J. Res. Natl. Bur. Stand.* **4** (1952) 263–265; doi:10.6028/jres.049.027.

[6] A. Y. Ivanitskiy, *Stable methods for solving systems of linear equations and inequalities with interval coefficients: candidate dissertation of physical and mathematical sciences* (Moscow University Press, Moscow, 1998) (in Russian).

[7] A. Y. Ivanitskiy, F. P. Vasilyev and V. A. Morozov, "Estimation of speed of convergence of residual method for linear programming problems", *Zh. Vychisl. Mat. Mat. Fiz.* **38** (1990) 1257–1262.

[8] L. V. Kantorovich, "Mathematical methods of organizing and planning production" (English translation), *Manag. Sci.* **6** (1960) 366–422; doi:10.1287/mnsc.6.4.366.

[9] N. Karmakar, "A new polynomial-time algorithm for linear programming", *Combinatorica* **4** (1984) 373–395; doi:10.1007/BF02579150.

[10] L. Khachiyan, "Polynomial algorithms in linear programming", *Dokl. Akad. Nauk SSSR* **244** (1979) 1093–1096; doi:10.1016/0041-5553(80)90061-0.

[11] A. S. Leonov, *Solving ill-posed inverse problems*, (URSS, Moscow, 2009) (in Russian).

[12] V. A. Morozov, "Stable numerical methods for solving simultaneous systems of linear algebraic equations", *Zh. Vychisl. Mat. Mat. Fiz.* **24** (1984) 179–186, available at https://www.sciencedirect.com/sdfe/pdf/download/eid/1-s2.0-0041555384901290/first-page-pdf.

[13] Y. Nesterov and A. Nemirovskiy, *Interior-point polynomial algorithms in convex programming*, (John Wiley, New York, 1983); doi:10.1137/1.9781611970791.

[14] J. Renegar, "Some perturbation theory for linear programming", *Math. Program.* **65** (1994) 73–91; doi:10.1007/BF01581690.

[15] A. N. Tikhonov, "On some problems of optimal planning", *Zh. Vychisl. Mat. Mat. Fiz.* **6** (1966) 81–89.

[16] A. N. Tikhonov, V. G. Karmanov and T. L. Rudneva, "On the stability of linear programming", in: *Collection of works of Research Computing Center of Moscow University "Computational methods and programming"*, Volume 12, (Moscow University Press, Moscow, 1969) 3–9 (in Russian).

[17] A. N. Tikhonov, A. A. Ryutin and G. M. Agayan, "A stable method of solving a linear programming problem with approximate data", *Dokl. Akad. Nauk SSSR* **272** (1983) 1058–1063, available at http://mi.mathnet.ru/eng/dan9923.

[18] V. N. Titarenko and A. Yagola, "The problems of linear and quadratic programming for ill-posed problems on some compact sets", *J. Inverse Ill Posed Probl.* **11** (2003) 311–328; doi:10.1515/156939403769237079.

[19] R. J. Vanderbei, *Linear programming: foundations and extensions*, Volume 196 of *Int. Ser. in Oper. Res. & Manag. Sci.*, (Springer, New York, 2014); doi:10.1007/978-1-4614-7630-6.

[20] F. P. Vasilyev, *Methods for optimizations*, (Faktorial Press, Moscow, 2002).

[21] F. P. Vasilyev and A. Y. Ivanitskiy, *In-depth analysis of linear programming*, (Springer, Dordrecht, 2001); doi:10.1007/978-94-015-9759-3.

[22] F. P. Vasilyev, V. A. Morozov and M. D. Yashimovich, "Estimate of speed of convergence of regularization method for linear programming problems", *Zh. Vychisl. Mat. Mat. Fiz.* **29** (1989) 631–635; doi:10.1016/0041-5553(89)90032-3.

[23] J. von Neumann, "Zur Theorie der Gesellschaftsspiele", *Math. Ann.* **100** (1928) 295–320; doi:10.1007/BF01448847.