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THE RUDIN–KEISLER ORDERING OF P-POINTS UNDER $\mathfrak{b} = \mathfrak{c}$

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Abstract. M. E. Rudin (1971) proved, under CH, that for each P-point p there exists a P-point q strictly RK-greater than p. This result was proved under $\mathfrak{p} = \mathfrak{c}$ by A. Blass (1973), who also showed that each RK-increasing ω-sequence of P-points is upper bounded by a P-point, and that there is an order embedding of the real line into the class of P-points with respect to the RK-ordering. In this paper, the results cited above are proved under the (weaker) assumption that $\mathfrak{b} = \mathfrak{c}$. A. Blass asked in 1973 which ordinals can be embedded in the set of P-points, and pointed out that such an ordinal cannot be greater than $\mathfrak{c}^+$. In this paper it is proved, under $\mathfrak{b} = \mathfrak{c}$, that for each ordinal $\alpha < \mathfrak{c}^+$, there is an order embedding of α into P-points. It is also proved, under $\mathfrak{b} = \mathfrak{c}$, that there is an embedding of the long line into P-points.

§1. Introduction. In [10], M. E. Rudin proved that, under CH, for each P-point p there exists a P-point q strictly RK-greater than p. A. Blass showed the same [1] assuming that $\mathfrak{p} = \mathfrak{c}$;[1] moreover, he proved that if $\mathfrak{p} = \mathfrak{c}$, then each RK-increasing ω-sequence of P-points is upper bounded by a P-point, and there exists an order-embedding of the real line into the class of P-points with respect to the RK-ordering. Since then, the RK-ordering of P-points has been thoroughly investigated; however, most of the obtained results were proved under $MA_{\sigma\text{–centr}}$, or stronger assumptions,[2] usually with complicated proofs and using sophisticated techniques. We prove the results mentioned above under $\mathfrak{b} = \mathfrak{c}$. Perhaps more importantly, we present a method of proof that turns out be effective in the study of P-points under $\mathfrak{b} = \mathfrak{c}$. The ideas used in the present paper were originally presented in an unpublished paper [12], where the RK-ordering concerned the ultrafilters in the classes of the so-called P-hierarchy, the first class of which coincides with that of P-points. The method is based on the use of contours and quasi-subbases, which enables us to employ surprisingly concise arguments, in contrast with the approaches of some other papers on similar topics.

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[1]Actually, all results from [1] quoted in this paper were stated under MA, but the proofs also work under $\mathfrak{p} = \mathfrak{c}$, as pointed out by A. Blass in [2]; definitions of $\mathfrak{p}, \mathfrak{c}$ and few other cardinal invariants are recalled on page 3.

[2]Note that A. Blass asks [1, Question 5]: What can be proved about P-points without using MA?

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After a scrutiny of mechanisms underlying our proofs, we introduce an apparently new cardinal invariant $\mathfrak{q}$, the use of which enables us to weaken the set-theoretic assumptions of most of our results. Finally, we show that $\mathfrak{q}$ is an instance of a general method of constructing useful variants of cardinal invariants.

In a recent paper, D. Raghavan and S. Shelah [8] proved (under $\mathfrak{p} = \mathfrak{c}$) that there is an order-embedding of $\mathcal{P}(\omega)/\mathrm{fin}$ into the set of P-points ordered by $\geq_{RK}$, and gave a short review of earlier results concerning embeddings of different orders into the class of P-points.

A. Blass also asked [1, Question 4] which ordinals can be embedded in the set of P-points, and pointed out that such an ordinal cannot be greater than $\mathfrak{c}^+$. We show that under $\mathfrak{b} = \mathfrak{c}$ each ordinal less than $\mathfrak{c}^+$ is order-embeddable into P-points. A recent paper by B. Kuzeljević and D. Raghavan [6] answers the question of the embedding of ordinals into P-points under MA.

§2. Tools. A free ultrafilter u is a *P-point* if and only if, for each partition $(V_n)_{n<\omega}$ of ω, there exists a set $U \in u$ such that either $U \subset V_n$ for some $n < \omega$ or else $U \cap V_n$ is finite for all $n < \omega$. A filter $\mathcal{F}$ is said to be *Rudin–Keisler greater* (*RK-greater*) than a filter $\mathcal{G}$ (written as $\mathcal{F} \geq_{RK} \mathcal{G}$) if there exists a map h such that $G \in \mathcal{G}$ if and only if $h^{-1}(G) \in \mathcal{F}$. Let

$$\mathcal{W} = \{W_n : n < \omega\} \tag{1}$$

be a *partition* of a subset of ω into infinite sets. A filter $\mathcal{K}$ is called a *contour* if there exists a partition $\mathcal{W}$ such that $K \in \mathcal{K}$ if and only if there is a cofinite set $I \subset \omega$ such that $K \cap W_n$ is cofinite on W_n for each $n \in I$. We call $\mathcal{K}$ a contour of $\mathcal{W}$, and denote $\mathcal{K} = \int \mathcal{W}$.[3]

A fundamental property used in the present paper is the following reformulation of [11, Proposition 2.1].

PROPOSITION 2.1. *A free ultrafilter is a P-point if and only if it does not include a contour.*

As usual, $\mathfrak{c}$ denotes the cardinality of the continuum. If $f, g \in {}^{\omega}\omega$, then we say that f *dominates* g (and write $f \geq^* g$) if $f(n) \leq g(n)$ for almost all $n < \omega$. We say that a family $\mathcal{F}$ of ${}^{\omega}\omega$ functions is *unbounded* if there is no $g \in {}^{\omega}\omega$ that dominates all functions $f \in \mathcal{F}$. The minimal cardinality of an unbounded family is the *bounding number* $\mathfrak{b}$. We also say that a family $\mathcal{F} \subset {}^{\omega}\omega$ is *dominating* if, for each $g \in {}^{\omega}\omega$, there is some $f \in \mathcal{F}$ that dominates g. The *dominating number* $\mathfrak{d}$ is the minimal cardinality of a dominating family. The *pseudointersection number* $\mathfrak{p}$ is the minimal cardinality of a free filter without *pseudointersection*, which is a set almost included in each element of the filter. Finally, the *ultrafilter number* $\mathfrak{u}$ is the minimal cardinality of a base of a free ultrafilter. It is well known that $\mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$, and $\mathfrak{p} \leq \mathfrak{b} \leq \mathfrak{u} \leq \mathfrak{c}$, and that there are models for which $\mathfrak{p} < \mathfrak{b}$ (see, for example, [5]).

We say that a family $\mathcal{A}$ is *centered* (has *fip*) if the intersection of any finite subfamily is nonempty; a family $\mathcal{A}$ is *strongly centered* (has *sfip*) if the intersection of any finite subfamily is infinite. If $\mathcal{A}$ and $\mathcal{B}$ are families of sets, then we say that $\mathcal{A}$ and $\mathcal{B}$ are *compatible* if $\mathcal{A} \cup \mathcal{B}$ is centered. If $\mathcal{A} = \{A\}$ we say that A is compatible with $\mathcal{B}$.

[3] See [3] and the last section of [4] for a systematic presentation of contours.

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If a family $\mathcal{A}$ is centered, then we denote by $\langle \mathcal{A} \rangle$ the filter generated by $\mathcal{A}$. Let A be an infinite subset of ω. A filter $\mathcal{F}$ on ω is said to be *cofinite on A* whenever $U \in \mathcal{F}$ if and only if $A \setminus U$ is finite. A filter $\mathcal{F}$ is said to be *cofinite* if it is cofinite on some $A \subset \omega$. It is well-known that a filter is free on A if and only if it includes the cofinite filter of A.

2.1. A relation between sets and functions. Let $\mathcal{W}$ be a partition (1). For each $n < \omega$, let $\left(w_k^n\right)_{k<\omega}$ be an increasing sequence such that

$$W_n = \{w_k^n k < \omega\}.$$

For each $f \in {}^{\omega}\omega$ and $m < \omega$, let

$$E_{\mathcal{W}}(f, m) = \{w_k^n f(n) \leq k, m \leq n\}. \tag{2}$$

If $F \in \int \mathcal{W}$, then, by definition, there exists the least $n_F < \omega$ such that $W_n \setminus F$ is finite for each $n \geq n_F$. Now, for each $n \geq n_F$, there exists a minimal $k_n < \omega$ such that $w_k^n \in F$ for each $k \geq k_n$. Let $f f_F$ denote the set of those functions f for which

$$n \geq n_F \Longrightarrow f(n) = k_n. \tag{3}$$

Then $E_{\mathcal{W}}(f, n_F)$ is the same for each $f \in f f_F$, and it is the largest set of the form (2) included in F. Sure enough, $E_{\mathcal{W}}(f_F, n_F) \in \int \mathcal{W}$.

