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REDUCING THE CLASSICAL MULTIPLIERS ℓ^∞, c_0 AND bv_0

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For $R \in \{bv_0, c_0, \ell^\infty\}$ a multiplier of FK spaces, the classical sectional convergence theorems permit the reduction of R to any of its dense barrelled subspaces as a simple consequence of the Closed Graph Theorem. (Cf. the Bachelis/Rosenthal reduction of $R = \ell^\infty$ to its dense barrelled subspace m_0.) A natural modern setting permits the reduction of R to any of the larger class of dense $\beta\varphi$ subspaces. Bennett and Kalton's FK setting remarkably reduced $R = \ell^\infty$ to any of its dense subspaces. This extreme reduction also obtains in the modern $\beta\varphi$ setting since, surprisingly, every dense subspace of ℓ^∞ is a $\beta\varphi$ subspace. Moreover, the standard results, including the Bennett/Kalton reduction, easily follow from their $\beta\varphi$ versions and the Closed Graph Theorem. Our two supporting papers find relevant "Non-barrelled dense $\beta\varphi$ subspaces" and study "Generalized sectional convergence and multipliers". Here we specialize the $\beta\varphi$ approach to ordinary, particularly unconditional, sectional convergence.

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1. Introduction

Two companion papers [**7**, **8**] establish the $\beta\varphi$ approach for a generalized notion of sectional convergence. For ordinary sectional convergence, the approach proves simple and quite general, and quickly includes the standard multiplier space results upon application of the Closed Graph Theorem. Simplicity and unification are achieved with some knowledge of locally convex spaces [**4**, **10**].

We briefly recall (cf. [**7**, **8**]) that a locally convex sequence space is a *K space* if it is continuously included in the space of all sequences $\omega = \mathbb{K}^{\mathbf{N}}$ with its usual product topology, where $\mathbb{K}$ denotes the (real or complex) scalar field, and is an *FK* or *BK space* if it is, in addition, a Fréchet or Banach space, respectively. A *K space* that densely contains the space φ of eventually zero sequences is an *AD space*.

If the canonical unit vectors e_n form a⟨n unconditional⟩ Schauder basis for a K space S, then S is said to be an ⟨*unconditional*⟩ *AK space* and to *have* ⟨*unconditional*⟩ *AK*. Schauder basis results in a locally convex space may be translated into statements about the associated AK space of coefficients. If S has AK, then $x \in S$ implies $x = \lim_n P_n x$ in S, where each $P_n x$ agrees with x on the first n coordinates and is 0 thereafter, proving S is an AD space.

Among the treasures of Schauder basis/K space theory are three intensely studied multiplier results historically linked (cf. [7]) to Toeplitz, Köthe, Kadec, Pelczynski, Yamazaki and Garling. (The multiplication ab of two sequences a and b is coordinatewise, so that, for each j, $(ab)(j) = a(j)b(j)$. For sets A and B of sequences,

$AB = \{ab : a \in A, b \in B\}$, and [**AB**] denotes the smallest sequence space containing AB and φ.) These three results have a unified K space presentation:

I. *Let R be $\langle$either ℓ^∞ or $c_0\rangle$ (bv_0). An AD FK space S has $\langle$unconditional AK$\rangle$ (AK) if and only if $RS \subset S$.*

Since the space c of convergent sequences is spanned by c_0 and $(1, 1, \ldots)$, we have $cS \subset S$ if and only if $c_0S \subset S$. Reducing $R = \ell^\infty$ to c_0 does not change the conclusion on S, but further reduction to $bv_0 = \{x \in c_0 : \sum_n |x(n) - x(n+1)| < \infty\}$ does. Bachelis and Rosenthal [**1**] complemented Kadec and Pelczynski [**5**] by reducing $R = \ell^\infty$ to its dense subspace m_0 of finite range sequences. Now m_0 is known to be barrelled, and a general principle applies:

(**a**). *If S is any FK space and Q is a dense barrelled subspace of a K space R, then $RS \subset S$ if (and only if) $QS \subset S$.*

Proof. Given $s \in S$ and the definition of K space, the linear mapping $q \mapsto qs$ from the barrelled space Q into the Fréchet space S has a closed graph and is therefore continuous, and may be continuously extended to R by density of Q and completeness of S, implying $RS \subset S$ by the definition of K space. $\square$

Reductions of ℓ^∞, c_0 and bv_0 are immediate:

II. *Let R be a dense barrelled subspace $\langle$of either ℓ^∞ or $c_0\rangle$ (of bv_0). An AD FK space S has $\langle$unconditional AK$\rangle$ (AK) if and only if $RS \subset S$.*

In the case of ℓ^∞, Bennett and Kalton [**2**] succeeded with dramatically weaker hypotheses on both R and S which: (i) do not require R to be barrelled, only dense, and (ii) require $S \supset \varphi$ to be merely separable instead of AD. In the cases of c_0 and bv_0, neither relaxation is possible: (i) take $R = \varphi$ and take S as in [**7**, Example 3.7]; (ii) take $S = c$.

