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THE HESSIAN MAP IN THE INVARIANT THEORY OF REFLECTION GROUPS

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§ 1. Introduction

Let V be a complex vector space of dimension l. Let S be the C-algebra of polynomial functions on V. Let Der_S be the S-module of derivations of S and let $\Omega_S = \mathrm{Hom}_S(\mathrm{Der}_S, S)$ be the dual S-module of differential 1-forms. Let $\{e_i\}$ be a basis for V and let $\{x_i\}$ be the dual basis for V^*. Then $\{D_i = \partial/\partial x_i\}$ and $\{dx_i\}$ are bases for Der_S and Ω_S as S-modules. If $f \in S$, define a map $\mathrm{Hess}(f)\colon \mathrm{Der}_S \to \Omega_S$ by

$$\mathrm{Hess}(f)\colon \theta \longrightarrow \sum \theta(D_i f) dx_i \qquad \theta \in \mathrm{Der}_S . \tag{1.1}$$

Then $\mathrm{Hess}(f)$ is an S-module homomorphism which does not depend on the choice of basis for V. Let $\mathbf{H}(f)$ denote the the matrix of the map $\mathrm{Hess}(f)$ with respect to the pair of bases $\{D_i\}$ and $\{dx_i\}$. Then $\mathbf{H}(f)$ is the usual Hessian matrix of second partial derivatives of f.

Let $G \subset GL(V)$ be a finite unitary reflection group and let $R = S^G$ be the subalgebra of G-invariant polynomials. Both Der_S and Ω_S are G-modules. If $f \in R$ then $\mathrm{Hess}(f)$ induces a homomorphism again denoted $\mathrm{Hess}(f)\colon \mathrm{Der}_S^G \to \Omega_S^G$. If G has a real form, so that G is a Coxeter group, then it has a non-degenerate invariant quadratic form f. Nondegeneracy implies $\det \mathbf{H}(f) \neq 0$. Since $\mathbf{H}(f)$ is a matrix with entries in C it is invertible in $M_l(C)$ so $\mathrm{Hess}(f)\colon \mathrm{Der}_S^G \to \Omega_S^G$ is an isomorphism [15]. The situation is more complicated for unitary reflection groups which do not have a real form. We show that if $\mathrm{Hess}(f)$ is an isomorphism then f is an invariant form of minimal positive degree but this minimality condition is not sufficient. In Theorem (5.10) of this paper we characterize those irreducible unitary reflection groups for which an invariant form f_1 of minimal positive degree induces an isomorphism $\mathrm{Hess}(f_1)\colon \mathrm{Der}_S^G \to \Omega_S^G$.

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Shephard [17, 18] introduced the notion of a regular complex polytope $\mathscr{P}$ and showed that its symmetry group $G = \operatorname{Aut}(\mathscr{P})$ is a finite irreducible unitary reflection group. A regular convex polytope in $\boldsymbol{R}^l$ defines a regular complex polytope in $V = \boldsymbol{C}^l$ by scalar extension; if $\mathscr{P}$ arises in this way we say that $\mathscr{P}$ has a real form. If $\mathscr{P}$ has a real form then G is a finite Coxeter group.

(1.2) DEFINITION. A *Shephard group* is the symmetry group of a regular complex polytope.

Shephard groups have been classified and studied in the work of Shephard [17, 18], Shephard and Todd [19], Coxeter [2, 3, 4] and Koster [10]. There are irreducible unitary reflection groups which are not Shephard groups. In Table 1 we list the Shephard groups which are not Coxeter groups and some relevant information about each group. If $l \geq 3$ our arguments depend in part on this classification. Our main result is:

(1.3) THEOREM. *Let $G \subset GL(V)$ be a Shephard group and let $f_1 \in R$ be a G-invariant form of minimal positive degree. Then the map* $\operatorname{Hess}(f_1)$: $\operatorname{Der}_S^G \to \Omega_S^G$ *is an isomorphism.*

By Chevalley's theorem there exist forms $f_1, \cdots, f_l \in R$ called *basic invariants* such that $R = \boldsymbol{C}[f_1, \cdots, f_l]$. Let $d_i = \deg f_i$ and number the f_i so that $d_1 \leq \cdots \leq d_l$. Let $m_i = d_i - 1$ be the exponents of G. The R-module Ω_S^G is free of rank l with basis $df_1, \cdots, df_l$. The R-module Der_S^G is free of rank l with a basis of homogeneous elements $\theta_1, \cdots, \theta_l$ called *basic derivations*. Let K be the quotient field of R. Since $K \otimes \operatorname{Der}_S^G$ and $K \otimes \Omega_S^G$ are vector spaces over K of the same finite dimension l, it suffices in (1.5) to prove that $\operatorname{Hess}(f_1)$ is surjective. However, our argument uses injectivity in the proof of surjectivity. To prove that $\operatorname{Hess}(f_1)$ is injective it suffices to show that $\det \mathbf{H}(f_1) \neq 0$. The vanishing of the Hessian determinant has an interesting history which we outline briefly in Section 4. To prove that $\operatorname{Hess}(f_1)$ is surjective it suffices, since the df_i generate Ω_S^G as R-module, to prove that $df_i \in im \operatorname{Hess}(f_1)$. Let $\theta_E = \sum x_i D_i$ be the Euler derivation. Since $\theta_E(D_i f_1) = m_1(D_i f_1)$ it follows from (1.1) that $\operatorname{Hess}(f_1)\theta_E = m_1 df_1$. We prove that for given basic invariants $f_1, \cdots, f_l$ there exist basic derivations $\theta_1, \cdots, \theta_l$ such that

$$\operatorname{Hess}(f_1)\theta_i = m_1 df_i \qquad 1 \leq i \leq l\,. \tag{1.4}$$

This forces $\theta_1 = \theta_E$. The assertion (1.4) and hence (1.3) is equivalent to

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a statement about matrices over S. To give the matrix formulation we make the following definitions. If $f_1, \cdots, f_l \in S$ define the Jacobian matrix $\mathbf{J} = \mathbf{J}(f_1, \cdots, f_l)$ by $\mathbf{J}_{ij} = D_i f_j$. If $\theta_1, \cdots, \theta_l \in \mathrm{Der}_S$ define a matrix $\mathbf{Q} = \mathbf{Q}(\theta_1, \cdots, \theta_l)$ by $\mathbf{Q}_{ij} = \theta_j(x_i)$. Thus $\theta_j = \sum \mathbf{Q}_{ij} D_i$. The matrices $\mathbf{J}(f_1, \cdots, f_l)$ and $\mathbf{Q}(\theta_1, \cdots, \theta_l)$ depend on the chosen basis $\{e_i\}$. It will be convenient to use the notation $J(f_1, \cdots, f_l) = \det \mathbf{J}(f_1, \cdots, f_l)$, $Q(\theta_1, \cdots, \theta_l) = \det \mathbf{Q}(\theta_1, \cdots, \theta_l)$ and $H(f) = \det \mathbf{H}(f)$.

(1.5) THEOREM. *Let $G \subset GL(V)$ be a Shephard group and let $f_1, \cdots, f_l$ be a set of basic invariants, with f_1 of minimal positive degree. Then there exist basic derivations $\theta_1, \cdots, \theta_l$ such that*

$$\mathbf{H}(f_1)\mathbf{Q}(\theta_1, \cdots, \theta_l) = m_1 \mathbf{J}(f_1, \cdots, f_l)\,.$$

The equivalence of (1.3) and (1.5) is clear via (1.4). Note that the formula (1.5) contains the assertion $H(f_1) \neq 0$ because $J(f_1, \cdots, f_l) \neq 0$ by the algebraic independence of $f_1, \cdots, f_l$.