Conversely, for every function $f \in {}^{\omega}\omega$, we define a family $\mathcal{W}_f$ of subsets of ω as follows: $F \in \mathcal{W}_f$ if there is $n_F < \omega$ such that $F = E_{\mathcal{W}}(f_F, n_F)$. Therefore, we can state the following.

PROPOSITION 2.2. *The family* $\bigcup_{f \in {}^{\omega}\omega} \mathcal{W}_f$ *is a base of* $\int \mathcal{W}$.

2.2. Quasi-subbases. We say that a family $\mathcal{A}$ is *finer* than $\mathcal{B}$ if $\langle \mathcal{B} \rangle \subset \langle \mathcal{A} \rangle$. Moreover, $\mathcal{A}$ is called a *quasi-subbase* of (a filter) $\mathcal{F}$ if there exists a countable family $\mathcal{C}$ such that $\langle \mathcal{A} \cup \mathcal{C} \rangle = \mathcal{F}$. Accordingly, $\mathcal{A}$ is *quasi-finer* than $\mathcal{B}$ if there exists a countable family $\mathcal{C}$ such that $\mathcal{A} \cup \mathcal{C}$ is centered and $\mathcal{B} \subset \langle \mathcal{A} \cup \mathcal{C} \rangle$. Finally we say that a family is a P^+ *-family* if it is quasi-finer than no contour.

If $\mathcal{W}$ is a partition (1), then for each $i < \omega$, let

$$\widetilde{W}_i = \bigcup_{n \geq i} W_n \text{ and } \widetilde{\mathcal{W}} = \{\widetilde{W}_i : i < \omega\}.$$

PROPOSITION 2.3. *Let* $\mathcal{W}$ *be a partition and let* $\mathcal{A}$ *be a centered family. Then the following are equivalent*:

(1) $\mathcal{A}$ *is quasi-finer than* $\int \mathcal{W}$,

(2) *there exists a set* D *such that* $\mathcal{A} \cup \widetilde{\mathcal{W}} \cup \{D\}$ *is centered, and* $\int \mathcal{W} \subset \langle \mathcal{A} \cup \widetilde{\mathcal{W}} \cup \{D\} \rangle$.

PROOF. The implication $2 \Rightarrow 1$ is evident. We will show $1 \Rightarrow 2$. Suppose the contrary, and let $\mathcal{B}$ be a countable family of sets such that $\int \mathcal{W} \subset \langle \mathcal{A} \cup \mathcal{B} \rangle$. Taking finite intersections $\bigcap_{i \leq n} B_i$ instead of B_n, we obtain a decreasing sequence so that, without loss of generality, we can assume that $\mathcal{B} = \{B_n\}_{n<\omega}$ is decreasing. Since (2) is false, for each n there exists $A_n \in \int \mathcal{W}$ such that $A_n \notin \langle \mathcal{A} \cup \widetilde{\mathcal{W}} \cup \{B_n\} \rangle$.

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Without loss of generality, for each n there is $k(n) \geq n$ such that $A_n \cap W_i$ is empty for all $i < k(n)$ and $W_i \setminus A_n$ is finite for all $i \geq k(n)$. Define $A_\infty = \bigcup_{i<\omega} \left(\bigcap_{\{n:k(n)\leq i\}} A_n \cap W_i \right)$ and note that $A_\infty \in \int \mathcal{W}$.

We will show that $A_\infty \notin \langle \mathcal{A} \cup \mathcal{B} \cup \widetilde{\mathcal{W}} \rangle \supset \langle \mathcal{A} \cup \mathcal{B} \rangle$. For this purpose, it suffices to show that $A_\infty \notin \langle \mathcal{A} \cup \widetilde{\mathcal{W}} \cup \{B_n\} \rangle$ for each $n < \omega$. Indeed, note that $A_\infty \subset A_n \cup \widetilde{W}^c_{k(n)}$ for each $n < \omega$. From $(A_n \cup \widetilde{W}^c_{k(n)}) \cap \widetilde{W}_{k(n)} \subset A_n \notin \langle \mathcal{A} \cup \widetilde{\mathcal{W}} \cup \{B_n\} \rangle$, we infer that $(A_n \cup \widetilde{W}^c_{k(n)}) \notin \langle \mathcal{A} \cup \widetilde{\mathcal{W}} \cup \{B_n\} \rangle$, and so $A_\infty \notin \langle \mathcal{A} \cup \widetilde{\mathcal{W}} \cup \{B_n\} \rangle$. ⊣

Remark 2.4. Let $\mathcal{A}$ be finer than $\int \mathcal{W}$. Then there exists a partition $\mathcal{V}$ such that $\int \mathcal{V} \subset \langle \mathcal{A} \cup \widetilde{\mathcal{V}} \rangle$.

Proof. Take V_i to be the i-th infinite $W_j \cap D$, with D from Proposition 2.3(2).

Theorem 2.5. *Let $(\mathcal{A}_\alpha)_{\alpha<\beta<\mathfrak{b}}$ be an increasing sequence of P^+-families. Then, $\bigcup_{\alpha<\beta} \mathcal{A}_\alpha$ is a P^+-family.*

Proof. By way of contradiction, $\{\mathcal{A}_\alpha\}_{\alpha<\beta<\mathfrak{b}}$ be as in the assumptions, and let $\int \mathcal{W}$ be a contour and $\mathcal{B}$ be a countable family of sets such that $\int \mathcal{W} \subset \langle \mathcal{A} \cup \mathcal{B} \rangle$. By Proposition 2.3 and Remark 2.4, without loss of generality, we may assume that $\mathcal{B} = \{B_n\}_{n<\omega} = \widetilde{\mathcal{W}}$ is decreasing. Let $\mathcal{C}_\alpha = \langle \mathcal{A}_\alpha \cup \mathcal{B} \rangle$. Clearly $\mathcal{C}_\alpha$ does not include $\int \mathcal{W}$ for each $\alpha < \beta$. Thus, for each $\alpha < \beta$, there exists a set $D_\alpha \in \int \mathcal{W}$ such that $D_\alpha \notin \mathcal{C}_\alpha$. Let $g_\alpha \in ff_{D_\alpha}$ for each $\alpha < \beta$. As $\beta < \mathfrak{b}$, the family $\{g_\alpha\}_{\alpha<\beta}$ is bounded by some function g. Let $G \in \mathcal{W}_g$. We will show that $G \notin \bigcup_{\alpha<\beta} \mathcal{C}_\alpha$, hence $G \notin \mathcal{C}_\alpha$ for each $\alpha < \beta$. Suppose not, and let α_0 be a witness. By construction, there exists $n_0 < \omega$ such that $G \subset B^c_{n_0} \cup D_{\alpha_0}$. As $(D_{\alpha_0} \cup B^c_{n_0}) \cap B_{n_0} \subset D_{\alpha_0} \notin \mathcal{C}_{\alpha_0}$, it follows that $D_{\alpha_0} \cup B^c_{n_0} \notin \mathcal{C}_{\alpha_0}$, and so $G \notin \mathcal{C}_{\alpha_0}$. ⊣

The anonymous referee noticed that Corollary 2.6 easily follows from [7, Proposition 2.28] by A. R. D. Mathias by argument pointed out in Mathias' proof. One can also prove Corollary 2.6 inductively using Theorem 2.5 and Proposition 2.1.

Corollary 2.6 [7] ($\mathfrak{b} = \mathfrak{c}$). *If $\mathcal{A}$ is a strongly centered P^+-family of subsets of ω, then there exists a P-point p such that $\mathcal{A} \subset p$.*

Let us recall a well-known theorem (see, for example, [1, Corollary 1]).

Theorem 2.7. *Let u be an ultrafilter. If $f(u) =_{RK} u$, then there exists $U \in u$ such that f is one-to-one on U.*

§3. Applications: RK-ordering of P-points. M. E. Rudin [10] proved that, under CH, for each P-point p there exists a P-point q strictly RK-greater than p. Some years later, A. Blass [1, Theorem 6] proved this theorem under $\mathfrak{p} = \mathfrak{c}$.

The referee also noticed that Theorem 3.1 is easily derivable from [7, Proposition 2.28] by A. R. D. Mathias combined with [1, Theorem 6] by A. Blass. Nevertheless we present our original proof because its methods will be used in the sequel.