2. $\beta\varphi$ spaces

If s and t are scalar sequences, $\langle s, t\rangle$ denotes the sum $\sum_n s(n)t(n)$, provided the series converges. If s is a member of any K space S and $t \in \varphi$, the series, having only finitely many non-zero terms, must converge; moreover, by definition of K space, the linear functional $\langle \cdot, t\rangle$ is continuous, and we may view t as a member of the topological dual S' of S. A subset A of the algebraic dual E^* of a vector space E is *E-bounded* if it is $\sigma(E^*, E)$-bounded. If E is a sequence space and A is a set of sequences with $\{\langle s, t\rangle : t \in A\}$ defined and bounded for each $s \in E$, we also say A is *E-bounded*. We refer to the $\beta(S, \varphi)$ topology on any sequence space S as its *$\beta\varphi$ topology* [**6**], which has a base of neighbourhoods of zero all polar sets of the form

$$A^\circ = \{s \in S : |\langle s, t\rangle| \le 1 \text{ for all } t \in A\}$$

as A ranges through all S-bounded subsets of φ. With its $\beta\varphi$ topology, S is a *$\beta\varphi$ space*. Multiplication spaces of the form $[RS]$ are assumed to have the $\beta\varphi$ topology unless otherwise indicated. The familiar FK and BK spaces are (barrelled) $\beta\varphi$ spaces, including ω, $\ell^p(1 \le p \le \infty)$, c_0 and bv_0, as is the (non-metrizable) strong dual φ of ω. If in its relative topology a subspace of a K space is ⟨barrelled⟩ [a $\beta\varphi$ space], then we say the subspace is a *⟨barrelled⟩ [$\beta\varphi$] subspace*. The following facts (cf. [7, **8**]) are easily proved:

- *Every barrelled K space has a topology stronger than its $\beta\varphi$ topology.*
- *Every barrelled subspace of a $\beta\varphi$ space is a $\beta\varphi$ subspace.*
- ([**8**, Theorem 2.2]). *A subspace of φ, ω, c_0 or $\ell^p(1 < p < \infty)$ is a barrelled subspace if (and only if) it is a $\beta\varphi$ subspace.*

The previous fact requires

- ([**8**, Lemma 2.1]). *Let E be a locally convex space and let M be a subspace of E' such that each E-bounded subset of M is equicontinuous. If A is an E-bounded subset of E' and there exists an equicontinuous subset C of E' with $A \subset M + C$, then A is equicontinuous.*

Proof. For each $v \in A$ choose $v_1 \in M$ and $v_2 \in C$ such that $v = v_1 + v_2$, and set $B = \{v_1 : v \in A\}$. Being equicontinuous, C is E-bounded, and thus so is $B \subset A - C$. Therefore $B \subset M$ is equicontinuous, and so is $B + C$ and its subset A. □

The existence of non-barrelled dense $\beta\varphi$ subspaces [**8**] of ℓ^∞, ℓ^1 and bv_0 ensure that for two of the three classical multipliers, dense $\beta\varphi$ subspace reductions properly include the dense barrelled ones. Moreover,

(*) ([**8**, Theorem 3.1]). *Every dense subspace of ℓ^∞ is a $\beta\varphi$ subspace,*

and while isomorphisms do not generally preserve $\beta\varphi$-ness,

(**) ([**8**, Corollary 2.7]). *The $\beta\varphi$ subspaces of bv_0 are precisely the images of such subspaces under the canonical isomorphism from ℓ^1 onto bv_0.*

Another basic result is

(***) ([**7**, Corollary 3.5]). *Every barrelled AK space is a $\beta\varphi$ space.*

Unlike some metrizable barrelled AD spaces, every $\beta\varphi$ space S has a (unique) K space completion $\tilde{S}$ (cf. [**6**, **7**]). Familiar basis arguments show that *an AD $\beta\varphi$ space S has AK if and only if $bv_0 S \subset \tilde{S}$*: indeed, S has AK if and only if for each $x \in S$,