If $l = 2$ then our argument in Section 3 yields formulae relating Klein's vertex, face and edge forms of the regular polyhedra which we believe to be new. Let $\mathscr{P}$ be one of the three polyhedra: tetrahedron, octahedron, icosahedron. Let $\varphi_V = \varphi_V(\mathscr{P})$, $\varphi_F = \varphi_F(\mathscr{P})$, and $\varphi_E = \varphi_E(\mathscr{P})$ be the forms which Klein [9, Chapter 2] has associated with the vertices, faces, and edges of $\mathscr{P}$:

$\mathscr{P}$	φ_V	φ_F	φ_E
tetrahedron	Φ	Ψ	t
octahedron	t	W	χ
icosahedron	f	H	T

Klein showed that

$$\varphi_F \approx H(\varphi_V) \tag{1.6}$$

$$\varphi_E \approx J(\varphi_V, \varphi_F)\,. \tag{1.7}$$

Here and throughout this paper the symbol $\approx$ means that the forms are equal up to a nonzero constant multiple. It follows from our results in Section 3 applied to certain Shephard groups in two dimensions that with $q = 3, 4, 5$ for the tetrahedron, octahedron, icosahedron we have

$$\varphi_F^2 \approx J(\varphi_V, \varphi_E) \tag{1.8}$$

$$\varphi_V^{q-1} \approx J(\varphi_F, \varphi_E) \tag{1.9}$$

$$D_i\varphi_E \approx J(D_i\varphi_V, \varphi_F) \qquad i = 1, 2 \tag{1.10}$$

for each of the three polyhedra $\mathscr{P}$. The constant implied by $\approx$ in (1.10) is the same for $i = 1, 2$. Note that (1.7) follows from (1.10) and the Euler formula but that (1.10) is not a formal consequence of (1.7). Klein [9, pps. 62–63] also remarks that "apart from trivial exceptions" the vertex forms φ_V of the three polyhedra are characterized among all binary forms by the vanishing of their fourth transvectant. At the end of Section 3 we discuss the connection between this vanishing property and our work in this paper.

In Section 2 we prove some general lemmas about the matrices $\mathbf{Q}(\theta_1, \cdots, \theta_l)$, $\mathbf{J}(f_1, \cdots, f_l)$ and $\mathbf{H}(f)$. In Section 3 we prove (1.3) for $l = 2$. In Section 4 we prove (1.3) for $l \geq 3$. Our argument in case $l = 2$ does not use the classification. In case $l \geq 3$ it does. In Section 5 we deduce some consequences of (1.3). In particular there is an inclusion of ideals

$$(f_1, \cdots, f_l) \subseteq (D_1 f_1, \cdots, D_l f_1)\,. \tag{1.11}$$

This shows that if G is a Shephard group then an invariant form f_1 of minimal positive degree is nondegenerate. In the sequel [14] we use (1.5) to prove that if G is a Shephard group, then the complement of the union of its reflecting hyperplanes is a $K(\pi, 1)$ space. In [14] we also give, for all Shephard groups, explicit basic derivations $\theta_1, \cdots, \theta_l$ and basic invariants $f_1, \cdots, f_l$ which satisfy the matrix equation of (1.5).

We would like to thank Hiroaki Terao for many interesting discussions on arrangements of hyperplanes, which convinced us that our ideas on $S \otimes V$ are more naturally stated in terms of Der_S.

§ 2. General lemmas

In this section G is any finite irreducible unitary reflection group. In fact it would suffice in (2.1)–(2.18) to assume that G is any subgroup of $GL(V)$. We use the notation of Section 1. Let $\langle x, v\rangle$ denote the natural pairing $V^* \times V \to \boldsymbol{C}$ and let $\langle \omega, \theta\rangle$ denote the natural pairing $\Omega_S \times \mathrm{Der}_S \to S$. If $v \in V$ let $D_v \in \mathrm{Der}_S$ be the derivation defined by $D_v x = \langle x, v\rangle$ for $x \in V^*$. If $f \in S$ then $df \in \Omega_S$ is defined by $\langle df, \theta\rangle = \theta(f)$ for $\theta \in \mathrm{Der}_S$. In terms of the bases $\{D_i = \partial/\partial x_i\}$ and $\{dx_i\}$ we have $df = \sum (D_i f)dx_i$ and $\theta(f) = \sum \theta(x_i)D_i f$.

The spaces S, Der_S, and Ω_S have G-module structures. We define the G-actions and list some transformation formulas. Let $g \in G$, $v \in V$, $a \in S$,

$\theta \in \mathrm{Der}_S$, and $\omega \in \Omega_S$. The G-module structure in S is defined by

$$(2.1) \qquad (ga)(v) = a(g^{-1}v) .$$

It follows that

$$(2.2) \qquad g(D_v a) = D_{gv}(ga)$$

$$(2.3) \qquad d(ga) = g(da) .$$

The G-module structure in Der_S is defined by

$$(2.4) \qquad (g\theta)(a) = g(\theta(g^{-1}a)) .$$

It follows that

$$(2.5) \qquad gD_v = D_{gv}$$

$$(2.6) \qquad g(a\theta) = (ga)(g\theta) .$$

The G-module structure in Ω_S is defined by

$$(2.7) \qquad (g\omega)(\theta) = g(\omega(g^{-1}\theta)) .$$

It follows that

$$(2.8) \qquad g(a\omega) = (ga)(g\omega)$$

$$(2.9) \qquad \langle g\omega, g\theta\rangle = g\langle\omega, \theta\rangle .$$

We give $M_l(S)$ the G-module structure

$$(2.10) \qquad (g\mathbf{P})_{ij} = g(\mathbf{P}_{ij}) \qquad \mathbf{P} \in M_l(S) .$$

If $g \in G$ let $[g]$ denote the matrix for g in the basis $\{e_i\}$ so that $[g]_{ij} = \langle x_j, ge_i\rangle$. The following transformation rules are easy consequences of (2.1)–(2.10).

(2.11) If $f \in R$ then $g\mathbf{H}(f) = [g]^T\mathbf{H}(f)[g]$.

(2.12) If $f_1, \cdots, f_l \in R$ and $\mathbf{J} = \mathbf{J}(f_1, \cdots, f_l)$ then $g\mathbf{J} = [g]^T\mathbf{J}$.

(2.13) If $\theta_1, \cdots, \theta_l \in \mathrm{Der}_S^G$ and $\mathbf{Q} = \mathbf{Q}(\theta_1, \cdots, \theta_l)$ then $g\mathbf{Q} = [g^{-1}]\mathbf{Q}$.

(2.14) LEMMA. *If $f \in R$ then* $\mathrm{Hess}(f)\colon \mathrm{Der}_S \to \Omega_S$ *is a G-module homomorphism and thus induces a map* $\mathrm{Hess}(f)\colon \mathrm{Der}_S^G \to \Omega_S^G$.

Proof. Let $g \in G$. First suppose $f \in S$ is any polynomial. Let $h = \mathrm{Hess}(f)$. From (1.1) we get

$$(2.15) \qquad hD_v = \sum D_v(D_i f)dx_i = \sum D_i(D_v f)dx_i = d(D_v f)\,.$$

It follows from (2.15) and (2.1)–(2.8) that $g(hD_v) = g(d(D_v f)) = d(g(D_v f)) = d(D_{gv} gf)$ and $h(gD_v) = h(D_{gv}) = d(D_{gv} f)$ for all $v \in V$. Thus if $f \in R$ then $g(hD_v) = h(gD_v)$. Let $a \in S$. Since h is an S-module map we have $g(h(aD_v)) = g(a(hD_v)) = (ga)(g(hD_v)) = (ga)(h(gD_v)) = h((ga)(gD_v)) = h(g(aD_v))$. The result follows because $\mathrm{Der}_S = \sum SD_v$. □

The S-modules Der_S and Ω_S are graded as follows. Give $S = \oplus_{p \geq 0} S_p$ its usual grading so that $S_1 = V^*$. We call *nonzero* elements of S_p forms of degree p. Grade Der_S by

$$(2.16) \qquad \theta \in (\mathrm{Der}_S)_q \quad \text{if } \theta S_p \subseteq S_{p+q} \quad \text{for all } p \geq 0$$

and grade Ω_S by

$$(2.17) \qquad \omega \in (\Omega_S)_q \quad \text{if } \omega((\mathrm{Der}_S)_p) \subseteq S_{p+q} \quad \text{for all } p \geq 0\,.$$

Thus D_i has degree -1 and dx_i has degree $+1$. It follows from (2.14) and this grading that:

(2.18) LEMMA. *If $f \in R$ is a form of degree p then* $\mathrm{Hess}(f)$: $\mathrm{Der}_S^G \to \Omega_S^G$ *is a graded R-module map of degree p.* □

The main result of this paper concerns an invariant form of minimal positive degree. The significance of minimality is made clear by the following lemma.