Theorem 3.1. ($\mathfrak{b} = \mathfrak{c}$).[4] *If p is a P-point, then there exists a P-point q that is strictly Rudin–Keisler greater than p.*

[4]Note that all theorems in this section are proved, in fact, under (possibly) weaker assumptions, what we will discuss in detail in Section 4.

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PROOF. Let $f \in {}^{\omega}\omega$ be a finite-to-one function such that

$$\limsup_{n \in P} \operatorname{card}(f^{-1}(n)) = \infty$$

holds for all $P \in p$. We define a family $\mathcal{A}$ as follows: $A \in \mathcal{A}$ if and only if there exist $i < \omega$ and $P \in p$ such that $\operatorname{card}(f^{-1}(n) \setminus A) < i$ for each $n \in P$. Then, Theorem 2.7 ensures that the ultrafilters we are building are strictly RK-greater than p.

We claim that $\{f^{-1}(p)\} \cup \mathcal{A}$ is a P^+-family. Suppose not, and take a witness $\int \mathcal{W}$. From Remark 2.4, without loss of generality, we may assume that $\int \mathcal{W} \subset \langle f^{-1}(p) \cup \mathcal{A} \cup \widetilde{\mathcal{W}} \rangle$. Consider two cases:

Case 1: There exists a sequence $(B_n)_{n<\omega}$ and a strictly increasing $k \in {}^{\omega}\omega$ such that $B_n \subset W_{k(n)}$, $f(B_n) \notin p$, and B is compatible with $f^{-1}(p) \cup \mathcal{A} \cup \widetilde{\mathcal{W}}$, where $B = \bigcup_{n<\omega} B_n$. Take a sequence $(f(\bigcup_{i \le n} B_n))_{n<\omega}$. This is an increasing sequence, and it is clear that $\bigcup_{n<\omega} f(\bigcup_{i \le n} B_n) = f(B) \in p$. Make a partition of $f(B)$ by taking $f(\bigcup_{n \le i+1} B_n) \setminus f(\bigcup_{n \le i} B_n)$ for $i < \omega$. As p is a P-point, there exists some $P \in p$ such that $P \cap (f(\bigcup_{n \le i+1} B_n) \setminus f(\bigcup_{n \le i} B_n))$ is finite for all $i < \omega$, and thus $f^{-1}(P) \cap B_n$ is finite for all $i < \omega$. Therefore, $f^{-1}(P) \cap W_i \cap B$ is finite, and thus $(f^{-1}(P) \cap B)^c \in \int \mathcal{W}$, which means that $\int \mathcal{W}$ is not compatible with $\langle f^{-1}(p) \cup \{B\} \rangle$.

Case 2: Otherwise without loss of generality $f(W_i) \in p$ for each $i < \omega$, since we are not in Case 1. Define sets $V_1 = W_1$, $V_i = W_j \cap f^{-1}(\bigcap_{k<i} f(W_k))$, and note that $\bigcup_{i<\omega} V_i \in \langle \mathcal{A} \cup f^{-1}(p) \rangle$ since we are not in Case 1. Then, $(f(V_i))_{i<\omega}$ is a decreasing sequence, and because f is finite-to-one, $(f(V_i) \setminus f(V_{i+1}))_{i<\omega}$ is a partition of $f(V_1) \in p$. As p is a P-point, there exists $P \in p$ such that $P_i = (f(V_i) \setminus f(V_{i+1})) \cap P$ is finite for each $i < \omega$. Let $g : \omega \to \omega$ be defined by $g(i) = E\left(\frac{i+1}{2}\right)$, where $E(x)$ stands for the integer part of x. Let

$$R = \bigcup_{i<\omega} (f^{-1}(P_i) \cap \bigcup_{j \in [g(i),\ldots,i]} V_i).$$

Note that $R \cap V_i$ is finite for each $i < \omega$, and that

$$\limsup_{n \in \tilde{P}} \operatorname{card}(f^{-1}(n) \cap R) = \infty$$

for all $\tilde{P} \in p$ and $\tilde{P} \subset P$. Thus, $R^c \notin \langle \mathcal{A} \cup f^{-1}(p) \cup \widetilde{\mathcal{W}} \rangle$, although $R^c \in \int \mathcal{W}$, which completes Case 2.

To complete the proof of the theorem use Corollary 2.6. ⊣

The following two, probably known, facts will be needed for Theorem 3.4 that extends, under CH, Theorem 3.1.

FACT 3.2. *Let $\mathcal{A}$ be a centered family of subsets of ω such that $\mathcal{A} \cup \{F\}$ is not an ultrafilter subbase for any F compatible with $\mathcal{A}$. Let $\mathcal{F}$ be a countable family compatible with $\mathcal{A}$. Then $\mathcal{A} \cup \mathcal{F}$ is not an ultrafilter subbase.*

PROOF. Without loss of generality, we may assume that $(F_n)_{n<\omega}$ is a decreasing sequence of sets, such that $F_{n+1} \notin \langle \mathcal{A} \cup \{F_n\} \rangle$. Put $B_n = F_n \setminus F_{n+1}$ and define $B^1 = \bigcup_{n<\omega} B_{2n}$, $B^2 = \bigcup_{n<\omega} B_{2n+1}$. Clearly at least one of sets B^1, B^2 interact $\mathcal{A}$—say B_1 does. If $B^1 \notin \langle \mathcal{A} \cup \mathcal{F} \rangle$ then we are done. Suppose that $B^1 \in \langle \mathcal{A} \cup \mathcal{F} \rangle$, and denote by n_0, the minimal $n < \omega$ that $B^1 \in \langle \mathcal{A} \cup \{F_n\} \rangle$. But $F_{n_0+1} \cap B^1 = F_{n_0+2} \cap B^1$ and so $F_{n_0+2} \in \langle \mathcal{A} \cup \mathcal{F}_{n_0+1} \rangle$, which is a contradiction. ⊣

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FACT 3.3. *Let $\mathcal{Y}, \mathcal{Z}$ be centered families of subsets of ω, which are not ultrafilter subbases. If $h \in {}^{\omega}\omega$, then there exist sets Y and Z such that Y is compatible with $\mathcal{Y}$, Z is compatible with $\mathcal{Z}$, and $h(Z)$ is not compatible with Y.*

PROOF. Take any O such that O and O^c are compatible with $\mathcal{Y}$.
If $h^{-1}(O)$ is not compatible with $\mathcal{Z}$ then $Y = O$, $Z = (h^{-1}(O))^c$;
if $h^{-1}(O^c)$ is not compatible with $\mathcal{Z}$ then $Y = O^c$, $Z = h^{-1}(O)$;
if $h^{-1}(O)$ is compatible with $\mathcal{Z}$ and $h^{-1}(O^c)$ is compatible with $\mathcal{Z}$ then $Y = O$, $Z = (h^{-1}(O))^c$. ⊣

THEOREM 3.4 (CH). *If p is a P-point, then there exists a set $\mathfrak{U}$ of cardinality $\mathfrak{c}$ of Rudin–Keisler incomparable P-points with $u >_{RK} p$ for each $u \in \mathfrak{U}$.*

PROOF. First repeat the proof of Theorem 3.1 except for the last line. Then continue as follows.

We arrange all contours in a sequence $(\int \mathcal{W}_\alpha)_{\alpha<\mathfrak{b}}$ and ${}^{\omega}\omega$ in a sequence $(f_\beta)_{\beta<\mathfrak{b}}$. We will build a family $\{(\mathcal{F}^\beta_\alpha)_{\alpha<\mathfrak{b}}\}_{\beta<\mathfrak{b}}$ of increasing $\mathfrak{b}$-sequences $(\mathcal{F}^\beta_\alpha)_{\alpha<\mathfrak{b}}$ of filters such that:

1) Each $\mathcal{F}^\beta_\alpha$ is generated by $\mathcal{A}$ together with some family of sets of cardinality $< \mathfrak{b}$;
2) $\mathcal{F}^\beta_0 = \mathcal{A}$ for each $\beta < \mathfrak{b}$;
3) For each $\alpha, \beta < \mathfrak{b}$, there exists $F \in \mathcal{F}^\beta_{\alpha+1}$ such that $F^c \in \int \mathcal{W}_\alpha$;
4) For every limit α and for each β, let $\mathcal{F}^\beta_\alpha = \bigcup_{\gamma<\alpha} \mathcal{F}^\beta_\gamma$;
5) For each $\alpha, \gamma < \alpha, \beta_1, \beta_2 < \alpha$, there exists a set $F \in \mathcal{F}^{\beta_1}_{\alpha+1}$ such that $(f_\gamma(F))^c \in \mathcal{F}^{\beta_2}_{\alpha+1}$.

