Bull. Aust. Math. Soc. **94** (2016), 296–303
doi:10.1017/S0004972715001835

MORE ON THE ARENS REGULARITY OF $B(X)$

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(Received 30 November 2015; accepted 9 December 2015; first published online 1 March 2016)

Abstract

We focus on a question raised by Daws ['Arens regularity of the algebra of operators on a Banach space', *Bull. Lond. Math. Soc.* **36** (2004), 493–503] concerning the Arens regularity of $B(X)$, the algebra of operators on a Banach space X. Among other things, we show that $B(X)$ is Arens regular if and only if X is ultrareflexive.

2010 *Mathematics subject classification*: primary 47L10; secondary 46H25, 47L50.

Keywords and phrases: Arens products, algebra of operators, reflexive space, superreflexive, weak operator topology, ultrapower.

1. Introduction

The second dual A^{**} of a Banach algebra A can be made into a Banach algebra with two, in general, different (Arens) products, each extending the original product of A [1]. A Banach algebra A is said to be Arens regular when the Arens products coincide. For example, every C^*-algebra is Arens regular [2]. For an explicit description of the properties of these products and the notion of Arens regularity, one may consult [3].

For the Banach algebra $B(X)$ of bounded operators on a Banach space X, Daws [5, Theorem 1] showed that if X is superreflexive, then $B(X)$ is Arens regular. He also conjectured the validity of the converse. To the best of our knowledge, this has not yet been resolved. However, it is known that the Arens regularity of $B(X)$ necessitates the reflexivity of X (see [7, Theorems 2, 3] or [3, Theorem 2.6.23]).

In Section 2, we provide some preliminaries related to ultrapowers and superreflexivity. In Section 3 we prove Theorem 3.1 from which we show that the reflexivity of X is equivalent to the *wo*-compactness of $\mathrm{Ball}(B(X))$. This motivates us to introduce the notion of an ultrareflexive space and compare it with superreflexivity. Section 4 is devoted to the main result of the paper (Theorem 4.4) stating that $B(X)$ is Arens regular if and only if X is ultrareflexive.

2. Preliminaries

Let X be a Banach space, I an indexing set and $\mathcal{U}$ an ultrafilter on I. We define the ultrapower $X_{\mathcal{U}}$ of X with respect to $\mathcal{U}$ to be the quotient space

$$X_{\mathcal{U}} = \ell^\infty(X, I)/\mathcal{N}_{\mathcal{U}},$$

where $\ell^\infty(X, I)$ is the Banach space

$$\ell^\infty(X, I) = \Big\{(x_\alpha)_{\alpha\in I} \subseteq X : \|(x_\alpha)\| = \sup_{\alpha\in I} \|x_\alpha\| < \infty\Big\},$$

and $\mathcal{N}_{\mathcal{U}}$ is the closed subspace

$$\mathcal{N}_{\mathcal{U}} = \Big\{(x_\alpha)_{\alpha\in I} \in \ell^\infty(X, I) : \lim_{\mathcal{U}} \|x_\alpha\| = 0\Big\}.$$

Then the norm $\|(x_\alpha)\|_{\mathcal{U}} := \lim_{\mathcal{U}} \|x_\alpha\|$ coincides with the quotient norm. We can identify X with a closed subspace of $X_{\mathcal{U}}$ via the canonical isometric embedding $X \hookrightarrow X_{\mathcal{U}}$, sending $x \in X$ to the constant family (x). More information about ultrapowers can be found in [6].

A Banach space X is called superreflexive if every finitely representable Banach space in X is reflexive. We recall that a Banach space Y is finitely representable in X if each finite dimensional subspace of Y is $(1 + \epsilon)$-isomorphic to some subspace of X, for each $\epsilon > 0$. For example, every Banach space is finitely representable in c_0, and every finitely representable Banach space in ℓ^2 is a Hilbert space. In the language of ultrapowers, Y is finitely representable in X if and only if Y is isometrically isomorphic to a subspace of $X_{\mathcal{U}}$ for some ultrafilter $\mathcal{U}$ on X [6, Theorem 6.3]. It follows that a Banach space is superreflexive if and only if all of its ultrapowers are reflexive. Further, X is superreflexive if and only if X^* is superreflexive [6].