(2.19) LEMMA. *Let f be an invariant form. If* $\mathrm{Hess}(f)$: $\mathrm{Der}_S^G \to \Omega_S^G$ *is an epimorphism then f is an invariant form of minimal positive degree.*

Proof. Suppose $f \in R_p$ and suppose that $R_q \neq 0$ for some positive integer q. Then $(\Omega_S^G)_q \supseteq dR_q \neq 0$. Since $\mathrm{Hess}(f)$ is a surjective map of degree p we have $(\mathrm{Der}_S^G)_{q-p} \neq 0$. Since G is an irreducible group we have $(\mathrm{Der}_S^G)_{-1} = 0$. Thus $q - p \geq 0$. □

Each reflection in G fixes a hyperplane in V. Let $\mathscr{A} = \mathscr{A}(G)$ be the set of these hyperplanes and let $n = |\mathscr{A}|$. If $H \in \mathscr{A}$ let $\alpha_H \in V^*$ be a linear form with kernel H. The subgroup of G fixing H is cyclic. Let s_H be a generator of this cyclic group and let e_H be its order. Let m be the number of reflections in G. Then

$$(2.20) \qquad m = \sum_{H \in \mathscr{A}} (e_H - 1)\,, \qquad n = \sum_{H \in \mathscr{A}} 1.$$

(2.21) LEMMA. *Let $f_1, \cdots, f_l$ be basic invariants for G. The R-module Ω_S^G is free of rank l with basis $df_1, \cdots, df_l$. The R-module* Der_S^G *is free of rank l with a basis of homogeneous elements.*

Proof. If M is a vector space over $\boldsymbol{C}$ we give $S \otimes M$ the S-module structure $a(b \otimes m) = ab \otimes m$, the G-module structure $g(a \otimes m) = ga \otimes gm$ and the grading $(S \otimes M)_p = S_p \otimes M$. Define S-module isomorphisms α: $S \otimes V \to \mathrm{Der}_S$ and β: $S \otimes V^* \to \Omega_S$ by $\alpha(a \otimes v) = aD_v$ and $\beta(a \otimes x) = adx$. It follows from (2.1)–(2.8) that both α and β are G-module homomorphisms and thus, by restriction, define R-module isomorphisms

$$\alpha\colon (S \otimes V)^G \longrightarrow \mathrm{Der}_S^G \quad \text{and} \quad \beta\colon (S \otimes V^*)^G \longrightarrow \Omega_S^G . \tag{2.22}$$

These maps are homogeneous with degrees: $\deg \alpha = -1$ and $\deg \beta = +1$. It is shown in [20, Lemma 2; 13, (2.3)] that if M is any G-module of finite dimension, then $(S \otimes M)^G$ is a free R-module of rank equal to $\dim_C M$. Apply this with $M = V$ and $M = V^*$. The assertions of the lemma follow from the R-module isomorphisms (2.22). □

We call a homogeneous R-basis for Der_S^G a set of *basic derivations*. Let $\{\theta_1, \cdots, \theta_l\}$ be a set of basic derivations. Define integers n_i by $\deg \theta_i = n_i - 1$. Our definition of the integers n_i here agrees with the definition of the coexponents n_i in [13] because the isomorphism α in (2.22) has degree -1. It follows from [13, (2.3) ff.] that the integers n_i do not depend on the choice of basic derivations. We agree to number the n_i so that $n_1 \leq n_2 \leq \cdots \leq n_l$.

(2.23) LEMMA. (i) $1 = n_1 < n_2$. (ii) *If $\theta_E = \sum x_i D_i$ is the Euler derivation then $\theta_1 = \theta_E$.*

Proof. Since G is an irreducible group we have $(\mathrm{Der}_S^G)_{-1} = 0$. Thus it suffices to show that $(\mathrm{Der}_S^G)_0 = \boldsymbol{C}\theta_E$. Let α be the isomorphism in (2.22). Then $(\mathrm{Der}_S^G)_0 = \alpha((S_1 \otimes V)^G) = \alpha((V^* \otimes V)^G)$. Since G is irreducible we have $(V^* \otimes V) = \boldsymbol{C} \sum x_i \otimes e_i$. The assertion follows since $\alpha(\sum x_i \otimes e_i) = \theta_E$.

Define polynomials J and Q by

$$J = \prod_{H \in \mathscr{A}} \alpha_H^{e_H - 1} \qquad Q = \prod_{H \in \mathscr{A}} \alpha_H . \tag{2.24}$$

It follows from (2.20) that $\deg J = m$ and $\deg Q = n$. It is known [19, (8.3); 13, (3.11)] that

$$\sum m_i = m \quad \text{and} \quad \sum n_i = n . \tag{2.25}$$

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If $g \in G$ let $\delta(g) = \det g$. Recall the notation

$$J(f_1, \cdots, f_l) = \det \mathbf{J}(f_1, \cdots, f_l) \quad \text{and} \quad Q(\theta_1, \cdots, \theta_l) = \det \mathbf{Q}(\theta_1, \cdots, \theta_l)\,.$$

(2.26) LEMMA. *Let $f_1, \cdots, f_l$ be basic invariants. Then* (i) $gJ(f_1, \cdots, f_l) = \delta(g)J(f_1, \cdots, f_l)$, (ii) $J(f_1, \cdots, f_l) \approx J$, (iii) *If $a \in S$ and $ga = \delta(g)a$ for all $g \in G$ then $a \in RJ$.*

Proof. These facts are known: (i) follows from (2.12); (ii) is proved in [19, (5.2)]; (iii) is given in [1, 5.5, Prop. 6]. □

(2.27) LEMMA. *Let $\theta_1, \cdots, \theta_l$ be basic derivations. Then* (i) $gQ(\theta_1, \cdots, \theta_l) = \delta(g^{-1})Q(\theta_1, \cdots, \theta_l)$, (ii) $Q(\theta_1, \cdots, \theta_l) \approx Q$, (iii) *If $a \in S$ and $ga = \delta(g^{-1})a$ for all $g \in G$ then $a \in RQ$.*

Proof. (i) follows from (2.13). We prove (ii) and (iii) simultaneously. Suppose $a \in S$ and $ga = \delta(g^{-1})a$ for all $g \in G$. Let $H \in \mathscr{A}$ and let $v \in H$. Then $a(v) = a(s_H v) = (s_H^{-1}a)(v) = \delta(s_H)a(v)$. Since $\delta(s_H) \neq 1$ we have $a(v) = 0$. Thus α_H divides a. This is true for all $H \in \mathscr{A}$ so Q divides a. In particular with $a = Q(\theta_1, \cdots, \theta_l)$ it follows from (i) that Q divides $Q(\theta_1, \cdots, \theta_l)$. We know from [13, (2.5)] applied to $M = V^*$, and the isomorphism α of (2.22), that $Q(\theta_1, \cdots, \theta_l) \neq 0$. From (2.24) and (2.25) we have $\deg Q(\theta_1, \cdots, \theta_l) = \sum n_i = n = \deg Q$. This proves (ii) and hence (iii). □

Note that although the matrices $\mathbf{J}(f_1, \cdots, f_l)$ and $\mathbf{Q}(\theta_1, \cdots, \theta_l)$ depend on choice of $f_1, \cdots, f_l$ and $\theta_1, \cdots, \theta_l$ their determinants $J(f_1, \cdots, f_l) \approx J$ and $Q(\theta_1, \cdots, \theta_l) \approx Q$ are nonzero polynomials uniquely determined up to a constant multiple.

The next lemma is an analog of a result of Saito [16, Theorem 1.8. ii].

(2.28) LEMMA. *A set $\eta_1, \cdots, \eta_l$ of homogeneous elements of* Der_S^G *is a set of basic derivations if and only if $Q(\eta_1, \cdots, \eta_l) \approx Q$.*

Proof. If $\eta_1, \cdots, \eta_l$ is a set of basic derivations then (2.27. ii) shows that $Q(\eta_1, \cdots, \eta_l) \approx Q$. Conversely suppose $Q(\eta_1, \cdots, \eta_l) \approx Q$. Let $\theta_1, \cdots, \theta_l$ be a set of basic derivations. Write $\eta_i = \sum b_{ji}\theta_j$ and let $B = [b_{ij}]$. Then $Q(\eta_1, \cdots, \eta_l) = \det(B)Q(\theta_1, \cdots, \theta_l)$. The hypothesis and (2.27. ii) imply $\det(B) \in C^*$ so the conclusion follows. □

(2.29) LEMMA. *If $f \in R$ then* (i) $gH(f) = \delta(g)^2H(f)$ *and* (ii) $H(f)Q \in RJ$.