The existence of such families follows by a standard induction with sub-inductions using Theorem 2.5 and Fact 3.3 for Condition 5. It follows from the proof of Fact 3.2 that $\mathcal{F}^\beta_\alpha$ is not an ultrafilter subbase for each α and β. It suffices now to take for each $\beta < \mathfrak{c}$, any ultrafilter extending $\bigcup_{\beta<\mathfrak{c}} \mathcal{F}^\beta_\alpha$ and note that, by Proposition 2.1, it is a P-point. ⊣

A. Blass [1, Theorem 7] also proved that, under $\mathfrak{p} = \mathfrak{c}$, each RK-increasing sequence of P-points is upper bounded by a P-point. By $\mathrm{Level}_n(T)$ we denote level n in the tree T.

THEOREM 3.5 ($\mathfrak{b} = \mathfrak{c}$). *If $(p_n)_{n<\omega}$ is an RK-increasing sequence of P-points, then there exists a P-point u such that $u >_{RK} p_n$ for each $n < \omega$.*

PROOF. For each $n < \omega$ we let f_n to be a finite-to-one function that witnesses $p_{n+1} >_{RK} p_n$. Consider a sequence $(T_n)_{n<\omega}$ of disjoint trees such that for each $n < \omega$

1) the root of T_n is equal to $n \in \omega$;
2) $\mathrm{Level}_m(T_n) = \{f_m^{-1}(k) : k \in \mathrm{Level}_{m-1}(T_n)\}$ for $1 \le m \le n$;
3) $\mathrm{Level}_m(T_n) = \emptyset$ for $m > n$.

Since $L_\infty = \bigcup_{n<\omega} \max T_n$ is countably infinite we treat it as ω as well as $L_m = \bigcup_{n<\omega} \mathrm{Level}_m(T_n)$. Let $g_m : L_\infty \setminus \bigcup_{k<m} L_k \to L_m$ be a function defined by order of the trees T_n.

On L_∞ we define a family of sets: $\mathcal{B} = \bigcup_{n<\omega} g_m^{-1}(p_m)$.

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To conclude it suffices, by Corollary 2.6, to show that $\mathcal{B}$ is a P^+-family, thus by Theorem 2.5 it suffices to prove that $g_m^{-1}(p_m)$ is a P^+-family, for any m. But this is an easier version of the fact which we established in the proof of Theorem 3.1. ⊣

In [1], A. Blass asked (Question 4) which ordinals could be embedded in the set of P-points, noticing that such an ordinal could not be greater than $\mathfrak{c}^+$. The question was also considered by D. Raghavan and S. Shelah in [8] and answered, under MA, by B. Kuzeljević and D. Raghavan in a recent paper [6].

We prove that, under $\mathfrak{b} = \mathfrak{c}$, there is an order embedding of each ordinal less than $\mathfrak{c}^+$ into P-points. To this end, we need some (probably known) facts associated with the following definition: we say that a subset A of ${}^\omega\omega$ is *sparse* if $\lim_{n<\omega} |f(n) - g(n)| = \infty$ for each $f, g \in A$ such that $f \neq g$.

FACT 3.6. *There exists a strictly $<^*$-increasing sparse sequence $\mathcal{F} = (f_\alpha)_{\alpha<\mathfrak{b}} \subset {}^\omega\omega$ of nondecreasing functions such that $f_\alpha(n) \leq n$ for each $n < \omega$ and $\alpha < \mathfrak{b}$.*

PROOF. First, we build, from the definition of $\mathfrak{b}$, an $<^*$-increasing sparse sequence $(g_\alpha)_{\alpha<\mathfrak{b}} \subset {}^\omega\omega$ of nondecreasing functions that fulfill the following condition: if $\alpha < \beta < \mathfrak{b}$, then $g_\alpha(n) > g_\beta(2n) + n$ for almost all $n < \omega$. Then a $\mathfrak{b}$-sequence $(f_\alpha)_{\alpha<\mathfrak{b}}$ defined by $f_\alpha(m) = m - \max\{n : g_\alpha(n) < m\}$ is as desired. ⊣

FACT 3.7. *If an ordinal number α can be sparsely embedded in ${}^\omega\omega$, under identity, as nondecreasing functions, then α can be sparsely embedded in ${}^\omega\omega$, under any function f that converges to ∞.*

PROOF. Let $h < f$ be a nondecreasing function such that $h(n+1) - h(n) \leq 1$. Let $(g_\beta)_{\beta<\alpha}$ be an embedding of α. Define $(f_\beta)_{\beta<\alpha}$ by: $f(\alpha)(k) = g_\alpha(n)$ if and only if $h(k) = n$. ⊣

FACT 3.8. *If an ordinal number α can be sparsely embedded in ${}^\omega\omega$ as nondecreasing functions that are less than any function $f \in {}^\omega\omega$, then α can be sparsely embedded as nondecreasing functions between any sparse pair of functions $g <^* h \in {}^\omega\omega$.*

PROOF. Without loss of generality, we assume f to be nondecreasing. Let $(f_\beta)_{\beta<\alpha}$ be an embedding of α under f. Clearly, it suffices to prove that there is an embedding under s defined by $s(n) = h(n) - g(n)$ if $h(n) \geq g(n)$ and $s(n) = 0$ otherwise.

Define a sequence $(k(n))_{n<\omega}$ by $k(0) = \min\{m : s(i) \geq f(0) \text{ for all } i \geq m\}$, $k(n+1) = \min\{m : m > k(n) \;\&\; s(i) \geq f(n+1) \text{ for all } i \geq m\}$. Finally define g_α as follows: $g_\alpha(n) = f_\alpha(m)$ if and only if $k(n) \leq m < k(n+1)$. ⊣

FACT 3.9. *For each $\gamma < \mathfrak{b}^+$, there exists a strictly $<^*$-increasing sparse sequence $\mathcal{F} = (f_\alpha)_{\alpha<\gamma} \subset {}^\omega\omega$ of nondecreasing functions.*

PROOF. Facts 3.6 and 3.8 clearly imply that the first ordinal number which cannot be embedded as a sparse sequence in ${}^\omega\omega$ under id_ω is equal to α or to $\alpha + 1$ where α is a limit number. Facts 3.6 and 3.8 also imply that the set of ordinals less than α is closed under $\mathfrak{b}$ sums.

Indeed, let β be the minimal ordinal number $< \mathfrak{b}^+$ that may not be embedded under identity as an $<^*$-increasing sparse sequence. Clearly $\mathrm{cof}(\beta) \leq \mathfrak{b}$. Take an increasing sequence $(\alpha_\delta)_{\delta<\mathrm{cof}\,\beta}$ that converges to β. Clearly for each $\alpha < \beta$ there is $(g_\eta^\alpha)_{\eta<\alpha}$—an embedding of α into ${}^\omega\omega$ as a sparse sequence under identity. By Fact

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3.8 for each $\alpha < \beta$ there is an $<^*$-increasing sparse sequence of $(f^\alpha_\eta)_{\eta<\alpha}$ such that $f_\alpha <^* f^\alpha_\eta <^* f_{\alpha+1}$ (for $f_\alpha, f_{\alpha+1}$ from the proof of Fact 3.6). Now $(f^\alpha_\eta)_{\alpha<\mathrm{cof}(\beta),\eta<\alpha}$ with lexicographic order is a required embedding of β.

Thus, this number is not less than $\mathfrak{b}^+$. ⊣

THEOREM 3.10 ($\mathfrak{b} = \mathfrak{c}$). *For each $\gamma < \mathfrak{b}^+$, for each P-point p there exists an RK-increasing sequence $\{p_\alpha : \alpha < \gamma\}$ of P-points such that $p_0 = p$.*

PROOF. Note that $\mathrm{cof}(\gamma) \leq \mathfrak{b}$. Consider a set of pairwise disjoint trees T_n such that each T_n has a minimal element, each element of T_n has exactly n immediate successors, and each branch has the highest ω.