From [6, Section 7], there is a canonical isometry $J : (X^*)_{\mathcal{U}} \to (X_{\mathcal{U}})^*$ defined by

$$\langle J((f_\alpha)_{\mathcal{U}}), (x_\alpha)_{\mathcal{U}}\rangle = \lim_{\mathcal{U}} \langle f_\alpha, x_\alpha\rangle, \quad ((f_\alpha)_{\mathcal{U}} \in (X^*)_{\mathcal{U}}, (x_\alpha)_{\mathcal{U}} \in X_{\mathcal{U}}),$$

which is a surjection if and only if $X_{\mathcal{U}}$ is reflexive (where $\mathcal{U}$ is countably incomplete). In particular, when X is superreflexive, J is an isometric isomorphism.

As $\mathrm{Ball}(X^{**})$ is w^*-compact, we can define a norm-decreasing map $\sigma : X_{\mathcal{U}} \to X^{**}$ by

$$\sigma((x_\alpha)_{\mathcal{U}}) = w^* - \lim_{\mathcal{U}} \kappa_X(x_\alpha), \quad ((x_\alpha)_{\mathcal{U}} \in X_{\mathcal{U}}),$$

where κ_X is the canonical embedding of X into X^{**}.

Proposition 2.1 [6, Proposition 6.7]. *Let X be a Banach space. Then there exist an ultrafilter $\mathcal{U}$ and a linear isometric embedding $K : X^{**} \to X_{\mathcal{U}}$ such that $\sigma \circ K$ is the identity on X^{**} and $K \circ \kappa_X$ is the canonical embedding of X into $X_{\mathcal{U}}$. Thus $K \circ \sigma$ is a norm-one projection of $X_{\mathcal{U}}$ onto $K(X^{**})$.*

https://doi.org/10.1017/S0004972715001835 Published online by Cambridge University Press

Note that the ultrafilter $\mathcal{U}$, used in the above proposition, is countably incomplete. Indeed, $\mathcal{U}$ is the ultrafilter induced by refining the order filter on the set

$$I = \{(M, N, \varepsilon) : M \subseteq X^{**} \text{ is finite}, \ N \subseteq X^* \text{ is finite}, \ \varepsilon > 0\}.$$

Set $I_n = I_{(M,N,n^{-1})} = \{(M_0, N_0, \varepsilon) \in I : M \subseteq M_0, N \subseteq N_0 \text{ and } \varepsilon \leq n^{-1}\}$. Then $I_{n+1} \subseteq I_n$ and $\bigcap_{n=1}^{\infty} I_n = \emptyset$, so $\mathcal{U}$ is countably incomplete.

There are several criteria for the Arens regularity of a Banach algebra, from which we quote the following, which we will use frequently. For a proof see [3, 5].

Proposition 2.2. *For every Banach algebra A the following assertions are equivalent.*

(1) *A is Arens regular.*
(2) *For each $\lambda \in A^*$ the operator $a \mapsto \lambda \cdot a : A \longrightarrow A^*$ is weakly compact.*
(3) *For each $\lambda \in A^*$ there exist a reflexive space Z and a pair of bounded linear maps $\phi : A \to Z$ and $\psi : A \to Z^*$ such that $\langle \lambda, ab \rangle = \langle \psi(a), \phi(b) \rangle$ for all $a, b \in A$.*

We remark that, in Proposition 2.2, we can choose Z and ϕ so that $\|\lambda\| \leq \|\phi\|$.

