Can. J. Math., Vol. XXX, No. 2, 1978, pp. 238–242

THE STABLE HOMEOMORPHISM CONJECTURE IN DIMENSION FOUR—AN EQUIVALENT CONJECTURE

BJORN FRIBERG

1. Introduction. The *stable homeomorphism conjecture* in dimension n, SHC (n), says that every orientation preserving homeomorphism of S^n is stable, i.e. can be written as the composition of homeomorphisms, each of which are the identity on some open set. This is equivalent to the homeomorphism being isotopic to the identity [**6**]. Call a homeomorphism *k-stable* if it is isotopic to a homeomorphism which is the identity on $S^k \subset S^n$. The main results are:

1) $SCH(n)$ for $n \leqq 3$ has long been known.

2) Cernavskii [**1**] showed that every homeomorphism of S^n is $(n - 3)$-stable.

3) Cernavskii [**2**] showed that for $n > 4$, every homeomorphism of S^n which is $(n - 2)$-stable is stable if it preserves orientation.

4) Kirby [**6**] showed that $SHC(n)$ is true for all $n > 4$.

5) This author [**4**] showed that orientation preserving homeomorphisms of S^n which are $(n - 2)$-stable are stable, for all n.

Thus the remaining question is whether every homeomorphism of S^4 is 2-stable, i.e., is isotopic to a homeomorphism which is fixed on S^2, or equivalently using standard techniques, fixed on a 2-disk $D^2 \subset S^4$.

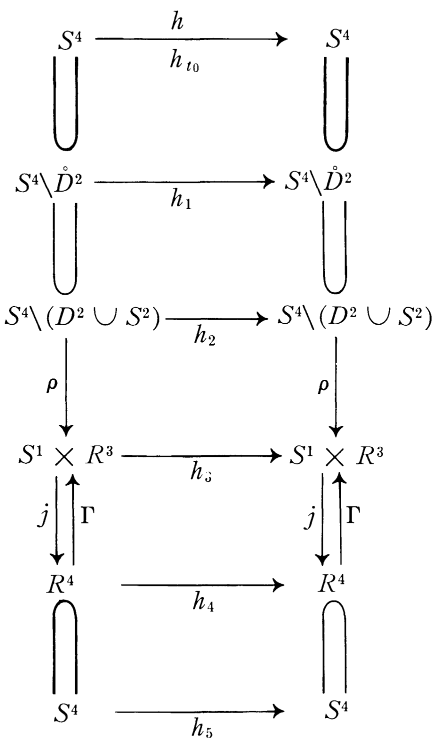
The object of this paper is to show that it is sufficient to consider a weaker problem. We prove that $SCH(4)$ is equivalent to what we call the *pseudo-isotopy conjecture* in dimension 4, $PIC(4)$, which states that every homeomorphism of S^4 which is fixed on S^1 is pseudo-isotopic to a homeomorphism of $S^4\backslash\mathring{D}^2$, fixed on S^1, the boundary of D^2. By *pseudo-isotopy* we mean an (almost) isotopy which fails to be a homeomorphism (of the whole domain) at the last stage. We do not make the usual requirement that the last stage define a map of the whole domain.

2. Preliminaries and notation. $H(X)$ will denote the space of homeomorphisms of the manifold X, with the compact-open topology. For $h \in H(X)$, an isotopy of h will be a path in $H(X)$ starting at h, or equivalently, a level preserving map $X \times I \rightarrow X \times I$ such that each level gives a homeomorphism of X, with the 0-level giving h. A pseudo-isotopy of h will be a half open path in $H(X)$, starting at h, such that the limit converges to a homeomorphism of a subset of X.

Received July 12, 1976 and in revised form, June 14, 1977. This research was supported in part by N.R.C. Grant A8735.

We denote the k-sphere by S^k, and we regard S^4 as $S^1 * S^2$, i.e., the image of the identification map: $S^1 \times [-\infty, \infty] \times S^2 \to S^1 * S^2$ with $\{s\} \times \{-\infty\} \times S^2$ identified with $\{s\} \in S^1$, and $S^1 \times \{\infty\} \times \{z\}$ identified with $z \in S^2$. Let π denote projection of S^4 on $[-\infty, \infty]$, induced by projection of $S^1 \times [-\infty, \infty] \times S^2$ on $[-\infty, \infty]$ under the identification. Let D^2 be a 2-disk in S^4 given by $S^1 * \{z_0\}$, for some $z_0 \in S^2$. We consider subspaces of S^4 as follows: $S^4 \backslash (D^2 \cup S^2) \subset S^4 \backslash \mathring{D}^2 \subset S^4$. We note that $S^4 \backslash (D^2 \cup S^2)$ is homeomorphic to $S^1 \times R^3$ by a homeomorphism $\rho : S^4 \backslash (D^2 \cup S^2) \to S^1 \times R^3$. Now $S^1 \times R^3$ embeds in R^4 in a natural way, i.e., by the embedding $j : S^1 \times R^3 \to R^4 \backslash R^2 = (R^2 \backslash \{0\}) \times R^2$ defined by $j(s, r, z) = ((\eta(r), s), z)$, $s \in S^1$, $r \in R$, $z \in R^2$, where $\eta : R \to (0, \infty)$ is a homeomorphism and $(\eta(r), s)$ are polar coordinates in $R^2 \backslash \{0\}$.