Let $\{f_\alpha\}_{\alpha<\gamma} \subset {}^\omega\omega$ be a sparse, strictly $<^*$-increasing sequence, the existence of which is demonstrated by Fact 3.9. For each $\alpha < \gamma$, define

$$X_\alpha = \bigcup_{n<\omega} \mathrm{Level}_{f_\alpha(n)} T_n.$$

For each $\alpha < \beta \leq \gamma$, define

$$f^\beta_\alpha : \bigcup_{\{n<\omega : f_\alpha(n) < f_\beta(n)\}} \mathrm{Level}_{f_\beta(n)} T_n \to \bigcup_{\{n<\omega : f_\alpha(n) < f_\beta(n)\}} \mathrm{Level}_{f_\alpha(n)} T_n$$

that agrees with the order of trees T_n for $n < \omega$ such that $f_\alpha(n) < f_\beta(n)$. Note that $\mathrm{dom}\, f^\beta_\alpha$ is cofinite on X_β for each $\alpha < \beta$.

Let $p = p_0$ be a P-point on $X_0 = \bigcup_{n<\omega} \mathrm{Level}_0 T_n$. We proceed by recursively building a filter p_β on X_β. Suppose that p_α are already defined for $\alpha < \beta$. If β is a successor, then it suffice to repeat a proof of Theorem 3.1 for $P_{\beta-1}$ and $f^\beta_{\beta-1}$.

So suppose that β is limit. Let $R \subset \beta$ be cofinite with β and of order type less than or equal to $\mathfrak{b}$. Define a family

$$\mathcal{C} = \bigcup_{\alpha\in R} \{(f^\beta_\alpha)^{-1}(p_\alpha)\},$$

which is obviously strongly centered.

Clearly each filter that extends $\mathcal{C}$ is RK-greater than each p_α for $\alpha < \beta$. But we need a P-point extension. Thus, by Corollary 2.6 it suffices to prove that $\mathcal{C}$ is a P^+-family. Thus, by Theorem 2.5 it suffices to prove that $\bigcup_{\gamma\in R, \gamma\leq\alpha} \{(f^\beta_\gamma)^{-1}(p_\gamma)\} \subset (f^\beta_\alpha)^{-1}(p_\alpha)$ is a P^+-family, for each $\alpha \in R$. But it is (an easier version of) what we did in the proof of Theorem 3.1. ⊣

By Theorems 3.5 and 3.10 the following natural question arises:

QUESTION 3.11. *What is the least ordinal α such that there exists an unbounded embedding of α into the set of P-points?*[5]

A. Blass [1, Theorem 8] also proved that, under $\mathfrak{p} = \mathfrak{c}$, there is an order-embedding of the real line into the set of P-points. We will prove the same fact, but under $\mathfrak{b} = \mathfrak{c}$. Our proof is based on the original idea of set X defined by A. Blass. Therefore, we quote the beginning of his proof, and then use our machinery.

[5]A recent paper by D. Raghavan and J. L. Verner [9] showed that under $\diamondsuit$, the cardinal ω_1 is the answer, but we still do not know the answer in terms of cardinal invariants which, as we suppose, play an important role in this domain.

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THEOREM 3.12 ($\mathfrak{b} = \mathfrak{c}$). *There exists an order-embedding of the real line into the set of P-points.*

PROOF. ——————— (beginning of quotation) ———————

Let X be a set of all functions $x : \mathbb{Q} \to \omega$ such that $x(r) = 0$ for all but finitely many $r \in \mathbb{Q}$; here $\mathbb{Q}$ is the set of rational numbers. As X is denumerable, we may identify it with ω via some bijection. For each $\xi \in \mathbb{R}$, we define $h_\xi : X \to X$ by

$$h_\xi(x)(r) = \begin{cases} x(r) \text{ if } r < \xi, \\ 0 \text{ if } r \geq \xi. \end{cases}$$

Clearly, if $\xi < \eta$, then $h_\xi \circ h_\eta = h_\eta \circ h_\xi = h_\xi$. The embedding of $\mathbb{R}$ into P-points will be defined by $\xi \to D_\xi = h_\xi(D)$ for a certain ultrafilter D on X. If $\xi < \eta$, then

$D_\xi = h_\xi(D) = h_\xi \circ h_\eta(D) = h_\xi(D_\eta) \leq D_\eta$.

We wish to choose D in such a way that

(a) $D_\xi \ncong D_\eta$ (therefore, $D_\xi < D_\eta$ when $\xi < \eta$), and

(b) D_ξ is a P-point.

Observe that it will be sufficient to choose D such that

(a') $D_\xi \ncong D_\eta$ when $\xi < \eta$ and both ξ and η are rational, and

(b') D is a P-point.

Indeed, (a') implies (a) because $\mathbb{Q}$ is dense in $\mathbb{R}$. If (a) holds, then $D_{\xi-1} < D_\xi$, so D_ξ is a nonprincipal ultrafilter $\leq D$; hence (b') implies condition (b).

Condition (a') means that, for all $\xi < \eta \in \mathbb{Q}$ and all $g : X \to X$, $D_\eta \neq g(D_\xi) = gh_\xi(D_\eta)$. By Theorem 2.7, this is equivalent to $\{x : gh_\xi(x) = x\} \notin D_\eta$, or by our definition of D_η,

$$\{x : gh_\xi(x) = h_\eta(x)\} = h_\eta^{-1}\{x : gh_\xi(x) = x\} \notin D. \tag{a''}$$

We now proceed to construct a P-point D satisfying (a") for all $\xi < \eta \in \mathbb{Q}$ and for all $g : X \to X$; this will suffice to establish the theorem.

——————— (end of quotation) ———————

List all pairs (ξ, η), $\xi < \eta \in \mathbb{Q}$ in the sequence $(\xi_i, \eta_i)_{i<\omega}$. For each $g \in {}^X X$, $\xi < \eta \in \mathbb{Q}$, define $A_{g,\xi,\eta} = \{x \in X : gh_\xi(x) \neq h_\eta(x)\}$, $\mathcal{A}_i = \mathcal{A}_{\xi_i,\eta_i} = \{A_{g,\xi_i,\eta_i} : g \in {}^X X\}$, and $\mathcal{A} = \bigcup_{i<\omega} \mathcal{A}_i$. Clearly, $\mathcal{A}$ is strongly centered.

We claim that $\mathcal{A}$ is a P^+-family.

Indeed, by Theorem 2.5, it suffices to prove that for each $n < \omega$, $\bigcup_{i<n} \mathcal{A}_i$ is a P^+-family. Suppose not and take (by Remark 2.4) the witnesses i_0 and $\int \mathcal{W}$ such that $\int \mathcal{W} \subset \langle \bigcup_{i<i_0} \mathcal{A}_i \cup \widetilde{\mathcal{W}} \rangle$. For each $n < \omega$, consider the condition (S_n):

$$\exists x_n : \forall i < i_0 : \left(x_n \in h_{\xi_i}(\widetilde{W}_1) \ \& \ \mathrm{card}\,(h_{\eta_i}(h_{\xi_i}^{-1}(x_n)) \cap h_{\eta_i}(\widetilde{W}_n)) > n \right).$$

Case 1: S_n is fulfilled for all $n < \omega$. Then, for each $n < \omega$, $j < n$, choose $x_n^j \in \widetilde{W}_n$ such that $h_{\xi_i}(x_n^j) = x_n$ and $h_{\eta_i}(x_n^{j_0}) \neq h_{\eta_i}(x_n^{j_1})$ for $j_0 \neq j_1$. Define $E = \bigcup_{n<\omega} \bigcup_{j \leq n} \{x_n^j\}$. Clearly $E^c \in \int \mathcal{W}$, but $E \not\subset \bigcup_{i<i_0} \bigcup_{g \in \mathcal{G}} (A_{g,\xi_i,\eta_i}) \cup \bigcup_{l<m} W_l$ for any choice of finite family $\mathcal{G} \subset {}^X X$ and for any $m < \omega$.

Case 2: S_n is not fulfilled for some $n_0 < \omega$. Then, there exist functions $\{g_{n,i}\}_{n \leq n_0, i<i_0} \subset {}^X X$ such that $\widetilde{W}_1 \subset \bigcup_{n \leq n_0} \bigcup_{i<i_0} (A_{g_{n,i},\xi_i,\eta_i}) \cup \bigcup_{n \leq n_0} W_n$, i.e., $\int \mathcal{W}\neg$ is not compatible with $\langle \bigcup_{i<i_0} \mathcal{A}_i \cup \widetilde{\mathcal{W}} \rangle$.

Corollary 2.6 completes the proof. ⊣

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The *long line* is defined as $\mathbb{L} = \omega_1 \times (0, 1]$ ordered lexicographically. If $f : Y \to \omega$, then the *support* of f is defined as

$$\operatorname{supp}(f) = \{y \in Y : f(y) \neq 0\}.$$

LEMMA 3.13 ($\mathfrak{b} = \mathfrak{c}$). *For each P-point p, there exists an order-embedding of the real line into the set of P-points above p.*

PROOF. We will combine ideas form proofs of Theorems 3.1 and 3.12 with some new arguments.