$\{P_nx : n \in \mathbb{N}\}$ is bounded in S (cf. [7, Theorems 3.2, 3.3]). Let $x \in S$. If $\{P_nx : n \in \mathbb{N}\}$ is bounded in S and $r \in bv_0$, then $[r(1)]x + \sum_{n=1}^{\infty}[r(n+1) - r(n)][x - P_nx]$ is absolutely Cauchy and must converge to rx in $\tilde{S}$; on the other hand, if $\{P_nx : n \in \mathbb{N}\}$ is not bounded, then some S-bounded $A \subset \varphi$ is not uniformly bounded on $\{P_nx : n \in \mathbb{N}\}$ and induction yields $r \in bv_0$ such that the equicontinuous A is not bounded at rx, implying rx is not in $\tilde{S}$.

The $\beta\varphi$ hypothesis on S embraces (***) but is technically neither weaker nor stronger than FK-ness. The $\beta\varphi$ analogue to (a) is simpler to state and prove:

(b). *For R, S $\beta\varphi$ spaces, $RS \subset \tilde{S}$ (if and) only if $\tilde{R}\tilde{S} \subset \tilde{S}$.*

The proof is the same as for (a), except that in place of the Closed Graph Theorem we use

(c). (Cf. [7, Theorem 2.1]). *If R and T are sequence spaces and w is a sequence such that $wR \subset T$, then the map $r \mapsto wr$ from R into T is continuous with respect to the $\beta\varphi$ topologies on R and T.*

Proof. For B a T-bounded subset of φ, the equality $\langle r, wu\rangle = \langle wr, u\rangle$ for $r \in R$ and $u \in B$ implies that wB is an R-bounded subset of φ whose polar $(wB)^\circ$ in R is the pre-image of the polar B° in T. □

The same simplicity shines in Theorem 5.1 of [7]:

(d). *If R is an AK $\beta\varphi$ space and S is any sequence space, then $[RS]$ has AK.*

Proof. Continuity yields $rs = (\lim P_nr)s = \lim(P_nr)s = \lim P_n(rs)$. □

3. $\beta\varphi$ reductions

There are $\beta\varphi$ reductions of bv_0 in both the classical and $\beta\varphi$ settings.

(e). (Cf. [7, Theorems 5.7, 5.8]). *If R is a dense $\beta\varphi$ subspace of bv_0 and $S \supset \varphi$ is any $\langle\beta\varphi$ space$\rangle$ (FK space), the following statements are equivalent:*

1. *S has AK;*
2. *S has AD and $\langle RS \subset \tilde{S}\rangle$ $(RS \subset S)$;*
3. *$[RS]$ is a dense $\beta\varphi$ subspace $\langle$of $\tilde{S}\rangle$ (of S);*
4. *$[RS]$ is a dense barrelled subspace $\langle$of $\tilde{S}\rangle$ (of S).*

Proof. Consider the case of S a $\beta\varphi$ space. We have already sketched the equivalence of 1 and 2, and [4 ⇒ 3] is obvious. But Theorem 5.7 of [7] completely covers the $\beta\varphi$ case via (**).

Now we assume the parenthetical options, with S an FK space. Denote the barrelled space S with its coarser $\beta\varphi$ topology by T. If 1 holds, then (***) yields $S = T = \tilde{T}$. Indeed, if any one of 1–4 holds, the previous $\beta\varphi$ case implies $[RS] = [RT]$ is a dense barrelled subspace of T, implying T itself is barrelled, so that $T = S$ by the Closed Graph Theorem. □

A series in an $AD\ \beta\varphi$ space S is unconditionally convergent if and only if it is subseries convergent in $\tilde{S}$. Thus if S has unconditional AK, then $rS \subset \tilde{S}$ for each r with range $\{0,1\}$; $m_0 S \subset \tilde{S}$. It is very easy to see that m_0 is a dense subspace and therefore, by (*), a $\beta\varphi$ subspace of ℓ^∞. [In fact, m_0 is a barrelled subspace of ℓ^∞, but the proof (Grothendieck) takes some work.] We conclude from (b), then, that $\ell^\infty S \subset \tilde{S}$. Conversely, if $\ell^\infty S \subset \tilde{S}$, then (b) yields $\ell^\infty \tilde{S} \subset \tilde{S}$, so that $bv_0 \tilde{S} \subset \tilde{S} = \tilde{\tilde{S}}$ implies, by (e), that $\tilde{S}$ has AK, and then because $m_0 \tilde{S} \subset \tilde{S}$, expansions are subseries convergent in $\tilde{S}$, which implies $\tilde{S}$ has unconditional AK. We have proved