3. The main diagram. We will consider a diagram of homeomorphisms as follows:



h is orientation preserving

h is fixed on S^1

h is isotopic to h_{t_0}

$h_1 = h_{t_0}$ on U_1

$h_2 = h_1$ on U_2

$h_3 = \rho \circ h_2 \circ \rho^{-1}$

$h_4 \circ j = j \circ h_3$ on U_3

$h_5 = h_4$ on R^4

Constructing h_1: Let h be an orientation preserving homeomorphism of S^4 fixed on S^1. Let h_1' be the homeomorphism given by $PIC(4)$, assuming it true. For t_0 close to 1, h_{t_0} will be close to h_1' on a compact neighborhood in $S^4 \backslash D^2$. By local contractability [**3**], h_1' can be isotoped to agree with h_{t_0} on an open set $U_1 \subset S^4 \backslash D^2$. Call the resulting homeomorphism h_1.

Constructing h_2: Let $T_r = \{S^4\backslash \mathring{D}^2\} \cap \{\pi^{-1}[-\infty, r]\}$, where $\pi : S^4 \to [-\infty, \infty]$ denotes projection. Let $\{r_i\}$ $i = 0, 1, 2, \ldots$ be defined so that $r_0 = 0$, $1 + r_{i+1} \leqq r_i$, and

$$T_{r_{2i+2}} \subset h_1(T_{r_{2i+1}}) \subset T_{r_{2i}}.$$

The existence of the r_i follows immediately from the continuity of $h_1(h_1^{-1})$ and the fact that every neighborhood of S^1 in $S^4\backslash \mathring{D}^2$ contains some T_r.

Let Φ_t be an isotopy (of the identity) of $S^4 = S^1 * S^2$ such that Φ_t is fixed on S^1 and off T_0, commutes with projections on S^1 and S^2, and $\Phi_1(T_{-i}) = T_{r_i}$ $i = 0, 1, 2, \ldots$. Note that Φ_t restricts to $S^4\backslash \mathring{D}^2$. Then $h_{1+t} = \Phi_{2t}^{-1} \circ h_1 \circ \Phi_{2t}$ defines an isotopy of h_1 to $h_{3/2}$ satisfying

$$(**)\ T_{2n-2} \subset h_{3/2}\,(T_{2n-1}) \subset T_{2n}$$

for all $n \leqq 0$. Now, using standard techniques, we can define h_t, $3/2 \leqq t < 2$ so that $\lim_{t\to 2} h_t$ exists as a homeomorphism h_2 of $(S^4\backslash \mathring{D}^2)\backslash S^2$ and that $(**)$ holds for all n. For details see [**3** p. 85] or [**4**, p. 400]. Basically the image (under $h_{3/2}$) of some $\dot{T}_{2n-1}$ $(S^4\backslash \mathring{D}^2 \cap \pi^{-1}\{2n - 1\})$ is slid, first along the "natural" rays and then along the "curved" rays provided by an appropriate coordinate system until the images of two of the $\dot{T}_{2n-1}$ agree, i.e. commutes with the corresponding translation along the "natural" rays. It is then easy to continue the isotopy until the images of all the $\dot{T}_{2n-1}$ $n > 0$ agree, and $(**)$ holds for all n.

We restrict h_2 to $S^4\backslash(D^2 \cup S^2)$ and remark that the above pseudo-isotopy can easily be modified to insure that $h_2 = h_1$ on an open subset U_2 of $S^4\backslash(D^2 \cup S^2)$ and $h_2(x_0) = x_0$ for some x_0, losing perhaps the inclusion $(**)$ for a finite number of n. In any case we have that $|\pi \circ h_2(x) - \pi(x)| \leqq M$, for some M and all $x \in S^4\backslash(D^2 \cup S^2)$.