Proof. Assertion (i) follows from (2.11). It follows from (i) and (2.27. i, ii) that $g(H(f)Q) = \delta(g)H(f)Q$. Now (2.26. iii) shows that $H(f)Q \in RJ$. □

(2.30) LEMMA. *Let* $f \in R_p$ *and suppose* $H(f) \neq 0$. *Then* (i) $l(p-2) = m - n + d \geq m - n$ *where* d *is the degree of an invariant form*; (ii) *If* $l(p-2) = m - n$ *then* $H(f) \approx J/Q$.

Proof. Since $H(f) \neq 0$ it follows from (2.29. ii) that $H(f) = aJ/Q$ for some form $a \in R$ of degree say d. Then (i) follows by comparing degrees. If $l(p-2) = m - n$ then $d = 0$ so a is constant. This proves (ii).

§ 3. The two-dimensional case

In this section we assume that $\dim V = 2$.

(3.1) LEMMA. *Let* $f \in S$ *be a form. If* $H(f) = 0$ *then* f *is a power of a linear form.*

Proof. This is part of 19th century invariant theory [6, p. 235]. Since there is an easy argument [21, Ex. 3.3.14] we give it here. By the Euler formula we have $\begin{vmatrix} D_{11}f & D_{12}f \\ D_1 f & D_2 f \end{vmatrix} = 0$. Let x be an indeterminate and define a homomorphism $\pi\colon S \to \boldsymbol{C}[x]$ by $\pi(x_1) = x$ and $\pi(x_2) = 1$. Note that no form lies in the kernel of π. Let $D = d/dx$. Then $\pi(D_1 u) = D(\pi u)$ for all $u \in S$. Let $u_i = \pi(D_i f)$ for $i = 1, 2$. Then $\begin{vmatrix} Du_1 & Du_2 \\ u_1 & u_2 \end{vmatrix} = 0$. If $u_1 = 0$ then $f \in \boldsymbol{C}x_2^d$ where $d = \deg f$ and the assertion is clear. If $u_1 \neq 0$ then $D(u_2/u_1) = 0$ so $u_2/u_1 \in \boldsymbol{C}$. Say $u_2 = cu_1$. Then $\pi(D_2 - cD_1)f = 0$ so $(D_2 - cD_1)f = 0$ and the assertion follows. □

(3.2) LEMMA. *If* $G \subset GL(V)$ *is an irreducible group and* $f \in S$ *is a* G*-invariant form of positive degree then* $H(f) \neq 0$.

Proof. This follows from (3.1). □

(3.3) LEMMA. *Let* $G \subset GL(V)$ *be an irreducible unitary reflection group. Then* $2(d_1 - 2) = m - n + d$ *where* $d \in \{0, d_1, d_2\}$.

Proof. Lemma (3.2) shows $H(f) \neq 0$ so we may apply (2.30) to conclude that $2(d_1 - 2) = m - n + d$ where d is the degree of an invariant form. Since $R = \boldsymbol{C}[f_1, f_2]$ we may write $d = a_1 d_1 + a_2 d_2$ where a_1, a_2 are non-negative integers. Since $m \geq n$ we have $(a_1 - 2)d_1 + a_2 d_2 \leq -4$. Thus $a_1 = 0, 1$. If $a_1 = 0$ then $a_2 = 0, 1$ because $d_1 \leq d_2$. If $a_1 = 1$ then $a_2 = 0$ because $d_1 \leq d_2$. □

(3.4) *Remark.* All three possibilities $d = 0, d_1, d_2$ do in fact occur.

By (2.25) we have $m = d_1 + d_2 - 2$ so the formula in the lemma is equivalent to $m + n = 2d_2 + d$ where $d \in \{0, d_1, d_2\}$. We observed in [13, Theorem 5.4] that $m + n \geq ld_l$ for all irreducible reflection groups. Thus Lemma (3.3) gives a case free argument for this inequality if $l = 2$.

(3.5) THEOREM. *If $G \subset GL(V)$ is a Shephard group then $2(d_1 - 2) = m - n$ and thus $m + n = 2d_2$.*

Proof. If we can show that

$$n < d_2 + 2 \tag{3.6}$$

then, in the notation of (3.3), we have $d = n + d_1 - d_2 - 2 < d_1$, so $d = 0$ by minimality of d_1 and the theorem is proved. Since G is a Shephard group, it follows from work of Coxeter [4, pps. 94–5] that G may be generated by two reflections s_1, s_2 and there exist positive integers p_1, p_2, q such that

$$s_1^{p_1} = 1 \qquad s_2^{p_2} = 1$$
$$s_1 s_2 s_1 \cdots = s_2 s_1 s_2 \cdots$$

where there are q terms on both sides of the last equation. Let $H_i \in \mathscr{A}$ be the hyperplane fixed by s_i and let $\mathcal{O}_i \subseteq \mathscr{A}$ be the G-orbit of H_i in the natural action of G on $\mathscr{A}$. Let $\mathcal{O}$ be any orbit of G on $\mathscr{A}$. If $\mathcal{O} \notin \{\mathcal{O}_1, \mathcal{O}_2\}$ it follows from [21, Theorem 4.3.4. i] that there exists a homomorphism $\lambda\colon G \to C^*$ such that $\lambda(s_i) = 1$ for $i = 1, 2$ and $\lambda(s) \neq 1$ for any reflection s which fixes a hyperplane $H \in \mathcal{O}$. This contradicts the fact that s_1, s_2 generate G. Thus $\mathcal{O} \in \{\mathcal{O}_1, \mathcal{O}_2\}$ so $\mathscr{A} = \mathcal{O}_1 \cup \mathcal{O}_2$.

Let E_i be the cyclic subgroup fixing H_i and let $e_i = |E_i|$. Then $e_i \geq p_i$. (In fact equality holds but we do not need this.) Suppose $\mathcal{O}_1 = \mathcal{O}_2$. Then $m = (e_1 - 1)|\mathcal{O}_1| = (e_1 - 1)n$. If $e_1 = 2$ then $p_1 = 2 = p_2$ so G is a dihedral group. In this case $d_1 = 2$ and $m = n$ so (3.5) is clear. If $e_1 \geq 3$ then $m \geq 2n$ so $d_1 + d_2 - 2 \geq 2n$ and (3.6) is clear. Thus we may assume $\mathcal{O}_1 \neq \mathcal{O}_2$. If q is odd then $s_1 s_2 s_3 \cdots = s_1 s_2 s_3 \cdots$ shows that s_1 and s_2 are conjugate in G so $\mathcal{O}_1 = \mathcal{O}_2$, a contradiction. Thus q is even. Let $F_i = \{g \in G \mid gH_i \subseteq H_i\}$ and let Z be the center of G. Since $ZE_i \subseteq F_i$ and $Z \cap E_i = 1$ we have $|F_i| \geq |Z| e_i$. Thus

$$|G\colon Z| \geq e_i |G\colon F_i| = e_i |\mathcal{O}_i| \qquad (i = 1, 2)\,. \tag{3.7}$$

Coxeter [4, p. 154] has shown that

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$$d_1 = 2h/q\,, \qquad d_2 = h \tag{3.8}$$

where h is the order of $s_1 s_2$. Since q is even it follows that d_1 divides d_2. It is known [22, (3.3)] that $|Z| = \text{g.c.d.}\,(d_1, d_2)$. In our case this says $|Z| = d_1$. By [19, (5.1)] we have $|G| = d_1 d_2$ and thus $|G\colon Z| = d_2$. It follows from (3.7) that

$$2d_2 \geq e_1|\mathcal{O}_1| + e_2|\mathcal{O}_2| = m + n = d_1 + d_2 - 2 + n$$

which proves (3.6). □

We prove a general lemma about binary forms which helps us to prove (1.5). Let f, φ be binary forms of degrees d, n. If r is a non-negative integer, the r-th transvectant $(f, \varphi)^r$ of f and φ is defined [6, p. 46] by

$$(f, \varphi)^r = \frac{(d-r)!\,(n-r)!}{d! \qquad n!} \sum_{k=0}^{r} (-1)^k \binom{r}{k} (D_1^{r-k} D_2^k f)(D_1^k D_2^{r-k} \varphi)\,. \tag{3.9.i}$$

It is convenient to omit the numerical factor and write as in [21, p. 57]

$$\tau_r(f, \varphi) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} (D_1^{r-k} D_2^k f)(D_1^k D_2^{r-k} \varphi). \tag{3.9. ii}$$

Thus $\tau_1(f, \varphi) = J(f, \varphi)$ and $\tau_2(f, f) = 2H(f)$.