3. Weak operator compactness and reflexivity

Let X and Y be two Banach spaces and let τ be a locally convex topology on Y induced by a separating family $\{p_\gamma\}_{\gamma \in \Gamma}$ of seminorms. Then τ induces a τo-topology on $B(X, Y)$ which is induced by the family $\{x \otimes p_\gamma\}_{x \in X, \gamma \in \Gamma}$ of seminorms, where $(x \otimes p_\gamma)(T) = p_\gamma(T(x))$ for all $x \in X$, $T \in B(X, Y)$. Consequently, $T_\alpha \xrightarrow{\tau o} T$ if and only if $T_\alpha(x) \xrightarrow{\tau} T(x)$, for each $x \in X$. For example, in the case when $\tau = w$ is the weak topology on Y, the τo-topology on $B(X, Y)$ is the weak operator topology (wo-topology) on $B(X, Y)$. The next result relates $\mathrm{Ball}(Y)$ and $\mathrm{Ball}(B(X, Y))$.

Theorem 3.1. *Let X and Y be two Banach spaces and let τ be a locally convex topology on Y which is weaker than the norm topology and induced by the seminorms $\{p_\gamma\}$ such that $\|y\| \leq \sup_{\|p_\gamma\| \leq 1} |p_\gamma(y)|$, for all $y \in Y$. Then* $\mathrm{Ball}(Y)$ *is τ-compact if and only if* $\mathrm{Ball}(B(X, Y))$ *is τo-compact.*

Proof. Suppose $\mathrm{Ball}(B(X, Y))$ is τo-compact and fix $f \in X^*$ with $\|f\| = 1$ and $f(x_0) = 1$, for some $x_0 \in X$. Then the operator $\Psi : Y \to B(X, Y)$, where $\Psi(y)(x) = f(x)y$ is an isometry. Moreover, $y_\alpha \xrightarrow{\tau} y$ if and only if $\Psi(y_\alpha) \xrightarrow{\tau o} \Psi(y)$. If $\{y_\alpha\}$ is a net in $\mathrm{Ball}(Y)$, then $\{\Psi(y_\alpha)\}$ is a net in $\mathrm{Ball}(B(X, Y))$, so it has a τo-convergent subnet, say, $\{\Psi(y_{\alpha_\beta})\}$. Then $\{y_{\alpha_\beta}\}$ is τ-convergent in Y: that is, $\mathrm{Ball}(Y)$ is τ-compact.

For the converse, define the operator $\Phi : B(X, Y) \to \Pi_{x \in S_X} Y$ by $\Phi(T) = (Tx)_{x \in S_X}$. Obviously, Φ is one-to-one and $T_\alpha \xrightarrow{\tau o} T$ if and only if $\Phi(T_\alpha) \xrightarrow{\Pi\tau} \Phi(T)$. If $\{T_\alpha\}$ is a net in $\mathrm{Ball}(B(X, Y))$, then $\{\Phi(T_\alpha)\}$ is a net in $\Pi_{x \in S_X} \mathrm{Ball}(Y)$. By the Tychonoff theorem, $\Pi_{x \in S_X} \mathrm{Ball}(Y)$ is compact in the product τ-topology, so $\{\Phi(T_\alpha)\}$ has a subnet $\{\Phi(T_{\alpha_\beta})\}$ convergent in the product τ-topology and we can define an operator $T : X \to X$ by $T(x) = \tau - \lim_\beta T_{\alpha_\beta}(x)$. For each $x \in S_X$, since $|p_\gamma(T_{\alpha_\beta}(x))| \leq \|p_\gamma\| \, \|T_{\alpha_\beta}(x)\|$,

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we have $|p_\gamma(T(x))| \le \|p_\gamma\| \liminf \|T_{\alpha_\beta}(x)\|$. It follows that $\|T(x)\| \le \sup_{\|p_\gamma\|\le 1} |p_\gamma(T(x))| \le \liminf \|T_{\alpha_\beta}(x)\| \le 1$. Thus $\|T\| \le 1$ and $T_{\alpha_\beta} \xrightarrow{\tau o} T$, as claimed. □

The next result is an immediate consequence.

