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A THEOREM ON INVOLUTIONS ON CYCLIC PEANO SPACES

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The purpose of this note is to prove that an involution f on a cyclic Peano space S leaves some simple closed curve in S setwise invariant.

We shall first define the required terms. A *Peano space* is a locally compact, connected and locally connected metric space. A connected space is called *cyclic* if it has no cut-point. An *involution* on a space is a periodic mapping whose period is 2; it is necessarily a homeomorphism. A mapping $f: X \to X$ is said to leave a subset E of S *setwise invariant* if $f(E)=E$. These definitions may be found, for example, in [2].

We shall use the following lemma, which is a variation of lemma 1 of [1].

LEMMA. *If U, V are disjoint nonempty open sets in a cyclic Peano space S, then there are two disjoint arcs ab, cd in S such that $a, c \in A$ and $b, d \in B$.*

An arc whose endpoints are a, b will generally be denoted by ab. If A and B are closed sets, we say that *ab is an arc from A to B* if $ab \cap A=\{a\}$ and $ab \cap B=\{b\}$.

THEOREM. *An involution f on a cyclic Peano space S leaves some simple closed curve in S setwise invariant.*

Proof. Let f be an involution on a cyclic Peano space S. Since $f(x) \neq x$ for some point x in S, it follows that there is a nonempty region R in S such that $R \cap f(R)=0$. By the lemma, there are two disjoint arcs ab and cd in S such that $a, c \in R$ and b, $d \in f(R)$.

In the first case suppose that one of these arcs is disjoint from its image, say $ab \cap f(ab)=0$. Let pq be an arc in R from ab to $f(ab)$. Then f leaves the simple closed curve $pq \cup qf(p) \cup f(pq) \cup f(q)p$ setwise invariant, where $qf(p) \subset f(ab)$ and $f(q)p \subset ab$.

In the second case suppose that both of the arcs meet their images. First consider ab. Let m be the first point on ab in the order a, b such that $am \cap f(am) \neq 0$, where $am \subset ab$. Then $am \cap f(am)$ contains just the two points $m, f(m)$. If $m \neq f(m)$ then the subarcs of am and $f(am)$ from m to $f(m)$ form a simple closed curve which is left setwise invariant under f. So suppose that $m=f(m)$. Also, let n be the first point on cd in the order c, d such that $cn \cap f(cn) \neq 0$, and suppose that $n=f(n)$. Then $am \cup f(am)$ and $cn \cup f(cn)$ are setwise invariant arcs under f. If $am \cap f(cn)=0$, then $am \cup f(am)$ and $cn \cup f(cn)$ are disjoint, and the construction of an arc pq in R from am to cn shows, as in the first case, that there is a simple closed curve which is setwise invariant under f. So suppose that $am \cap f(cn) \neq 0$. Let r be the first point on am in the order a, m which lies on $f(cn)$. Then $r \neq m, n$ so that $ar \cup rf(c)$ and

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$f(a)f(r) \cup f(r)c$ are disjoint arcs, where $ar \subset am$, $rf(c) \subset f(cn)$, $f(a)f(r) \subset f(am)$ and $f(r)c \subset cn$. Further $ar \cup rf(c)$ and $f(a)f(r) \cup f(r)c$ are images of each other under f, and ar and $f(r)c$ both meet R. Thus the construction of an arc pq in R from ar to $f(r)c$ again shows that there is a simple closed curve which is left setwise invariant by f.

REMARK. The well known cyclic connectivity theorem of [1] can be used to prove this theorem, in which case the region R is replaced by a point and the construction of the arc pq in each case becomes unnecessary. But use of the cyclic connectivity theorem does not change the ideas of the proof, and eliminates only the trivial constructions of the arc pq. On the other hand, the proof of the cyclic connectivity theorem is based upon the theory of cyclic elements, none of which is required in the above proof. Thus in our proof we have avoided the cyclic connectivity theorem and used only the lemma and in so doing have kept the proof at its most elementary level.

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