(f). *An $AD\ \beta\varphi$ space S has unconditional AK if and only if $\ell^\infty S \subset \tilde{S}$.*

Now all the unconditional AK reductions, including Bennett and Kalton's [**2**], will follow from

Theorem 3.1. *Let $R \subset \ell^\infty$ be a sequence space whose $\beta\varphi$ completion $\tilde{R}$ contains c_0. For any $\langle AD\ \beta\varphi$ space$\rangle$ $(AD\ FK$ space$)$ S the following are equivalent:*

(i) *S has unconditional AK;*

(ii) $\langle RS \subset \tilde{S}\rangle$ $(RS \subset S)$*;*

(iii) $[RS]$ *is a dense $\beta\varphi$ subspace* $\langle$*of* $\tilde{S}\rangle$ (*of* S)*;*

(iv) $[RS]$ *is a dense barrelled subspace* $\langle$*of* $\tilde{S}\rangle$ (*of* S).

Note. We cannot delay the AD hypothesis until (ii), as we did in (e) 2, nor can we relax it to merely require $S \supset \varphi$ be separable, since $cc = c$ is non-AK. If a sequence space T satisfies $c_0 \subset T \subset \ell^\infty$, then T is a $\beta\varphi$ subspace of ℓ^∞ (Uniform Boundedness Principle). Thus the hypothesis on R is equivalent to: "Let R be a $\beta\varphi$ subspace of ℓ^∞ whose closure contains c_0".

Proof. As in the proof of (e), the case with S a $\beta\varphi$ space implies, via the Closed Graph Theorem, the case with S an FK space, and so we consider only the former.

[(i) $\Rightarrow$ (ii)] follows from (f), and [(iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii)] is trivial. It remains to show that (ii) $\Rightarrow$ [(i) $\wedge$ (iv)]. Assume (ii) holds. By (b), $\tilde{R}\tilde{S} \subset \tilde{S}$, so that $\tilde{S}$ is an $AD\ \beta\varphi$ space with $bv_0 \tilde{S} \subset \tilde{S} = \tilde{\tilde{S}}$, which implies $\tilde{S}$ has AK by [2 $\Rightarrow$ 1] of (e). In particular, $r \in \tilde{R}$ and $s \in \tilde{S}$ imply the sequence $(P_n rs)_n$ converges to rs in $\tilde{S}$. Since $[RS]$ contains φ it is dense in the AD space $\tilde{S}$, and to prove (iv) we must show that if $G \subset \tilde{S}'$ is $[RS]$-bounded, then G is equicontinuous. First we prove (i). Given $s \in S$, for each $g \in G$ we compose continuous functions to obtain $h_{s,g} \in \tilde{R}'$ defined by $h_{s,g}(r) = g(rs)$. If $r \in \tilde{R}$ is, in fact, an element of R, then $rs \in RS$ and $\{h_{s,g}(r) : g \in G\} = \{g(rs) : g \in G\}$ is bounded; i.e., $A = \{h_{s,g} : g \in G\}$ is R-bounded.

A has a stronger property. For any $r \in \tilde{R}$,

$$\begin{aligned} h_{s,g}(r) &= g(rs) = \lim_n g(P_n(rs)) \\ &= \lim_n g((P_n r)s) = \lim_n h_{s,g}(P_n r) = \langle r, (h_{s,g}(e_n))_n \rangle, \end{aligned}$$

and $c_0 \subset \tilde{R} \subset \ell^\infty$ implies the $\beta\varphi$ topologies on R and $\tilde{R}$ are given by the sup norm, so we may identify each $h_{s,g}$ with the sequence $(h_{s,g}(e_n))_n \in \ell^1 \subset \varphi + C$, where C is the unit ball in ℓ^1. Since C acts equicontinuously on any subspace of ℓ^∞, so does A on R ([**8**, Lemma 2.1], proved above), and thus on $\tilde{R}$, by density. Given $w \in \ell^\infty$, then, A is uniformly bounded on the bounded subset $\{P_n w : n \in \mathbb{N}\}$ of $\tilde{R}$, so that $B = \{g(P_n w\cdot) \in S' : g \in G,\ n \in \mathbb{N}\}$ is bounded at s and represents an S-bounded subset of φ. Since the $\beta\varphi$ space S has AK, there exists M such that $m > M$ implies