Constructing h_3 *and* h_4. Let $h_3 = \rho \circ h_2 \circ \rho^{-1}$, where ρ is the natural homeomorphism from $S^4\backslash(D^2 \cup S^2)$ to $S^1 \times R^3$ induced by the inclusion of $S^1 \times R \times (S^2\backslash\{z_0\})$ in $S^1 \times [-\infty, \infty] \times S^2$ after identifying R^2 with $S^2\backslash\{z_0\}$. Note that $h_3(\rho(x_0)) = \rho(x_0)$ and $|\pi' \circ h_3(x) - \pi'(x)| \leqq M$, all $x \in S^1 \times R^3$. Here π' denotes projection of $S^1 \times R^3$ on the first R factor. The construction of h_4 from h_3 is standard. Let $\Gamma : R^4 \to S^1 \times R^3$ be a covering map such that:

1) $\Gamma = e \times \mathrm{id}\,|_{R^3}$ off a compact neighborhood of $\rho(x_0)$, where $e : R \to S^1$ is an ∞ cyclic cover.

2) $\Gamma \circ j = \mathrm{id}$ on a neighborhood N of $\rho(x_0)$. Let $h_4 : R^4 \to R^4$ be the unique homeomorphism satisfying $h_4(j \circ \rho(x_0)) = j \circ \rho(x_0)$. We note that $h_4 \circ j = j \circ h_3$ on a neighborhood U_3 of $\rho(x_0)$, and that $|\pi'' \circ h_4(x) - \pi''(x)| \leqq M'$ for some M', all $x \in R^4$, where $\pi'' : R^4 \to R$ denotes projection on the second R factor of R^4.

4. Bounded homeomorphisms. We call a homeomorphism $h : R^n — R^n$ *k-bounded* (by M) if $|\pi_k \circ h(x) - \pi_k(x)| \leqq M$, for all x, where $\pi_k : R^n \to R^k$

denotes projection on $R^k \subset R^n$. We recall that in the previous section, h_4 was was 1-bounded (with respect to the second R factor). In [**5**] it is shown that $(n-2)$-bounded orientation preserving homeomorphisms of R^n are isotopic to the identity. We sketch the proof here. Suppose $h(z_1, z_2) = (z_1', z_2')$, with $||z_2' - z_2|| \leqq M$ for all $(z_1, z_2) \in R^2 \times R^{n-2}$. For a continuous function $\beta : R^{n-2} \to [0, \infty)$, set $C_\beta = \{(z_1, z_2)|z_1 \leqq \beta(z_2)\}$. If β is the constant function $\beta(z) = r$, we write C_r. Using the $(n-2)$-boundedness of h, one defines β_i, for $i = 0, 1, 2, \ldots$ so that $\beta_0 = 0$, $1 + \beta_i(z) \leqq \beta_{i+1}(z)$, and

$$C_{\beta_{2i}} \subset h(C_{\beta_{2i+1}}) \subset C_{\beta_{2i+2}}.$$

Let θ_t be an isotopy (of the identity) of $R^2 \times R^{n-2}$ which commutes with projection on R^{n-2} and for which $\theta_1(C_i) = C_{\beta_i}$. The isotopy $h_t = \theta_t^{-1} \circ h \circ \theta_t$ then satisfies

$$C_{2i} \subset h_1(C_{2i+1}) \subset C_{2i+2}.$$

Next, using standard techniques, one defines a pseudo-isotopy h_t, $1 \leqq t < 2$ so that $\lim_{t\to 2} h_t = h_2$ exists as a homeomorphism on $(R^2 \backslash \{0\}) \times R^{n-2}$; and h_2 is bounded with respect to the radial factor of R^2 (regarded as R instead of $(0, \infty)$). Adjusting h_2 to fix some point and taking an infinite cyclic cover (as in the construction of h_4 in the main diagram), one obtains an $(n-1)$-bounded homeomorphism g which agrees with h_1 (and h) on an open set. Thus h is isotopic to g, while orientation preserving $(n-1)$-bounded homeomorphisms of R^n are easily seen to be isotopic to the identity [**4**]. The same method as in Lemma 1 gives an isotopy of g to a bounded homeomorphism of R^n. The well known Alexander isotopy now completes the isotopy to the identity.