(3.10) Lemma. *Suppose f, φ are semi-invariant forms for G with characters λ, μ. Then $\tau_r(f, \varphi)$ is 0 or is a semi-invariant form with character $\delta^r \lambda\mu$.*

Proof. Define the Cayley operator $\Omega\colon S \otimes S \to S \otimes S$ by $\Omega = D_1 \otimes D_2 - D_2 \otimes D_1$, and let $\pi\colon S \otimes S \to S$ be multiplication. If $g \in G$ then $\Omega(gf \otimes g\varphi) = \delta(g)\Omega(f \otimes \varphi)$, and π is a G-module homomorphism. The assertion follows since $\tau_r(f, \varphi) = \pi\Omega^r(f \otimes \varphi)$.

(3.11) Lemma. *Let f, $\varphi \in S$ be forms of positive degrees d, n. Suppose $H(f) \neq 0$ and $\tau_2(f, \varphi) = 0$. Let*

$$\mathbf{Q} = \begin{bmatrix} x_1 & -D_2\varphi \\ x_2 & D_1\varphi \end{bmatrix}$$

and let $\psi = (n + d - 2)^{-1} J(\varphi, f)$. Then f, ψ are algebraically independent and $\mathbf{H}(f)\mathbf{Q} = (d-1)\mathbf{J}(f, \psi)$.

Proof. Note that the hypotheses imply $n \geq 1$ and $d \geq 2$ so $n + d - 2 > 0$. Direct computation using the Euler formula gives

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$$\mathbf{H}(f)\mathbf{Q} = (d-1)\begin{bmatrix} D_1 f & \psi_1 \\ D_2 f & \psi_2 \end{bmatrix}$$

where ψ_1, ψ_2 are defined by

(i) $(d-1)\psi_1 = -(D_1^2 f)(D_2\varphi) + (D_1D_2f)(D_1\varphi)$,

(ii) $(d-1)\psi_2 = -(D_1D_2f)(D_2\varphi) + (D_2^2f)(D_1\varphi)$.

Since $H(f) \neq 0$ and $\det \mathbf{Q} = n\varphi \neq 0$, at least one of ψ_1, ψ_2 is not zero. Further computation gives $(d-1)(D_1\psi_2 - D_2\psi_1) = \tau_2(f, \varphi) = 0$. Thus ψ_1, ψ_2 are the partial derivatives of a form ψ of degree $n + d - 2$. Multiply (i) by x_1 and (ii) by x_2. The Euler formula gives

$$(d-1)(n+d-2)\psi = -(x_1D_1^2f + x_2D_1D_2f)(D_2\varphi) + (x_1D_1D_2f + x_2D_2^2f)(D_1\varphi).$$

Since D_1f, D_2f have degree $d-1$, application of the Euler formula to the right hand side proves $(n+d-2)\psi = J(\varphi, f)$. Thus $\mathbf{H}(f)\mathbf{Q} = (d-1)\mathbf{J}(f, \psi)$. Since $\det \mathbf{H}(f)\mathbf{Q} = nH(f)\varphi \neq 0$ it follows that $J(f, \psi) \neq 0$. This proves that f, ψ are algebraically independent. $\square$

(3.12) Corollary. *Let $f, \varphi \in S$ be forms of positive degree. Suppose $H(f) \neq 0$ and $\tau_2(f, \varphi) = 0$. Then $J(D_if, \varphi) \approx D_iJ(f, \varphi)$ for $i = 1, 2$.*

Proof. This follows by equating entries in the second column of the matrix equation in (3.11). $\square$

(3.13) Corollary. *Let $f, \varphi \in S$ be forms of positive degree. Suppose $H(f) \neq 0$ and $\tau_2(f, \varphi) = 0$. Then $J(f, J(\varphi, f)) \approx H(f)\varphi$.*

Proof. This follows from (3.11) by taking determinants. $\square$

(3.14) *Remark.* The assertion in (3.13) is true without the hypothesis $H(f) \neq 0$. It follows from a known identity [6, p. 78] for the Jacobian of a Jacobian: if f, φ, ψ are forms of degrees $d, n, p > 1$ then

$$J(J(f, \varphi), \psi) \approx \frac{d-n}{(d+n-2)}(f, \varphi)^2\psi + (f, \psi)^2\varphi - (\varphi, \psi)^2 f$$

where $(f, \varphi) = (f, \varphi)^1$ and $(f, \varphi)^2$ denote the transvectants defined in (3.9). Take $\psi = f$. The assertion follows since $(f, \varphi) \approx J(f, \varphi)$, $(f, f)^2 \approx H(f)$, and $(f, \varphi)^2 = 0$ by assumption.

(3.15) Theorem. *Let $G \subset GL(V)$ be a Shephard group. Let $\eta_1 = \theta_E$ be the Euler derivation and let f_1 be an invariant form of minimal positive degree. Define $\eta_2 \in \mathrm{Der}_S$ and $f_2 \in S$ by*

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$$\eta_2 = (-D_2Q)D_1 + (D_1Q)D_2 \tag{3.16}$$

$$d_2f_2 = J(Q, f_1)\,. \tag{3.17}$$

Then (i) η_1, η_2 *are basic derivations*, (ii) f_1, f_2 *are basic invariants, and* (iii) $\mathrm{Hess}\,(f_1)\eta_i = m_1 df_i$ *for* $i = 1, 2$.

Proof. Direct calculation shows that $\eta_2 \in \mathrm{Der}_S^G$. Since $Q(\eta_1, \eta_2) = nQ$ it follows from (2.28) that η_1, η_2 are basic derivations. By (2.27) Q is a semi-invariant with character δ^{-1}. Since f_1 is invariant it follows from (3.10) that $\tau_2(f_1, Q)$ is either 0 or a semi-invariant form of degree $(d_1 - 2) + (n - 2)$ with character δ. If $\tau_2(f_1, Q) \neq 0$ it follows from (2.26. iii) that $d_1 + n - 4 \geq m$ which contradicts Theorem (3.5). Thus

$$\tau_2(f_1, Q) = 0\,. \tag{3.18}$$

Since $H(f_1) \neq 0$ by (3.2), and $\tau_2(f_1, Q) = 0$ it follows from (3.11) that f_1 and f_2 are algebraically independent. In particular $f_2 \neq 0$. It follows from (3.5) that $\deg f_2 = \deg J(Q, f_1) = (n - 1) + (d_1 - 1) = d_2$. Since Q is a semi-invariant with character δ^{-1} and f_1 is an invariant, it follows from (3.10) that $J(Q, f_1)$ is a semi-invariant with character $\delta\delta^{-1} = 1$, so that f_2 is an invariant. Thus f_1, f_2 are algebraically independent invariant forms of degrees d_1, d_2 and hence are basic invariants. This proves (ii). Finally, since $n + d_1 - 2 = d_2$ the matrix equation in (3.11) implies $\mathbf{H}(f_1)\mathbf{Q}(\eta_1, \eta_2) = m_1\mathbf{J}(f_1, f_2)$. This proves (iii). □

To complete the proof of (1.3) in case $\dim V = 2$ let f_1, f_2 be the basic invariants constructed in (3.15). Since $df_i \in im\,\mathrm{Hess}\,(f_1)$ for $i = 1, 2$ and $\Omega_S^G = Rdf_1 + Rdf_2$ it follows that $\mathrm{Hess}\,(f_1)$ is surjective and hence by (3.2) it is an isomorphism.

