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ON THE WEDDERBURN THEOREM

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1. Introduction. In [**6**], Pierce studied the modules over a commutative regular ring R by using the representation of R as the global sections of a sheaf which we call the Pierce sheaf. When the stalks of the Pierce sheaf are regular, Magid gave a Galois theory and some properties for a central separable R-algebra [**4**, (2.4), (2.5), (2.6) and (2.7)]. When the stalks of the Pierce sheaf are semi-local, DeMeyer presented a Galois theory for a central separable R-algebra [**3**, sections 2 and 3] and the author characterized the finitely generated and projective modules over a central separable R-algebra in terms of the R-modules in [**7**] and [**8**]. By keeping the same assumption on R, the present paper gives a structural theorem for a finitely generated and projective R-module and extends the Wedderburn theorem in the context of a connected ring as given in [**2**] by DeMeyer. The author wishes to thank Professor DeMeyer for his suggestion of Theorem 3.4 and also to thank the referee for his proof of Theorem 3.1.

2. Preliminaries. Throughout the present paper, R is assumed to be a commutative ring with identity. We first describe the Pierce sheaf in [**6**]. Let $B(R)$ denote the Boolean algebra of idempotents of R, and let Spec $B(R)$ be the set of all prime ideals in $B(R)$ (hence they are maximal). For any e in $B(R)$, denote the set $\{x|x \text{ in Spec} B(R) \text{ and } (1 - e) \text{ in } x\}$ by U_e. Then $\{U_e\}$ form the basic open sets of a topology imposed on Spec $B(R)$. Spec $B(R)$ is a totally disconnected compact and Hausdorff topological space. Let $R_x = R/xR$ for x in Spec $B(R)$. Then a sheaf is defined whose base space is Spec $B(R)$ and whose stalks are R_x. Furthermore, R is isomorphic with the global sections of this sheaf. Also, denote $R_x \otimes_R M$ by M_x for an R-module M. Some facts given in [**9**] will be used in this paper, so they are listed below:

(1.A) For e in $B(R)$, the homomorphism $R \to Re$ establishes a homeomorphism $U_e \to \text{Spec}\, B(Re)$. Thus, if M is a finitely generated R-module satisfying $M_x = 0$ for all x in U_e, we may conclude from (2.11) in [**9**] that $eM = 0$.

(1.B) Let $\{U_e\}$ be an open cover of Spec $B(R)$. Then by compactness of Spec $B(R)$, there is a finite subcover $\{U_{e_i}, i = 1, 2, \ldots, n$ for some integer $n\}$ such that $\{U_{e_i}\}$ are disjoint. Thus $\{e_i\}$ are orthogonal and $\sum_i e_i = 1$ (see [**9**, (2.10) and the end of section 3]).

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3. Projective modules. In this section, a structural theorem for a finitely generated and projective R-module is proved; also a relation between the finitely generated projective property and the indecomposable property of an R-module is given.

THEOREM 3.1. *Assume R_x has the property that its projective modules are free for each x in* Spec $B(R)$. *Let M be a finitely generated and projective R-module. Then there is a decomposition of R, $R \cong \oplus \sum_{i=1}^{n} Re_i$ for some orthogonal idempotents e_i and an integer n such that Me_i is a free Re_i-module for each i.*

Proof. Since M is a finitely generated and projective R-module, there exists an R-module P such that $M \oplus P \cong R^n$, a free R-module of rank n for some integer n. Let $a_1, \ldots, a_n$ be a set of free generators of R^n, considered as global sections. For a point x of Spec $B(R)$, choose free generators $b_{1x}, \ldots, b_{rx}$ of M_x, and $b_{r+1x}, \ldots, b_{nx}$ of P_x. The b_{jx} can be considered as the values of local sections at x. Then it is possible to write

$$(*) \qquad a_{ix} = \sum_{j=1}^{n} r_{ijx} b_{jx}$$

where the matrix $[r_{ijx}]$ is invertible. Again, consider the r_{ijx} as the values of local sections at x. By the basic property of sheaves, there is a neighbourhood U_e of x in which (*) holds, and moreover the matrix $[r_{ijy}]$ is invertible for all y in this neighbourhood U_e. This implies that the elements $b_{1y}, \ldots, b_{ry}$ are free generators of M_y for all y's in the given neighbourhood U_e. Thus by using (1.A), we can show that Me is a free Re-module; that is, $Me \cong (Re)^r$. The proof is then completed by application of the partition property (1.B).