Again, let X be a set of all functions $x : \mathbb{Q} \to \omega$ such that $x(r) = 0$ for all but finitely many $r \in \mathbb{Q}$. Since X is infinitely countable, we treat it as ω.

Let p be a P-point on X such that for each $q \in \mathbb{Q}$ and for each $P \in p$ there exists $x \in P$ such that $\max \operatorname{supp}(x) < q$. Let $f \in {}^X X$ be a finite-to-one function such that $\limsup_{x \in P} \operatorname{card}(f^{-1}(x)) = \infty$ for all $P \in p$ and that $\max \operatorname{supp} x < \max \operatorname{supp} f(x)$. Again, we define a family $\mathcal{A}$ as follows: $A \in \mathcal{A}$ if and only if there exist $i < \omega$ and $P \in p$ such that $\operatorname{card}(f^{-1}(x) \setminus A) \leq i$ for each $x \in P$. For each $\xi \in \mathbb{R}$, we again define functions $h_\xi : X \to X$ by

$$h_\xi(x)(r) = \begin{cases} x(r) \text{ if } r < \xi, \\ 0 \text{ if } r \geq \xi. \end{cases}$$

List all rational numbers in ω-sequence $\mathbb{Q} = (q_i)_{i<\omega}$. Let $\mathcal{B}_i = h_{q_i}^{-1}(\mathcal{A} \cup f^{-1}(p))$ and let $\mathcal{B} = \bigcup_{i<\omega} \mathcal{B}_i$.

Our aim is to prove that $\mathcal{B}$ can be extended to such a P-point Q that $h_\xi(Q) \neq h_\eta(Q)$ for each $\xi \neq \eta \in \mathbb{Q}$ (and thus for each $\xi \neq \eta \in \mathbb{R}$). (4)

To this end, we add to $\mathcal{B}$ a family $\mathcal{C}$ defined as follows: list all pairs (ξ, η), $\xi < \eta \in \mathbb{Q}$ in the sequence $(\xi_i, \eta_i)_{i<\omega}$. For each $g \in {}^X X$, $\xi < \eta \in \mathbb{Q}$, define $C_{g,\xi,\eta} = \{x \in X : gh_\xi(x) \neq h_\eta(x)\}$, $\mathcal{C}_i = \mathcal{C}_{\xi_i,\eta_i} = \{C_{g,\xi_i,\eta_i} : g \in {}^X X\}$, and $\mathcal{C} = \bigcup_{i<\omega} \mathcal{C}_i$.

Thus to prove (4), it suffices by Corollary 2.6 to prove that $\mathcal{B} \cup \mathcal{C}$ is a P^+-family. Thus, by Theorem 2.5, in order to prove (4), it suffices to prove that:

$\mathcal{D}_i$ is a P^+-family for $\mathcal{D}_i$ defined for $i < \omega$ as follows:

$$\mathcal{D}_i = \bigcup_{j \leq i} \mathcal{B}_i \cup \bigcup_{j \leq i} \mathcal{C}_i. \tag{5}$$

First, to prove it, we notice that $\mathcal{D}_i$ is strongly centered. Indeed, define $q_m = \min\{q_j : j \leq i\}$, $q_M = \max\{q_j : j \leq i\}$, and $\xi_m = \min\{\xi_j : j \leq i\}$ and note that $h_{q_M}^{-1}(x) \subset h_{q_j}^{-1}(x)$ for each $j \leq i$ and for each x such that $\max \operatorname{supp}(x) < \min\{q_m, \xi_m\}$. It is easy to see that $h_{q_M}^{-1}(x) \cup \bigcup_{j \leq i} \mathcal{C}_i$ is strongly centered, hence $\mathcal{D}_i$ is strongly centered.

Fix i, and suppose that (5) does not hold. So take a witness $\int \mathcal{W}$. From Remark 2.4, without loss of generality, we may assume that $\int \mathcal{W} \subset \langle \mathcal{D}_i \cup \widetilde{\mathcal{W}} \rangle$.

Let $A_W \in \mathcal{A}$, $P_W \in p$, $n_W \in \omega$, $C_n^W \in \mathcal{C}$ for $n \leq n_W$, and $l_W \in \omega$. Define

$$W^*(A_W, P_W, C_1^W, \ldots, C_{n_W}^W, \widetilde{W_{l_W}}) = \bigcap_{n \leq n_W} C_n^W \cap \bigcap_{j \leq i} h_{\xi_j}^{-1}(A_W \cap P_W) \cap \widetilde{W_{l_W}}.$$

Define $W \in \mathcal{W}^-$ if and only if $W \in \int \mathcal{W}$ and W is co-finite or empty on each W_i. We will say that a set $W \in \mathcal{W}$ is *attainable* (by (n_A, P_W, n_W, l_W)) if there exist $A_W \in \mathcal{A}_{n_A}$, $\{C_k^W \in \bigcup_{j \leq i} \mathcal{C}_j : k \leq n_W\}$ such that the condition

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$W^*(A_W, P_W, C_1^W, \dots, C_{n_W}^W, \widetilde{W_{l_W}}) \subset W$ is satisfied. The complement (to $\widetilde{W_1}$) of the attainable set is called *removable* and sometimes we indicate which variables, sets, functions.

Since $\int \mathcal{W} \subset \langle \mathcal{D}_i \rangle$, thus each set $W \in \mathcal{W}^-$ is attainable.

Consider a sequence of possibilities:

1) l cannot be fixed, i.e., for each $l \in \omega$ there exists $W \in \mathcal{W}^-$ such that W is not removable by any (n_A, P_W, n_W, l);
2) l can be fixed, but n_A cannot, i.e., for each $W \in \mathcal{W}^-$, W is attainable by some (n_W^A, P_W, n_W^C, l), but for each n_A there exists $W' \in \mathcal{W}^-$ such that W' is not attainable by any $(n^A, P_{W'}, n_{W'}^C, l)$;
3) l and n^A can be fixed, but n_C cannot;
4) l, n^A and n^C can be fixed, but P cannot;
5) l, n^A, n^C, and P can be fixed.

Note that each set $W \in \mathcal{W}^-$ is attainable if and only if an alternative of cases 1) to 5) holds.

In case 1) for each l, let $W_l \subset \widetilde{W_l}$ and $W_l \in \mathcal{W}^-$ be a witness that l may not be fixed. Note that $\bigcup_{l<\omega} W_l \in \mathcal{W}^-$ and that $\bigcup_{l<\omega} W_l$ may not be removed by any (n_W^A, P_W, n_W^C, l_W).

In case 2) we proceed like in case 1). Note that if $l' \geq l$ and $n^{A'}$ and a set $W \in \mathcal{W}^-$ is not removable by any $(n^{A'}, P_W, n_W^C, l')$ then the set W is also not removable by any (n^A, P_W, n_W^C, l). Thus it suffices to consider cases when $l = n^A$. For each l, let $W_l \subset \widetilde{W_l}$ and $W_l \in \mathcal{W}^-$ be a witness that l and $n^A = l$ may not be fixed. Again note that $\bigcup_{l<\omega} W_l \in \mathcal{W}^-$ and that $\bigcup_{l<\omega} W_l$ may not be removed by any (n_W^A, P_W, n_W^C, l_W).

In case 3) we proceed just like in case 2), not using that n^A is fixed.

In case 4) for $k < \omega$, let X_k be the set of those $x \in X$ that for all $U \subset f^{-1}(x)$ such that $\operatorname{card}(f^{-1}(x) \setminus U) \leq n^A$, for all partitions of a set $\bigcup_{j=1}^{i}(\xi_j^{-1}(X) \cap \widetilde{(W_k)})$ on the sets $X_{m,n}$, for $m < n \leq i$, there exist m_0, n_0 such that $m_0 < n_0 \leq i$ and there exist $x_1, \dots, x_{n^C+1} \in X_{m_0,n_0}$ that for $\xi_{(min)} = \min\{\xi_{m_0}, \xi_{n_0}\}$, $\xi_{(max)} = \max\{\xi_{m_0}, \xi_{n_0}\}$ there is $\xi_{(min)}(x_r) = \xi_{(min)}(x_j)$ for $r, j \leq n^C + 1$ and $\xi_{(max)}(x_r) \neq \xi_{(max)}(x_j)$ for $r, j \leq n^C + 1$, $r \neq j$.