We close this section with some observations on the connection between our work and Klein's polyhedral invariants. Let G be the group $3[q]3$ in Coxeter's notation [4, p. 94]. It is known from Coxeter's work that $3[q]3$ is finite for $q = 3, 4, 5$. Since $p_i = 3$ for $i = 1, 2$ we have $\delta(s_i)^3 = \delta(s_i^3) = 1$ so $\delta^3 = 1$ and thus $\delta^4 = \delta$. Let $f = f_1$ be a G-invariant form of minimal positive degree. It follows from (3.10) that the fourth transvectant $(f, f)^4$ is 0 or it is a semi-invariant form with character δ. Since $\deg\,(f, f)^4 = 2d_1 - 8 < d_1 + d_2 - 2 = m$, it follows from (2.26. iii) that $(f, f)^4 = 0$. The binary forms f of degree $n \geq 4$ which satisfy $(f, f)^4 = 0$ were characterized by Wedekind [25, pps. 42–51]. He showed that either $f =$

$\alpha^{n-1}\beta$ where α and β are linear forms, allowing the possibility, $\alpha = \beta$, or that $n = 4, 6, 12$ and f lies in the $GL(V)$-orbit of the vertex form of one of the regular polyhedra with triangular faces: tetrahedron, octahedron, icosahedron. Since an irreducible group cannot have an invariant $\alpha^{n-1}\beta$ if $n \geq 3$, we may assume that $f \approx \varphi_V$ is a vertex form. There is something mysterious here because Wedekind's argument is a formal computation with the coefficients of a form f which satisfies $(f, f)^4 = 0$. This leads him to a Diophantine equation which produces the integers 4, 6, 12. One does not learn from his argument why the condition $(f, f)^4 = 0$ leads to a polyhedral group. The references given by Klein [9, pps. 62–63] have not helped us on this point.

Since $\delta^3 = 1$ we have $e_H = 3$ for all $H \in \mathscr{A}$ and thus from (2.24) and (2.30) we get $H(f) \approx J/Q = \prod \alpha_H^{e_H-2} = \prod \alpha_H = Q$. Thus from Klein's formula (1.8) we have $Q \approx H(\varphi_V) \approx \varphi_F$. It follows from (3.17) that $f_2 \approx J(f_1, Q) \approx J(\varphi_V, \varphi_F) \approx \varphi_E$ where the last equality is Klein's (1.7). Thus for the groups $3[q]3$ with $q = 3, 4, 5$ we have

$$f_1 \approx \varphi_V\,, \quad Q \approx \varphi_F\,, \quad f_2 \approx \varphi_E\,. \tag{3.19}$$

Formula (1.10) follows from (3.12) with $f = \varphi_V$ and $\varphi = \varphi_F$. From (3.19) and (2.26) we have $J(\varphi_V, \varphi_E) \approx J(f_1, f_2) \approx J = \prod \alpha_H^{e_H-1} = \prod \alpha_H^2 = Q^2 \approx \varphi_F^2$. This proves (1.8). It follows from (3.17) and (3.19) that $J(\varphi_F, \varphi_E) \approx J(Q, f_2) \approx J(Q, J(Q, f_1))$. Since Q is not a power of a linear form, it follows from (3.1) that $H(Q) \neq 0$. Since $\tau_2(f, Q) = 0$ we may replace f by Q and φ by f_1 in (3.13) to conclude that $J(Q, J(Q, f_1)) \approx H(Q)f_1 \approx H(Q)\varphi_V$. Thus the proof of (1.9) is reduced to showing that $H(Q) \approx f_1^{q-2}$. Since Q is a semi-invariant with character δ^{-1} and $H(Q) \neq 0$ it follows from (3.10) that $H(Q) \approx \tau_2(Q, Q)$ is a semi-invariant with character $\delta^2\delta^{-2} = 1$. Now (3.5) shows that $H(Q)$ is an invariant form of degree $2n - 4 < d_2$, and hence $H(Q) \approx f_1^\nu$ where $\nu = (2n-4)/d_1$. It follows from (3.5) and (3.8) that $2(n-2) = 2(d_2 - d_1) = (q-2)d_1$. This proves (1.9). The vanishing of $(f, Q)^2$ proved in (3.18) is equivalent, for the groups $3[q]3$, to the vanishing of $(f, f)^4$. This follows from the fact, remarked above, that $H(f) \approx Q$ and a formal identity [6, p. 52, Ex. 6]

$$(H(f), f)^2 = \frac{n-3}{2(2n-5)}(f, f)^4 f$$

valid for all binary forms f of degree n.

§ 4. The case $\dim V \geq 3$.

The argument in Section 3 shows that in case $l = 2$ Theorem (1.3) and hence Theorem (1.5) is a consequence of four facts: (i) $H(f) \neq 0$, (ii) $l(d_1 - 2) = m - n$, (iii) there is an explicit basis for Der_S^G in terms of Q and (iv) $\tau_2(f, Q) = 0$. Both (i) and (ii) may be stated for arbitrary l while (iii) and (iv) are special to $l = 2$. In this section we verify (i) and (ii) for Shephard groups with $l \geq 3$ and prove that (i) and (ii) imply Theorem (1.5). We could have used this method for $l = 2$ but it does not yield the explicit bases in (3.15) or the formulas (1.8)–(1.10) for Klein's polyhedral invariants.

Let $f \in S$ be any form. Hesse [7, 8] claimed that if $H(f) = 0$ then there exist linear forms $y_1, \cdots, y_{l-1} \in V^*$ such that $f \in \boldsymbol{C}[y_1, \cdots, y_{l-1}]$. Sylvester [23], when told about this theorem, wrote that "an hour's quiet reflection in bed $\cdots$ sufficed to disclose to me the true principle of the solution". Hesse's assertion is easy to prove if $l = 2$; see (3.1). It is much harder to prove if $l = 3, 4$. This was done by Gordan and Noether [5] who also showed that it is false if $l \geq 5$. See Noether's Note 30 in the appendix to Hesse's collected works [12], and the introduction to [5], for more history.

Suppose now that $G \subset GL(V)$ is any irreducible group. If f is a G-invariant form of positive degree, then the map $v \to \mathrm{Hess}(f)D_v = d(D_v f)$ is a nonzero G-module homomorphism from V into S. Since G is an irreducible group this map is a monomorphism. Thus the partial derivatives $D_1 f, \cdots, D_l f$ are linearly independent over $\boldsymbol{C}$. It follows that there do not exist linear forms $y_1, \cdots, y_{l-1}$ with $f \in \boldsymbol{C}[y_1, \cdots, y_{l-1}]$. Thus *if* Hesse's claim were correct we could conclude that $H(f) \neq 0$. As it stands, we know from Gordan and Noether that if $G \subset GL(V)$ is any irreducible group and $\dim V = 2, 3, 4$ then $H(f) \neq 0$ for every G-invariant form of positive degree. We do not know if an irreducible group in dimension ≥ 5 can have an invariant form f of minimal positive degree with $H(f) = 0$.

(4.1) Lemma. *Suppose* $\dim V \geq 3$. *If* $G \subset GL(V)$ *is a Shephard group and* f_1 *is a* G*-invariant form of minimal positive degree* d_1 *then* (i) $H(f_1) \neq 0$ *and* (ii) $l(d_1 - 2) = m - n$.

Proof. Suppose first that G is a Coxeter group. Since G is irreducible it has a unique invariant quadratic form f_1 which is nondegenerate. Thus $H(f_1) \neq 0$. This proves (i). Assertion (ii) holds since $d_1 = 2$ and $m = n$.

Suppose G is not a Coxeter group. Then (ii) follows from the numbers listed in Table 1. See Appendix. Assertion (i) is clear if $G = G(p, 1, l)$ where $f_1 = \sum x_i^p$. For the remaining groups $l = 3, 4$ so the assertion follows from the theorem of Gordan and Noether [5]. It is easy to check that $H(f_1) \neq 0$ directly using the invariants listed in Table 1. In fact $H(C_6) \approx \mathfrak{C}_{12}$ and Maschke [11, p. 339] remarks that $H(F_{12}) \approx F_{40}$. $\square$

We use Lemma (4.1) and the results in Section 2 to prove Theorem (1.7). If $a_1, \cdots, a_l$ are the columns of a matrix $\mathbf{A}$ we write $\mathbf{A} = [a_1|\cdots|a_l]$.