$$s - P_m s \in B^\circ \text{ (polar in } S).$$

Therefore $n \geq m > M$ and $g \in G$ imply

$$1 \geq |g(P_n w(s - P_m s))| = |g(P_n ws - P_n w P_m s)| = |g(P_n(ws) - P_m(ws))|,$$

which yields

$$\{P_n(ws) - P_m(ws) : n \geq m > M\} \subset G^\circ.$$

Hence $(P_n(ws))_n$ is Cauchy and converges in $\tilde{S}$, necessarily to ws, since $\tilde{S}$ is a K space. We conclude that $\ell^\infty S \subset \tilde{S}$, and (i) holds by (f).

For the special case $w = (1, 1, \ldots)$, B° is still a neighbourhood of 0 in S, and whenever $g \in G$ and $x \in B^\circ$ we have

$$|g(x)| = \lim_n |g(P_n x)| = \lim_n |g(P_n wx)| \leq 1,$$

so that $G^\circ \supset B^\circ$; i.e., G is equicontinuous on the dense subspace S, thus on $\tilde{S}$, and (iv) holds. □

Corollary 3.2. *If R is a dense $\beta\varphi$ [equivalently [**8**], barrelled] subspace of c_0 and $S \supset \varphi$ is any $\langle\beta\varphi$ space$\rangle$ (FK space), the following are equivalent:*

(i) *S has unconditional AK;*

(ii) *S has AD and $\langle RS \subset \tilde{S}\rangle$ $(RS \subset S)$;*

(iii) *$[RS]$ is a dense $\beta\varphi$ subspace $\langle$of $\tilde{S}\rangle$ (of S);*

(iv) *$[RS]$ is a dense barrelled subspace $\langle$of $\tilde{S}\rangle$ (of S).*

Note. The AD hypothesis on S delays to (ii), but cannot relax to mere separability and containment of φ since $c_0 c = c_0 \subset c$.

Proof. As above, the Closed Graph Theorem reduces the proof to the case of S a $\beta\varphi$ space. If any of (i)–(iv) holds, then S has AD and the Theorem applies. This is obvious for (i) and (ii). If (iii) holds, then by (b), $[c_0S]$ is a dense $\beta\varphi$ subspace of $\tilde{S}$ and has AK by (d). Thus $[c_0S]$, then $\tilde{S}$, and then S must have AD. Finally, since any barrelled subspace of a $\beta\varphi$ space is a $\beta\varphi$ subspace, (iv) likewise implies S has AD. □

For $R = \ell^\infty$ we relax the AD hypothesis on S via

Lemma 3.3. *If $S \supset \varphi$ is a $\beta\varphi$ space and $\ell^\infty S \subset \tilde{S}$, then S is an AD space if (and only if) S is separable.*

Proof. Suppose S is non-AD. We shall show that S is non-separable by showing $\tilde{S}$ is non-separable, and for this it suffices to find uncountably many disjoint neighbourhoods in $\tilde{S}$.

Since the $\beta\varphi$ space $\tilde{S}$ is non-AD, there exist $x \in \tilde{S}$ and an S-bounded subset A of φ such that $x + A^\circ$ misses φ. Thus $0 \notin x + A^\circ = x - A^\circ$, so that $x \notin A^\circ$, and there exists $y_1 \in A$ with $|\langle x, y_1\rangle| > 1$. Since $y_1 \in \varphi$, there exists $n_1 \in \mathbb{N}$ such that $y_1 = P_{n_1}y_1$. Now $P_{n_1}x$ is in φ, thus not in $x + A^\circ$, so there exists $y_2 \in A$ with $|\langle x - P_{n_1}x, y_2\rangle| > 1$; there exists $n_2 \in \mathbb{N}$ such that $y_2 = P_{n_2}y_2$. Of necessity, $n_2 > n_1$. We continue inductively to choose positive integers $n_1 < n_2 < \ldots$ and $y_1, y_2, \ldots \in A$ such that, defining $n_0 = 0$ and $P_{n_0} = 0$, we have

$$|\langle x - P_{n_{k-1}}x, y_k\rangle| > 1 \text{ and } P_{n_k}y_k = y_k \text{ for all } k \in \mathbb{N}.$$