LEMMA 1. h_4 *(in the main diagram) is isotopic to the identity.*

Proof. By the above remarks, it suffices to give an isotopy of h_4 to a 2-bounded homeomorphism of R^4. We introduce the following notation: if $\alpha : R^3 \to R$ is continuous, set

$$A_\alpha = \{(r, z) \in R \times R^3 | r \leqq \alpha(z)\};$$

if α is the constant map $: z \to r$, we write A_r. Set $\alpha_0 \equiv 0$, and define $\alpha_i : R^3 \to R$ so that $1 + \alpha_i(z) \leqq \alpha_{i+1}(z)$ and

$$A_{\alpha_{2i-1}} \subset h_4(A_{\alpha_{2i}}) \subset A_{\alpha_{2i+1}} \quad \text{for all } i.$$

The existence of the functions α_i is a simple exercise in elementary topology. Let Ψ_t be an isotopy (of the identity) of $R^1 \times R^3$ commuting with projection on R^3 so that $\Psi_1(i, z) = (\alpha_i(z), z)$. Then $h_{4+t} = \Psi_t^{-1} \circ h_4 \circ \Psi_t$ defines an isotopy of h_4 so that $A_{2i-1} \subset h_5(A_{2i}) \subset A_{2i+1}$ and h_5 is bounded with respect to the first two R factors, i.e., h_5 is 2-bounded.

5. The main theorem. We first prove the following:

LEMMA 2. (Common domain) *Let A be a sub-manifold of S^4. Let $h : S^4 \to S^4$*

https://doi.org/10.4153/CJM-1978-022-0 Published online by Cambridge University Press

and $g : A \to A$ be homeomorphisms which agree on a Euclidean neighborhood $N_1 \subset A$. Let N_2 be another Euclidean neighborhood in A. Then there exists an isotopy h_t of h to h_1 such that h_1 and g agree on N_2.

Proof. Without loss of generality, assume N_1 and N_2 are disjoint. Choose a Euclidean neighborhood U containing both N_1 and N_2 in A. Let ϕ_t be an isotopy (of the identity) of U with compact support and so that $\phi_1(N_2) = N_1$. Define isotopies (of the identity) of S^4 by:

$$\alpha_t = \begin{cases} h \circ \phi_t \circ h^{-1} & \text{on } U \\ \text{id} & \text{off } U \end{cases}$$

and

$$\beta_t = \begin{cases} g \circ \phi_t^{-1} \circ g^{-1} & \text{on } U \\ \text{id} & \text{off } U. \end{cases}$$

One easily verifies that the isotopy of h defined by $h_t = \beta_t \circ \alpha_t \circ h$ satisfies the conclusion of the lemma.

THEOREM 3. *$SHC(4)$ is equivalent to $PIC(4)$.*

Proof. $SHC(4) \Rightarrow PIC(4)$ is trivial. Assume $PIC(4)$. Let h be an orientation preserving homeomorphism of S^4, fixed on S^1. Construct the main diagram. By Lemma 1, h_4 is isotopic to the identity; hence so is h_5, the one point compactification of h_4. Applying Lemma 2 twice (to h_{t_0} and h_5) we get that h (since h_{t_0} is) is isotopic to g_1 with $g_1 = h_2$ on U_2; and that h_5 is isotopic to g_2 with $g_2 \circ j = j \circ h_3$ on $\rho(U_2)$. Let $f : S^4 \to S^4$ be a homeomorphism such that $f = j \circ \rho$ on $S^4 \backslash (D^2 \cup S^2)$. For example, f could be induced by identifying D^2 to $\{\infty\} \in S^4$. Then $f^{-1} \circ g_2 \circ f = g_1$ on U_2. Since g_2 is isotopic to the identity, so is $f^{-1} \circ g_2 \circ f$. Composing the inverse of this isotopy with g_1 gives an isotopy of g_1 (hence of h) to a homeomorphism which is the identity on U_2. But then h is stable.

REFERENCES

1. A. V. Cernavskii, *Homeomorphisms of R^n are k-stable for $k \leqq n - 3$*, Mat. Sb. *70* (112) (1966); Amer. Math. Soc. Transl. (2) *78* (1968) 605–606.

2. ——— *The k-stability of homeomorphisms and the union of cells*, Soviet Math. Dokl. *9* (1968), 729–732.

3. R. D. Edwards and R. C. Kirby, *Deformations of spaces of embeddings*, Ann. of Math. *93* (1971), 63–88.

4. B. Friberg, *Canonical isotopies in Euclidean space*, Israel J. Math. *16* (1973), 398–403.

5. ——— *Bounded homeomorphisms in Euclidean space*, to appear.

6. R. C. Kirby, *Stable homeomorphisms and the annulus conjecture*, Ann. of Math. *89* (1969), 575–582.

7. P. Wright, *A uniform generalized Schoenflies theorem*, Ann. of Math. *89* (1969), 292–304.

University of Saskatchewan,
Saskatoon, Saskatchewan