(4.2) LEMMA. *Let* $\mathbf{A} = [a_1|\cdots|a_l]$ *and* $\mathbf{B} = [b_1|\cdots|b_l]$ *be* $l \times l$ *matrices with coefficients in a field. If* $\mathbf{B}$ *is invertible then*

$$(\mathbf{B}^{-1}\mathbf{A})_{ij} = (\det \mathbf{B})^{-1} \det [b_1|\cdots|b_{i-1}|a_j|b_{i+1}|\cdots|b_l] .$$

Proof. Let $c_{ij} = (\mathbf{B}^{-1}\mathbf{A})_{ij}$ and let $c_j = [c_{1j}, c_{2j}, \cdots, c_{lj}]^T$. Then $[c_1|\cdots|c_l] = \mathbf{B}^{-1}\mathbf{A} = [\mathbf{B}^{-1}a_1|\cdots|\mathbf{B}^{-1}a_l]$. Thus $\mathbf{B}c_j = a_j$. Fix j and view this as a system of linear equations with given a_{ij} and unknowns c_{ij} for $i = 1, \cdots, l$. The assertion of the lemma is Cramer's rule. $\square$

Table 1

S & T	Coxeter	m_i	m	n_i	n	f_1	Q
$G(p,1,l)$	$p[4]2[3]\cdots 2$	$p-1, 2p-1, \cdots, lp-1$	$\frac{pl(l+1)}{2}-l$	$1, p+1, \cdots, (l-1)p+1$	$\frac{pl(l-1)}{2}+l$	$\sum x_i^p$	*
3	$[p]$	$p-1$	$p-1$	1	1	x^p	x
4	3[3]3	3, 5	8	1, 3	4	Φ	Ψ
8	4[3]4	7, 11	18	1, 5	6	W	t
16	5[3]5	19, 29	48	1, 11	12	H	f
5	3[4]3	5, 11	16	1, 7	8	t	W
10	4[4]3	11, 23	34	1, 13	14	χ	tW
18	5[4]3	29, 59	88	1, 31	32	T	Hf
20	3[5]3	11, 29	40	1, 19	20	f	H
6	3[6]2	3, 11	14	1, 9	10	Φ	Ψt
9	4[6]2	7, 23	30	1, 17	18	W	$t\chi$
17	5[6]2	19, 59	78	1, 41	42	H	fT
14	3[8]2	5, 23	28	1, 19	20	t	$W\chi$
21	3[10]2	11, 59	70	1, 49	50	f	HT
25	3[3]3[3]3	5, 8, 11	24	1, 4, 7	12	C_6	$\mathfrak{C}_{12}$
26	3[3]3[4]2	5, 11, 17	33	1, 7, 13	21	C_6	$C_9\mathfrak{C}_{12}$
32	3[3]3[3]3[3]3	11, 17, 23, 29	80	1, 7, 13, 19	40	F_{12}	F_{40}

* $\prod_{i\le i\le l} x_i \prod_{1\le i<j\le l} (x_i^p - x_j^p)$, $p \ge 3$.

(4.3) LEMMA. *Let $G \subset GL(V)$ be a unitary reflection group. Let $f_1, \cdots, f_l$ be basic invariants and let $\mathbf{J} = \mathbf{J}(f_1, \cdots, f_l)$. Suppose $\mathbf{A} \in M_l(S)$ and $g\mathbf{A} = [g]^T\mathbf{A}$ for all $g \in G$. Then there exists a matrix $\mathbf{C} \in M_l(R)$ such that $\mathbf{A} = \mathbf{JC}$.*

Proof. Write $\mathbf{A} = [a_1 | \cdots | a_l]$ and $\mathbf{J} = [b_1 | \cdots | b_l]$. Fix $g \in G$ and define ga_i by $g\mathbf{A} = [ga_1 | \cdots | ga_l]$. For simplicity of notation write $\mathbf{P} = [g]^T$. Since $g\mathbf{A} = \mathbf{PA}$ by assumption, and $g\mathbf{J} = \mathbf{PJ}$ by (2.12), we have $ga_i = \mathbf{P}a_i$ and $gb_i = \mathbf{P}b_i$. For each $1 \leq i, j \leq l$ define $\mathbf{M}(i, j) \in M_l(S)$ by

$$\mathbf{M}(i, j) = [b_1 | \cdots | b_{i-1} | a_j | b_{i+1} | \cdots | b_l] .$$

Then $g\mathbf{M}(i, j) = \mathbf{PM}(i, j)$. Thus $g(\det \mathbf{M}(i, j)) = (\det \mathbf{P})(\det \mathbf{M}(i, j)) = (\det g)(\det \mathbf{M}(i, j))$. By (2.26. iii) we have $\det \mathbf{M}(i, j) \in RJ$. By (2.26. ii) $\det \mathbf{J} \approx J \neq 0$ so $\mathbf{J}$ is invertible in $M_l(L)$ where L is the quotient field of S. Let $\mathbf{C} = \mathbf{J}^{-1}\mathbf{A}$. It follows from (4.2) that $\mathbf{C}_{ij} \approx J^{-1} \det \mathbf{M}(i, j)$. Since $\det \mathbf{M}(i, j) \in RJ$ we have $\mathbf{C}_{ij} \in R$. □

(4.4) *Remark.* Let $G \subset GL(V)$ be a unitary reflection group. Let $\theta_1, \cdots, \theta_l$ be basic derivations and let $\mathbf{Q} = \mathbf{Q}(\theta_1, \cdots, \theta_l)$. If $\mathbf{A} \in M_l(S)$ and $g\mathbf{A} = [g^{-1}]\mathbf{A}$ for all $g \in G$ then there exists a matrix $\mathbf{C} \in M_l(R)$ such that $\mathbf{A} = \mathbf{QC}$. The argument is similar to that in (4.3).

(4.5) PROPOSITION. *Let $G \subset GL(V)$ be a unitary reflection group. Let $f_1, \cdots, f_l$ be basic invariants and let $\theta_1, \cdots, \theta_l$ be basic derivations. If $f \in R$ then there exists $\mathbf{C} \in M_l(R)$ such that*

$$\mathbf{H}(f)\mathbf{Q}(\theta_1, \cdots, \theta_l) = \mathbf{J}(f_1, \cdots, f_l)\mathbf{C} .$$

If f is homogeneous then the nonzero entries of $\mathbf{C}$ are homogeneous.

Proof. Write $\mathbf{J} = \mathbf{J}(f_1, \cdots, f_l)$ and $\mathbf{Q} = \mathbf{Q}(\theta_1, \cdots, \theta_l)$. It follows from (2.11) and (2.13) that $g(\mathbf{H}(f)\mathbf{Q}) = [g]^T\mathbf{H}(f)\mathbf{Q}$ so we may apply (4.3) to find a matrix $\mathbf{C}$ in $M_l(R)$ which satisfies $\mathbf{H}(f)\mathbf{Q} = \mathbf{JC}$. The nonzero entries of a given column of $\mathbf{J}$ or $\mathbf{Q}$ are homogeneous polynomials all of which have the same degree. If f is homogeneous then all nonzero entries of $\mathbf{H}(f)$ have the same degree. Thus the nonzero entries of $\mathbf{C}$ are homogeneous. □

(4.6) *Remark.* Note that (2.29. ii) follows from (4.5) by taking determinants.

Now we may complete the proof of Theorem (1.3). By (4.1) we have

$H(f_1) \neq 0$ so $\operatorname{Hess}(f_1)$ is injective. Choose a set of basic derivations $\varphi_1, \cdots, \varphi_l$. To prove that $\operatorname{Hess}(f)$ is surjective we modify $\varphi_1, \cdots, \varphi_l$ to produce a new set $\theta_1, \cdots, \theta_l$ of basic derivations such that $\operatorname{Hess}(f_1)\theta_i = m_1 df_i$. To denote the dependence of $\mathbf{Q}$ on the basic derivations we write $\mathbf{Q}(\varphi)$ or $\mathbf{Q}(\theta)$ in this argument. By (4.5) there exists $\mathbf{C} \in M_l(R)$ such that $\mathbf{H}(f)\mathbf{Q}(\varphi) = \mathbf{JC}$. Let $C = \det \mathbf{C}$. Then $H(f)Q(\varphi) = JC$. Comparing degrees we get $l(d_1 - 2) + n = m + \deg C$. It follows from (4.1) that C is not zero. Thus $\mathbf{C}$ is invertible in $M_l(R)$. Define $\theta_1, \cdots, \theta_l \in \operatorname{Der}_S^G$ by $m_1\varphi_j = \sum \mathbf{C}_{kj}\theta_k$. Then $m_1\mathbf{Q}(\varphi) = \mathbf{Q}(\theta)\mathbf{C}$ and hence $\mathbf{H}(f)\mathbf{Q}(\theta) = m_1\mathbf{J}$. This proves that $\operatorname{Hess}(f)\theta_i = m_1 df_i$. Since f is homogeneous and the df_i are homogeneous, the θ_i may be replaced by suitable homogeneous components and thus may be assumed homogeneous. Since $Q(\theta) \approx Q(\varphi)$ it follows that $\theta_1, \cdots, \theta_l$ are basic derivations. $\square$

§5. Consequences of the Main Theorem

In this section we derive some consequences of Theorems (1.3) and (1.5). Write $\mathbf{J} = \mathbf{J}(f_1, \cdots, f_l)$ and $\mathbf{Q} = \mathbf{Q}(\theta_1, \cdots, \theta_l)$.