Clearly, (X_k) is a decreasing sequence. If there exists k such that $X_k \notin p$ then putting $l = k$ there exists a set $P = (X_k)^c$ such that all $W \in \mathcal{W}^-$ may be attained by (n^A, P, n^C, l) so we would be in case 5), so, without loss of generality, $X_k \in P$ for each $k < \omega$.

Thus take a partition of X by $(X_k \setminus X_{k+1})$. Since p is a P-point, and since $X_k \in p$ thus there exists $P_0 \in p$ such that $P_k = P_0 \cap (X_k \setminus X_{k+1})$ is finite for all $k < \omega$. For each $x \in P_k$ there exists a finite(!) set $K_{k,x} \subset \widetilde{(W_k)} \cap \bigcup_{j \leq i} h_{\xi_i}^{-1}$ that may not be removed by (n^A, X, n^C, k). (The proof that $K_{k,x}$ may be chosen finite is analogical, but easier, to that of case 5)). Take $K = \bigcup_{k<\omega, x \in P_k} K_{k,x}$ and notice that $\widetilde{(W_1)} \setminus K \in \mathcal{W}^-$ and that $\widetilde{(W_1)} \setminus K$ is not removable by (n^A, P, n^C, l) for any $P \in p$.

In case 5) arrange $\bigcup_{j \leq i} h_{\xi_j}^{-1}(P) \cap \widetilde{W_l}$ into a sequence $(x_k)_{k<\omega}$. Let $R(x) = \binom{\operatorname{card}(f^{-1}(x))}{\operatorname{card}(f^{-1}(x)) - n^C}$, where $\binom{n}{k}$ denotes a binomial coefficient, and let $(A_{x,r})_{r \leq R}$ be a sequence of all subsets of $f^{-1}(x)$ of cardinality equal to $\operatorname{card}(f^{-1}(x) - n^C)$.

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Consider a tree T, where the root is $\emptyset$ and on a level k the nodes are pairs of natural numbers j, r such that $j \leq i$ and $r \leq R(f(x_k))$ and, for each branch $\tilde{T}$ of T, $\pi_2(\tilde{T}(k_1)) = \pi_2(\tilde{T}(k_2))$ if $f(x_{k_1}) = f(x_{k_2})$, where $\tilde{T}(k)$ is an element of level k of a branch $\tilde{T}$ and π_2 is a projection on the second coordinate. We see j as a choice to which class $\mathcal{C}_j$ does a set $C^{(.)}_{(.)}$ belongs and we see r as a choice of one of sets $A_{f(x),r}$ that $C^{(.)}_{(.)}$ together with $A_{f(x),r}$ removes x_k.

Clearly, the complements of all finite sets belong to $\int \mathcal{W}$, so each finite set is removable. (6)

The maximal element of the branch $\tilde{T}$ has no successors if and only if there is $j \leq i$ such that there is no n sets in $\mathcal{C}_j$ that remove all x_t such that $\tilde{T}(k) = j$ and $f(x_k) \in A_{f(x_k),\pi_2(\tilde{T}(k))}$. It implies that the set $\{x_k : \tilde{T}(k) = j, f(x_k) \in A_{f(x_k),\pi_2(\tilde{T}(k))}\}$ contains more than n different elements, say $x^1, \dots, x^{n+1}$, such that $h_{\xi_j}(x_{s_1}) = h_{\xi_j}(x_{s_2})$ and $h_{\eta_j}(x_{s_1}) \neq h_{\eta_j}(x_{s_2})$ for $s_1 \neq s_2$, $s_1, s_2 \in \{1, \dots, n+1\}$.

By König Lemma if all branches are finite, then the height of the tree T is finite, and so there are irremovable finite sets in contrary to (6). Thus there is infinite branch and the whole set $\bigcap_{j \leq i} h^{-1}_{\xi_i}(P) \cap \widetilde{W}_l$ is removable. $\dashv$

As an immediate consequence of Lemma 3.13 (with the use of Theorem 3.5) we have the following:

THEOREM 3.14. ($\mathfrak{b} = \mathfrak{c}$). *For each P-point p, there exists an order embedding of the long line into the set of P-points above p.*

REMARK 3.15. Note that there is a potential chance to improve Theorem 3.14 in the virtue of Question 3.11, i.e., if, in some model, for each $\alpha < \kappa$ (for some cardinal invariant κ) each RK-increasing α -sequence of P-points is upper bounded by a P-point, then (in that model) if $\mathfrak{b} = \mathfrak{c}$, then, above each P-point, there is an order embedding of a κ-long-line into the set of P-points.

§4. Cardinal $\mathfrak{q}$. An inspection of our proofs indicates a possibility of refinement of most results with the aid of an, *a priori*, new cardinal invariant. We define $\mathfrak{q}$ to be the minimal cardinality of families $\mathcal{B}$, for which there exists a family $\mathcal{A}$ such that $\langle \mathcal{A} \cup \mathcal{B} \rangle$ includes a contour, and $\langle \mathcal{A} \cup \mathcal{C} \rangle$ includes no contour for every countable family $\mathcal{C}$.[6]

If $\mathbb{P}$ is a collection of families such that $\mathcal{P} \in \mathbb{P}$ whenever $\langle \mathcal{P} \rangle$ includes a contour, then $\mathfrak{q}$ fulfills

$$\mathfrak{q} = \min \{\mathrm{card}\,(\mathcal{B}) : \underset{\mathcal{A}}{\exists}\, \mathcal{A} \cup \mathcal{B} \in \mathbb{P} \wedge \underset{\mathcal{C}}{\forall} \big(\mathrm{card}\,(\mathcal{C}) \leq \aleph_0 \Longrightarrow \mathcal{A} \cup \mathcal{C} \notin \mathbb{P}\big)\}.$$

Each contour has a base of cardinality $\mathfrak{d}$, which, by the way, is the minimal cardinality of bases of contours [13, Theorem 5.2]. Therefore, taking into account Theorem 2.5, we have

THEOREM 4.1. $\mathfrak{b} \leq \mathrm{cof}\,(\mathfrak{q}) \leq \mathfrak{q} \leq \mathfrak{d}$.

[6]We may express this in terms of P^+-families: $\mathfrak{q}$ is the minimal cardinality of families $\mathcal{B}$, for which there exists a P^+-family $\mathcal{A}$ such that $\langle \mathcal{A} \cup \mathcal{B} \rangle$ includes a contour.

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Using the cardinal $\mathfrak{q}$, we are in a position to formulate stronger versions, if $\mathfrak{b} < \mathfrak{q}$ is consistent, of several of our theorems with almost unchanged proofs. Indeed, by the proof of Theorem 3.1 we get the following theorem:

THEOREM 4.2 ($\mathfrak{q} = \mathfrak{c}$). *For each P-point p there exists a P-point q strictly RK-greater than p.*

By the proof of Theorem 3.5, we have

THEOREM 4.3 ($\mathfrak{q} = \mathfrak{c}$). *If $(p_n)_{n<\omega}$ is an RK-increasing sequence of P-points, then there exists a P-point u such that $u >_{RK} p_n$ for each $n < \omega$.*

By the proof of Theorem 3.10, we get

THEOREM 4.4 ($\mathfrak{q} = \mathfrak{c}$). *For each P-point p, for each $\alpha < \mathfrak{b}^+$, there exists an order embedding of α into P-points above p.*

By the proof of Theorem 3.12, we obtain

THEOREM 4.5 ($\mathfrak{q} = \mathfrak{c}$). *Above each P-point, there exists an order-embedding of the real line in the set of P-points.*

By the proof of Theorem 3.14, we have

THEOREM 4.6 ($\mathfrak{q} = \mathfrak{c}$). *Above each P-point, there exists an order embedding of the long line into the set of P-points.*

A relative importance of the facts formulated above depends on answers to the following quest.

QUESTION 4.7. *Is $\mathfrak{q}$ equal to any already defined cardinal invariant? Is $\mathfrak{b} < \mathfrak{q}$ consistent? Is $\mathfrak{q} < \mathfrak{d}$ consistent?*

§5. Variants of invariants. The cardinal $\mathfrak{q}$ can be seen as an instance of cardinal invariants, which can possibly be defined in order to refine certain types of theorems, by scrutinizing the mechanisms underlying their proofs. In our approach, such cardinals represent "distances" between certain classes of objects. They carry some obvious questions about their relation to the usual cardinal invariants, and in particular to those that they are supposed to replace in potentially refined arguments.