COROLLARY 3.2. *Let M be a finitely generated projective and indecomposable R-module. If all of the stalks of R have the property that their projective modules are free, then $M \cong Re$ for some minimal idempotent e in R.*

COROLLARY 3.3. *Assume all of the stalks of R have the property that their projective modules are free. Let M, N be two finitely generated and projective R-modules. Then, $M \cong N$ if and only if* $\text{rank}_{R_x}(M_x) = \text{rank}_{R_x}(N_x)$ *for each x in* Spec $B(R)$.

The following theorem is due to F. DeMeyer. Keeping the above assumption on R, we have:

THEOREM 3.4. *Assume x in* Spec $B(R)$ *is a limit point. If M is a finitely generated and projective R-module with $M_x \neq 0$, then M is decomposable.*

Proof. Since M is a finitely generated and projective R-module, there is a basic open neighbourhood of x, U_e, such that Me is a free Re-module by the proof of Theorem 3.1. On the other hand, by hypothesis, $M_x \neq 0$, so $M_y \neq 0$ for all y in U_e. Noting that x is a limit point of Spec $B(R)$, we have at least one $y \neq x$ in U_e. Spec $B(R)$ is Hausdorff, so there exists a $U_{e'}$ of x such that

y is not in $U_{e'}$; and hence y is in $U_{1-e'}$. But then Spec $B(R)$ is covered by $\{U_{e'}, U_{1-e'}\}$ and

$$R \cong Re' \oplus R(1 - e'),$$

where Spec $B(Re') \cong U_{e'}$ and Spec $B(R(1 - e')) \cong U_{1-e'}$ by (1.A). This gives a decomposition of M, $M \cong Me' \oplus M(1 - e')$. Since $M_x = (Me')_x \neq 0$ and since $M_y = (M(1 - e'))_y \neq 0$, M is decomposable.

Remark. The technique in the above proof is similar to Theorem 1.3 in [**8**]. Also we note that for some x in Spec $B(R)$ there is no finitely generated projective and indecomposable R-module M with $M_x \neq 0$ (see the example given by D. Zelinsky in [**8**]). Furthermore, it is not hard to see that if for each x in Spec $B(R)$, there is a finitely generated projective and indecomposable R-module M with $M_x \neq 0$, then R is isomorphic with a finite direct sum of connected rings.

COROLLARY 3.5. *By keeping the above assumption on R, there is a one-to-one correspondence between the following sets of elements:*

(a) *The set of all isolated points x in* Spec $B(R)$.

(b) *The set of all minimal idempotents e in R.*

(c) *The set of all classes of the finitely generated projective and indecomposable R-modules M with $M_x \neq 0$.*

Proof. For (a) $\Rightarrow$ (c), it is not hard to see that R_x is a member in (c) if x is in (a) by Lemma 2.10 in [**5**]; (c) $\Rightarrow$ (a) is a consequence of the above theorem. (b) $\Leftrightarrow$ (c) is immediate because the idempotent e in (b) corresponds to M in (c), where $M_x \neq 0$ and $U_e = \{x\}$.

4. A general Wedderburn theorem. In this section, we shall show a general Wedderburn theorem. This is an extension of Corollaries 1 and 2 in [**2**].

THEOREM 4.1. *Assume R_x is semi-local for each x in* Spec $B(R)$. *Let A be a central separable R-algebra. Then there exists a central separable R-algebra D in the same class as A in the Brauer group of R such that*

$$A \otimes_R D^0 \cong \mathrm{Hom}_R(AE, AE)$$

for an idempotent E in A with (AE) a finitely generated projective and faithful R-module and

(1) *$D \cong EAE$ where D^0 is the opposite ring of D;*

(2) *there is a decomposition of R, $R \cong \oplus \sum_{i=1}^{n} Re_i$ for some orthogonal idempotents e_i in R and an integer n so that $A \cong \oplus \sum_{i=1}^{n} Ae_i$, $D \cong \oplus \sum_{i=1}^{n} De_i$ and $Ae_i \cong M_{k_i}(De_i)$, the full matrix ring of degree k_i over De_i for an integer k_i;*

(3) *for each i, $De_i \cong (Ee_i)A(Ee_i)$, and De_i is unique if $(Ee_i)_y$ is a minimal idempotent in $(Ae_i)_y$ for each y in* Spec $B(Re_i)$; *and $(Ee_i)_y$ is a minimal idempotent in $(Ae_i)_y$ for each y in* Spec $B(Re_i)$ *if and only if all idempotents in De_i are in Re_i;*