(5.1) Corollary. *Let $G \subset GL(V)$ be a Shephard group. Let $f_1, \cdots, f_l$ be basic invariants and let $\theta_1, \cdots, \theta_l$ be basic derivations such that $\mathbf{H}(f_1)\mathbf{Q} = m_1\mathbf{J}$. Then $\theta_i(f_j) = \theta_j(f_i)$.*

Proof. Since $\mathbf{H}(f_1)$ is symmetric so is $m_1\mathbf{J}^T\mathbf{Q} = \mathbf{Q}^T\mathbf{H}(f_1)\mathbf{Q}$. The assertion follows since $\theta_j f_i = \sum \theta_j(x_k)D_k f_i = (\mathbf{J}^T\mathbf{Q})_{ij}$. $\square$

(5.2) *Remark.* If follows from (2.9) that the natural S-bilinear pairing $\Omega_S \times \operatorname{Der}_S \to S$ restricts to an R-bilinear pairing Ψ: $\Omega_S^G \times \operatorname{Der}_S^G \to R$ for which $\Psi(df, \theta) = \theta(f)$. The matrix for Ψ with respect to the bases $f_1, \cdots, f_l$ and $\theta_1, \cdots, \theta_l$ is $\mathbf{J}^T\mathbf{Q}$. Each $f \in S$ defines a *symmetric* S-bilinear form $\operatorname{Der}_S \times \operatorname{Der}_S \to S$ given by $(\theta, \eta) \to \langle \operatorname{Hess}(f)\theta, \eta\rangle$. If $f \in R$ this form restricts to a symmetric R-bilinear form Φ_f: $\operatorname{Der}_S^G \times \operatorname{Der}_S^G \to R$ given by $\Phi_f(\theta, \eta) = \Psi(\operatorname{Hess}(f)\theta, \eta)$. The matrix for Φ_f with respect to the basis $\theta_1, \cdots, \theta_l$ is $\mathbf{Q}^T\mathbf{H}(f)\mathbf{Q}$. The two pairings Ψ and Φ_f are defined for all unitary reflection groups. If G is a Shephard group and $f = f_1$ then $\mathbf{H}(f_1)\mathbf{Q} = m_1\mathbf{J}$ shows that their matrices $m_1\mathbf{J}^T\mathbf{Q} = \mathbf{Q}^T\mathbf{H}(f_1)\mathbf{Q}$ are the same.

(5.3) Corollary. *Let $G \subset GL(V)$ be a Shephard group. Then $m_i - n_i = d_1 - 2$ is a non-negative integer independent of i.*

Proof. Choose $f_1, \cdots, f_l$ and $\theta_1, \cdots, \theta_l$ as in (1.4). Since $\deg df_i =$

$m_i + 1$, $\deg \theta_i = n_i - 1$ and $\deg \operatorname{Hess}(f_1) = d_1$ we have $n_i - 1 + d_1 = m_i + 1$. □

(5.4) COROLLARY. *Let $G \subset GL(V)$ be a Shephard group. An invariant form of minimal positive degree is unique up to a constant multiple.*

Proof. We have $m_1 + n_2 = m_2 + n_1$ from (5.3) and $1 = n_1 < n_2$ from (2.23). Thus $d_1 < d_2$. □

(5.5) COROLLARY. *Let $G \subset GL(V)$ be a Shephard group and let $f \in R$ be any invariant. Then $(D_1 f, \cdots, D_l f) \subseteq (D_1 f_1, \cdots, D_l f_1)$.*

Proof. Since $R = C[f_1, \cdots, f_l]$ it suffices to show this if $f = f_k$ is a basic invariant. Since $\mathbf{H}(f_1)\mathbf{Q} = m_1 \mathbf{J}$ we have

(5.6) $$\sum_j (D_i D_j f_1)(\theta_k x_j) = m_1 D_i f_k .$$

Multiply by x_i, sum over i and use the Euler formula. This gives

(5.7) $$\sum_j (D_j f_1)(\theta_k x_j) = d_k f_k .$$

Apply D_i to (5.7):

$$\sum_j (D_i D_j f_1)(\theta_k x_j) + \sum_j (D_j f_1)(D_i \theta_k x_j) = d_k (D_i f_k) .$$

By (5.6) the first sum is $m_1 D_i f_k$ so we have

$$\sum_j (D_j f_1)(D_i \theta_k x_j) = (d_k - m_1) D_i f_k .$$ □

(5.8) COROLLARY. *Let $G \subset GL(V)$ be a Shephard group and let $f_1, \cdots, f_l$ be basic invariants. Then* (i) *$(f_1, \cdots, f_l) \subset (D_1 f_1, \cdots, D_l f_1)$ and* (ii) *a form f_1 of minimal positive degree is nondegenerate.*

Proof. The first assertion follows from (5.5); in fact (5.7) expresses f_k as an S-linear combination of the $D_j f_1$. Since S is integral over R the origin is the only common zero of $f_1, \cdots, f_l$. It follows from (i) that the origin is the only common zero of $D_1 f_1, \cdots, D_l f_1$.

(5.9) *Remark.* There are irreducible unitary reflection groups for which the assertions in (5.3), (5.4), (5.5), and (5.8) are false. For example, in the groups numbered 7, 11, 19 by Shephard and Todd, all these assertions are false. Note also that there are finite irreducible Coxeter groups which are not the automorphism groups of regular polytopes and hence not Shephard groups.

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(5.10) THEOREM. *Let $G \subset GL(V)$ be a finite irreducible unitary reflection group. The following statements are equivalent*:

(i) *G is a Shephard group or a Coxeter group,*

(ii) Hess (f_1): $\mathrm{Der}_S^G \to \Omega_S^G$ *is an isomorphism,*

(iii) $m_i - n_i = d_1 - 2$ *for* $1 \leq i \leq l$,

(iv) $m - n = l(d_1 - 2)$,

(v) $H(f_1) \approx \prod_{H \in \mathscr{A}} \alpha_H^{e_H - 2}$.

Proof. If G is a Shephard group then (i) $\Rightarrow$ (ii) is Theorem (1.3), the main result of this paper. If G is a Coxeter group then f_1 is a nondegenerate quadratic form and (i) $\Rightarrow$ (ii) follows as we remarked in the Introduction. The argument used to prove Corollary (5.3) shows that (ii) $\Rightarrow$ (iii). The assertion (iii) $\Rightarrow$ (iv) follows from (2.25) and (iv) $\Rightarrow$ (v) follows from (2.30). Clearly (v) $\Rightarrow$ (iv) by comparing degrees. The assertion (iv) $\Rightarrow$ (i) follows from the information in [13, Table 2] *and* the list of Shephard groups in Table 1 of this paper. We expect a proof of (iv) $\Rightarrow$ (i), which does not use the classification, to be difficult. □

Appendix

In Table 1 we list those Shephard groups which are not Coxeter groups. Column 1 labels the groups as in Table VII of [19]; Shephard and Todd indicate which groups are symmetry groups of regular complex polytopes. Column 2 gives Coxeter's symbol for the groups as in [4]. The numbers in the columns headed m_i, m, n_i, n are given in [13, Table 2]. The column headed f_1 gives an invariant polynomial of degree $d_1 = m_1 + 1$. The column headed Q gives a polynomial $\prod_{H \in \mathscr{A}} \alpha_H$. The last two columns depend on choice of coordinates in V. If $l = 2$ we use the polynomials Φ, Ψ, t, W, χ, f, H, T of Klein [9]. In case $l = 3, 4$ we use the polynomials C_6, C_9, $\mathfrak{C}_{12}$, F_{12}, F_{40} of Maschke [11]. The invariants f_1 are listed by Shephard and Todd, who use different letters for these polynomials.

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