Let $\mathbb{S}$ and $\mathbb{T}$ be collections of families (of sets or functions, or possibly other objects) such that for each $\mathcal{S} \in \mathbb{S}$ there exists $\mathcal{T} \in \mathbb{T}$ such that $\mathcal{S} \subset \mathcal{T}$. For each $\mathcal{S} \in \mathbb{S}$, we define

$$\mathfrak{dist}(\mathcal{S}, \mathbb{T}) = \min\{\mathrm{card}\,(\mathcal{B}) : \mathcal{S} \cup \mathcal{B} \in \mathbb{T}\}.$$

Let $\mathfrak{D}(\mathbb{S}, \mathbb{T}) = \{\mathfrak{dist}(\mathcal{S}, \mathbb{T}) : \mathcal{S} \in \mathbb{S}\}$. As a set of cardinal numbers, $\mathfrak{D}(\mathbb{S}, \mathbb{T})$ is well ordered, hence we can define $\mathfrak{dist}_\beta(\mathbb{S}, \mathbb{T})$ to be the β-th element of $\mathfrak{D}(\mathbb{S}, \mathbb{T})$.[7]

[7] Equivalently, $\mathfrak{dist}_\alpha(\mathbb{S}, \mathbb{T})$ can be defined recursively by $\mathfrak{dist}_0(\mathbb{S}, \mathbb{T}) = 0$, and if $\mathfrak{dist}_\beta(\mathbb{S}, \mathbb{T})$ is already defined for all $\beta < \alpha$, then $\mathfrak{dist}_\alpha(\mathbb{S}, \mathbb{T})$ is equal to

$$\min\left\{\mathrm{card}\,(\mathcal{B}) : \underset{\mathcal{S}\in\mathbb{S}}{\exists}\left[\mathcal{S} \cup \mathcal{B} \in \mathbb{P} \wedge \underset{\mathcal{C}}{\forall}\,\underset{\beta<\alpha}{\forall}\left(\mathrm{card}\,(\mathcal{C}) \leq \mathfrak{dist}_\beta(\mathbb{S}, \mathbb{T}) \Longrightarrow \mathcal{S} \cup \mathcal{C} \notin \mathbb{P}\right)\right]\right\}.$$

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Moreover, if α is a limit ordinal, and the cardinals $\mathfrak{dist}_\beta(\mathbb{S},\mathbb{T})$ are defined for all $\beta<\alpha$, then $\mathfrak{dist}_{<\alpha}(\mathbb{S},\mathbb{T})=\sup_{\beta<\alpha}\mathfrak{dist}_\beta(\mathbb{S},\mathbb{T})$.

In particular, if $\mathbb{S}$ denotes the collection of compatible with a contour families of subsets of ω, and $\mathbb{T}$ stands for the collection of families including a subbases of a contour, then we write $\mathfrak{q}_\alpha=\mathfrak{dist}_\alpha(\mathbb{S},\mathbb{T})$. In order to show that $(\mathfrak{q}_\alpha)_\alpha$ are variants of $\mathfrak{q}$, we need the following Alternative Theorem [4, Theorem 3.1]. A relation $A\subset\{(n,k):n<\omega,k<\omega\}$ is called *transversal* if A is infinite, and $\{l:(n,l)\in A\}$ and $\{m:(m,k)\in A\}$ are at most singletons for each $n,k<\omega$. ⊣

THEOREM 5.1. *Let $(\mathcal{F}_n)_n$ and $(\mathcal{G}_k)_k$ be sequences of filters on a set X, and let*

$$\mathcal{F}:=\bigcup_{m<\omega}\bigcap_{n>m}\mathcal{F}_n,\ \textit{and}\ \mathcal{G}:=\bigcup_{m<\omega}\bigcap_{k>m}\mathcal{G}_k.$$

If $\mathcal{F}$ is compatible with $\mathcal{G}$, then the following alternative holds :
$\mathcal{F}_n$ is compatible with $\mathcal{G}_k$ for a transversal set of (n,k), or
$\mathcal{F}$ is compatible with $\mathcal{G}_k$ for infinitely many k, or
$\mathcal{F}_n$ is compatible with $\mathcal{G}$ for infinitely many n.

PROPOSITION 5.2. $\mathfrak{q}_0=0$, $\mathfrak{q}_1=1$, $\mathfrak{q}_2=\aleph_0$, $\mathfrak{q}_3=\mathfrak{q}$, *and* $\mathfrak{q}_\alpha\le\mathfrak{d}$ *for all* α.

PROOF. By taking $\mathcal{S}\in\mathbb{T}$ and $\mathcal{B}=\emptyset$, we infer that $\mathfrak{q}_0=0$.

To see that $\mathfrak{q}_1=1$, let A, B be disjoint countably infinite sets. Let $\mathcal{S}_A$ be a contour on A and let $\mathcal{S}_B$ be a cofinite filter on B. Define a filter $\mathcal{S}$ on $A\cup B$ by $S\in\mathcal{S}$ if and only if $S\cap A\in\mathcal{S}_A$ and $S\cap B\in\mathcal{S}_B$. Clearly $\mathcal{S}$ is not finer than a contour (since is RK-smaller than a cofinite filter $\mathcal{S}_B$), and $\mathcal{S}\cup\{A\}$ is a subbase of a contour.

Clearly, $\mathfrak{q}_2$ cannot be finite. To see that $\mathfrak{q}_2=\aleph_0$, let $\mathcal{W}=(W_n)_{n<\omega}$ be a partition of ω into infinite sets. We define a family $\mathcal{S}$ so that $S\in\mathcal{S}$ if and only if S is cofinite on each W_n. Suppose that there is a partition $\mathcal{V}=(V_n)_{n<\omega}$ such that $\int\mathcal{V}\subset\langle\mathcal{S}\cup S_0\rangle$ for some set S_0 such that $\mathcal{S}\cup S_0$ is strongly centered. Let $N_{fin}=\{n<\omega:\operatorname{card}(W_n\cap S_0)<\omega\}$ and $N_\infty=\omega\setminus N_{fin}$. Define $S_{fin}=S_0\cap\bigcup_{n\in N_{fin}}W_n$ and $S_\infty=S_0\cap\bigcup_{n\in N_\infty}W_n$. Since $S_{fin}^c\in\mathcal{S}$, without loss of generality, we can assume that $S_0=S_\infty$ and so without loss of generality we can assume that $S_0=\omega$.

Note also that $\bigcup_{n<\omega}V_n\cap W_i$ is infinite for infinitely many i, and so $\int\mathcal{W}$ is compatible with $\int\mathcal{V}$. Thus we meet the assumptions of Theorem 5.1, and in each of the three cases there exist $i,j<\omega$ such that $V_i\cap W_j$ is infinite. But $\omega\setminus V_i\in\int\mathcal{V}$ and thus $\omega\setminus(V_i\cap W_j)\in\mathcal{S}$, contrary to the definition of $\mathcal{S}$. On the other hand, by adding $\widetilde{\mathcal{W}}$ to $\mathcal{S}$, we obtain a subbase of $\int\mathcal{W}$.

That $\mathfrak{q}_3=\mathfrak{q}$ follows directly from the definition of $\mathfrak{q}$.

Finally, $\mathfrak{q}_\alpha\le\mathfrak{d}$ since each contour has a base of cardinality $\mathfrak{d}$, as we have shown in [13, Theorem 5.2].

If $\mathbb{S}$ is the collection of strongly centered families, but $\mathbb{T}$ is the collection of free ultrafilter subbases, then, taking $\mathcal{S}$ as an empty family, clearly $\mathfrak{dist}(\mathcal{S},\mathbb{T})=u$ and so $\mathfrak{dist}_\alpha(\mathbb{S},\mathbb{T})=\mathfrak{u}$ for some α, thus we obtain variants of $\mathfrak{u}$. By Fact 3.2,

FACT 5.3. $\mathfrak{u}_0=0$, $\mathfrak{u}_1=1$, $\mathfrak{u}_2\ge\aleph_1$.

By the proof of Theorem 3.4, we obtain:

THEOREM 5.4 ($\mathfrak{q}=\mathfrak{u}_2=\mathfrak{c}$). *If p is a P-point, then there is a set $\mathfrak{U}$ of Rudin–Keisler incomparable P-points such that* $\operatorname{card}\mathfrak{U}=\mathfrak{c}$ *and $u>_{RK}p$ for each $u\in\mathfrak{U}$.*

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A similar approach can be carried out for all other cardinal invariants. Its usefulness, however, depends on the way these cardinals are used in specific arguments.

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