(4) *there is exactly one isomorphic class of the finitely generated and projective left Ae_i-modules with the same Re_i-rank for each i.*

Proof. (1) Since $A_x(=R_x \otimes_R A)$ is a central separable R_x-algebra and since R_x is semi-local, there exists a unique central separable R_x-algebra D_x' in the same class as A_x in the Brauer group of R_x such that

$$A_x \otimes_{R_x} (D_x')^0 \cong \mathrm{Hom}_{R_x}(A_xE_x', A_xE_x')$$

where E_x' is a minimal left idempotent in A_x and $(D_x')^0$ is the opposite ring of D_x'. Also, $D_x' \cong E_x'A_xE_x'$ [**2**, Corollaries 1 and 2]. By [**4**, (1.6)], we have a central separable R-algebra D such that

$$(*) \qquad D_x \cong D_x' \cong E_x'A_xE_x'.$$

Furthermore, by [**9**, (2.12)], E_x' is lifted to an idempotent E in A so that $E_x = E_x'$. Hence

$$(**) \qquad E_x'A_xE_x' \cong (EAE)_x \text{ and } \mathrm{Hom}_{R_x}(A_xE_x', A_xE_x') \cong (\mathrm{Hom}_R(AE, AE))_x$$

by [**9**, (2.7)]. So, from (*) and (**), $A_x \otimes_{R_x} (D_x)^0 \cong (\mathrm{Hom}_R(AE, AE))_x$ and $D_x \cong (EAE)_x$; that is, $(A \otimes_R D^0)_x \cong (\mathrm{Hom}_R(AE, AE))_x$ and $D_x \cong (EAE)_x$. Thus, by [**4**, (1.7)], there is a basic open neighbourhood $U_{e'}$ of x such that $(A \otimes_R D^0)_y \cong (\mathrm{Hom}_R(AE, AE))_y$ for each y in $U_{e'}$, and there is a basic open neighbourhood of x, $U_{e''}$, such that $D_y \cong (EAE)_y$ for each y in $U_{e''}$. This gives $e'(A \otimes_R D^0) \cong e'(\mathrm{Hom}_R(AE, AE))$ and $e''D \cong e''(EAE)$. Denote the intersection of $U_{e'}$ and $U_{e''}$ by U_e. Then $e(A \otimes_R D^0) \cong e(\mathrm{Hom}_R(AE, AE))$ and $eD \cong e(EAE)$ as Re-algebras. Noting that eAE is a finitely generated projective and faithful Re-module, we have that eA and eD are in the same class in the Brauer group of Re. Let x vary over Spec $B(R)$ and cover Spec $B(R)$ with $\{U_e\}$. By (1.B), we have a finite subcover of Spec $B(R)$ with $\{U_{e_i}, i = 1, \ldots, n$ for some integer $n\}$, where $\{e_i\}$ are orthogonal and summing to 1. Then,

$$R \cong \oplus \sum_{i=1}^{n} Re_i, e_i(A \otimes_R D_i^{\,0}) \cong e_i\mathrm{Hom}_R(AE_i, AE_i) \text{ and } e_iD_i \cong e_i(E_iAE_i)$$

for each i. Consequently, let D be $\oplus \sum_{i=1}^{n} e_iD_i$ and $E = \sum_{i=1}^{n} e_iE_i$. Then $A \otimes_R D^0 \cong \mathrm{Hom}_R(AE, AE)$ and $D \cong EAE$. So, A and D are in the same class of the Brauer group of R.

(2) Since AE and D are finitely generated and projective R-modules from the proof of part (1), there are basic open neighbourhoods of x, $U_{e'}$ and $U_{e''}$, such that $e'D$ and $e''AE$ are free Re' and Re''-modules respectively. Denote the intersection of $U_{e'}$, $U_{e''}$ and U_e in part (1) by U_{e_0}. Then U_{e_0} is a basic open neighbourhood of x so that $e_0(A \otimes_R D^0) \cong e_0 \mathrm{Hom}_R(AE, AE)$ as Re_0-algebras. Noting that $(e_0AE)_y$ and $(e_0D)_y$ are free R_y-modules for each y in U_{e_0}, we have $(e_0\mathrm{D})_y \cong \oplus \sum_{j=1}^{s} (R_y)^j$, s-copies of R_y for some integer s.