Given any $D \subset \mathbb{N}$, define $\delta(D) \subset \mathbb{N}$ and $r_D \in \ell^\infty$ so that

$$\delta(D) = \{j \in \mathbb{N} : n_{k-1} < j \le n_k \text{ for some } k \in D\}, \text{ and}$$
$$r_D(j) = \begin{cases} 1, & \text{if } j \in \delta(D); \\ 0, & \text{if } j \notin \delta(D). \end{cases}$$

Now suppose C and D are distinct subsets of $\mathbb{N}$. Let q denote the least positive integer in $(C \cup D)\backslash(C \cap D)$. A routine check shows that

$$|\langle r_C \cdot x - r_D \cdot x, y_q\rangle| = |\langle \pm(P_{n_q}x - P_{n_{q-1}}x), y_q\rangle| = |\langle \pm(x - P_{n_{q-1}}x), y_q\rangle| > 1.$$

It follows that $r_C \cdot x - r_D \cdot x \notin A^\circ$, and $r_C \cdot x + \frac{1}{2}A^\circ$ misses $r_D \cdot x + \frac{1}{2}A^\circ$, so that there are $\mathfrak{c}$ disjoint neighbourhoods in $\tilde{S}$. □

Below, the FK version of [(i) $\Leftrightarrow$ (ii)] is the Bennett and Kalton reduction [2].

Corollary 3.4. *If R is a dense subspace of ℓ^∞ and $S \supset \varphi$ is any separable $\langle\beta\varphi$ space$\rangle$ (FK space), the following statements are equivalent:*

(i) *S has unconditional AK;*

(ii) $\langle RS \subset \tilde{S}\rangle$ $(RS \subset S)$;

(iii) $[RS]$ *is a dense* $\beta\varphi$ *subspace* $\langle$*of* $\tilde{S}\rangle$ (*of* S);

(iv) $[RS]$ *is a dense barrelled subspace* $\langle$*of* $\tilde{S}\rangle$ (*of* S).

Note. We cannot delay separability until (ii), since $\ell^\infty\ell^\infty = \ell^\infty$ is non-AK.

Proof. For S a $\beta\varphi$ space, use (*), (b), 3.3 and 3.1. □

Remark. Our reductions (e), 3.2 and 3.4 validate a simple and unified $\beta\varphi$ approach. However, it should be pointed out that some of the older FK versions put Rx with x a single point of S in place of our RS. Thus, for example, the Bennett and Kalton reduction [2] actually says that, in the above language, for $x \in S$, the series $\sum x(n)e_n$ converges unconditionally in S if and only if $Rx \subset S$. Many of our arguments have similar pointwise interpretations. To illustrate, the argument of 3.3 proves the following: *If* $S \supset \varphi$ *is a separable* $\beta\varphi$ *space and for some* $x \in S$ *we have* $\ell^\infty x \subset \tilde{S}$, *then* $[\ell^\infty x]$ *has* AD *in the topology induced by* S.

REFERENCES

1. G. F. Bachelis and H. P. Rosenthal, On unconditionally converging series and biorthogonal systems in Banach spaces, *Pacific J. Math.* **37** (1971), 1–5.

2. G. Bennett and N. J. Kalton, Addendum to FK spaces containing c_0, *Duke Math. J.* **39** (1972), 819–821.

3. D. J. H. Garling, On topological sequence spaces, *Proc. Cambridge Philos. Soc.* **63** (1967), 997–1019.

4. J. Horváth, *Topological Vector Spaces and Distributions, Vol. I.* (Reading, Addison-Wesley, 1966).

5. M. I. Kadec and A. Pelczynski, Basic sequences, biorthogonal sequences and norming sets in Banach and Fréchet spaces, *Studia Math.* **25** (1965), 297–323 (Russian).

6. W. Ruckle, Lattices of sequence spaces, *Duke Math. J.* **35** (1968), 491–503.

7. W. H. Ruckle and S. A. Saxon, Generalized sectional convergence and multipliers, *J. Math. Anal. Appl.* **193** (1995), 680–705.

8. S. A. Saxon, Non-barrelled dense $\beta\varphi$ subspaces, *J. Math. Anal. Appl.* **196** (1995), 428–441.

9. S. Saxon and A. Wilansky, The equivalence of some Banach space problems, *Colloq. Math.* **37** (1977), 217–226.

10. A. Wilansky, *Modern Methods in Topological Vector Spaces,* (New York, McGraw-Hill, 1978).

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