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But from the proof of part (1), $(e_0D)_x$ has no proper idempotents. Then $(e_0AE)_x$ is a free $(e_0D)_x$-module [**2**, Theorem 1]; that is,

$$(e_0AE)_x \cong \oplus \textstyle\sum_{j=1}^{k} (e_0D)_x{}^j,$$

k-copies of $(e_0D)_x$ for some integer k; that is,

$$(e_0AE)_x \cong \oplus \sum_{j=1}^{k} (e_0D)_x{}^j \cong \oplus \sum_{j=1}^{k} \left(\oplus \sum_{i=1}^{s} (R_x)^i\right)^j \cong \oplus \sum_{j=1}^{ks} (R_x)^j,$$

ks-copies of R_x. Since e_0AE and e_0D are free Re_0-modules and since $\operatorname{Spec} B(Re_0) \cong U_{e_0}$,

$$(e_0AE)_y \cong \oplus \sum_{j=1}^{ks} (R_y)^j \cong \oplus \sum_{j=1}^{k} (e_0D)_y{}^j$$

for each y in U_{e_0}. Thus $e_0AE \cong \oplus \sum_{j=1}^{k} (e_0D)^j$ as e_0D-modules [**4**, (1.7)]. $e_0A \otimes_{Re_0} e_0D^0 \cong \operatorname{Hom}_{Re_0}(e_0AE, e_0AE)$, so $e_0A \cong \operatorname{Hom}_{e_0D^0}(e_0AE, e_0AE)$ (for e_0A and e_0D are Morita equivalent); and so it is isomorphic with $M_k(e_0D)$, a full matrix ring of degree k over e_0D. Finally, by (1.B) again, we have a decomposition of R, $R \cong \oplus \sum_{i=1}^{n} e_iR$, $A \cong \oplus \sum_{i=1}^{n} e_iA$, $D \cong \oplus \sum_{i=1}^{n} e_iD$ and $e_iA \cong M_{k_i}(e_iD)$, a full matrix ring of degree k_i over e_iD for an integer k_i.

(3) Since $D \cong EAE$, $e_iD \cong (e_iE)A(e_iE)$ for each i. Now, if $(e_iE)_y$ is a minimal idempotent in $(e_iA)_y$, then $(e_iD)_y$ is unique with no proper idempotents by [**2**, Corollaries 1 and 2]. Thus e_iD is unique by [**4**, (1.7)].

Furthermore, if all idempotents in e_iD are in Re_i, then $(e_iD)_y$ has no proper idempotents. But $(e_iD)_y \cong E_yA_yE_y$; then E_y is a left minimal idempotent in A_y for each y in U_{ei} by [**2**, Corollaries 1 and 2]. Conversely, if E_y is a left minimal idempotent in $(e_iA)_y$ for each y in U_{ei}, then $(e_iD)_y \cong E_yA_yE_y$ is unique with no proper idempotents for each y in U_{ei} by the same corollaries. Thus all idempotents in e_iD are in Re_i. This follows because for an idempotent E in e_iD, Re_i is a submodule of $(Re_i + (Re_i)E)$. Noting that

$$(Re_i)_y = (Re_i + (Re_i)E)_y = (Re_i)_y + (Re_iE)_y$$

for each y in U_{ei}, we have that E_y is in $(Re_i)_y$; and hence $Re_i = (Re_i + Re_iE)$ by [**9**, (2.11)]. Thus E is in Re_i.

(4) Since e_iA and e_iD are in the same class in the Brauer group of Re_i; that is, $e_iA \cong \operatorname{Hom}_{e_iD0}(e_iAE, e_iAE)$ from part (2), it suffices to show the statement for the e_iR-central separable algebra e_iD by the Morita theorem. In fact, if M and N are two finitely generated and projective e_iD-modules with the same Re_i-rank, then $M_y \cong N_y$ for each y in U_{ei} as free R_y-modules. Thus $M \cong N \cong \oplus \sum_{j=1}^{k} (e_iD)^j$, k-copies of e_iD for some integer k, by the argument used in part (2). This completes the proof.

COROLLARY 4.2. *There is exactly one isomorphism class of the finitely generated and projective e_iD-modules with minimal rank over Re_i.*

Proof. We observe in fact, that they are isomorphic with e_iD.

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Remark. By [**4**, (1.6)], it is proved that for each central separable R_x-algebra A_x there is a central separable R-algebra D such that $D_x \cong A_x$. But it is not known whether there is a central separable R-algebra D such that $D_x \cong A_x$ and D_y has no proper idempotents for each y in some basic open neighbourhood of x when A_x has no proper idempotents. Suppose there was; then the central separable Re_i-algebra De_i in part (3) of the above theorem would be unique.

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