Proposition 3.2. *Let X be a Banach space. Then:*

(1) Ball$(B(X))$ *is wo-compact if and only if X is reflexive;*
(2) Ball$(B(X))$ *is so-compact if and only if X is finite dimensional; and*
(3) Ball(X^*) *is w^*-compact (Banach–Alaoglu).*

Proof. For (1) (respectively, (2)) we use Theorem 3.1 for $Y = X$ with τ as the weak (respectively, norm) topology. For (3) we use Theorem 3.1 for $Y = \mathbb{C}$ with τ as the usual topology. □

4. Arens regularity of $B(X)$ and ultrareflexivity

We commence with the next key lemma that will be used frequently in the subsequent work.

Lemma 4.1. *If X is reflexive then there is an (countably incomplete) ultrafilter $\mathcal{U}$ such that every $\lambda \in B(X)^*$ can be identified with $x_{\mathcal{U}} \otimes f_{\mathcal{U}}$ for some $x_{\mathcal{U}} \in \ell^2(X)_{\mathcal{U}}$ and $f_{\mathcal{U}} \in \ell^2(X^*)_{\mathcal{U}}$.*

Proof. The reflexivity of X implies that $B(X)^* \cong (X \widehat{\otimes} X^*)^{**}$. By Proposition 2.1, there exist an ultrafilter $\mathcal{U}$ and a linear isometric embedding

$$(X \widehat{\otimes} X^*)^{**} \hookrightarrow (X \widehat{\otimes} X^*)_{\mathcal{U}},$$

such that the composition $(X \widehat{\otimes} X^*) \hookrightarrow (X \widehat{\otimes} X^*)^{**} \hookrightarrow (X \widehat{\otimes} X^*)_{\mathcal{U}}$ coincides with the canonical embedding $(X \widehat{\otimes} X^*) \hookrightarrow (X \widehat{\otimes} X^*)_{\mathcal{U}}$. Consider $\lambda \in B(X)^* \hookrightarrow (X \widehat{\otimes} X^*)_{\mathcal{U}}$. Then $\lambda = (\lambda_\alpha)_{\mathcal{U}}$ with $\lambda_\alpha = \sum_{n=1}^\infty x_n^\alpha \otimes f_n^\alpha$, $\|\lambda_\alpha\| \le \sum_{n=1}^\infty \|x_n^\alpha\| \|f_n^\alpha\| \le \|\lambda\| + 1$ and $\|x_n^\alpha\| = \|f_n^\alpha\|$ for each $(\alpha, n) \in I \times \mathbb{N}$.

For each α, put $x^\alpha = (x_n^\alpha)_{n\in\mathbb{N}}$ and $f^\alpha = (f_n^\alpha)_{n\in\mathbb{N}}$. Then $x^\alpha \in \ell^2(X)$ and $f^\alpha \in \ell^2(X^*)$. Indeed

$$\|x^\alpha\|_2^2 = \sum_{n=1}^\infty \|x_n^\alpha\|^2 = \sum_{n=1}^\infty \|x_n^\alpha\| \|f_n^\alpha\| \le \|\lambda\| + 1.$$

Now set $x_{\mathcal{U}} = (x^\alpha)_{\mathcal{U}}$ and $f_{\mathcal{U}} = (f^\alpha)_{\mathcal{U}}$. Clearly $x_{\mathcal{U}} \in \ell^2(X)_{\mathcal{U}}$ and $f_{\mathcal{U}} \in \ell^2(X^*)_{\mathcal{U}}$. Then $\lambda = x_{\mathcal{U}} \otimes f_{\mathcal{U}}$, because

$$\begin{aligned}\lambda(T) = (\lambda_\alpha)_{\mathcal{U}}(T) = \lim_{\mathcal{U}} \lambda_\alpha(T) = \lim_{\mathcal{U}} \sum_{n=1}^\infty f_n^\alpha(T x_n^\alpha) \\ = ((x^\alpha)_{\mathcal{U}} \otimes (f^\alpha)_{\mathcal{U}})(T) = (x_{\mathcal{U}} \otimes f_{\mathcal{U}})(T).\end{aligned}$$ □

We recall that the superreflexivity of X is equivalent to that of $\ell^2(X)$, the Banach space of all two-summable sequences in X (see [5, Proposition 4]). So X is superreflexive if and only if Ball$(\ell^2(X)_{\mathcal{U}})$ is weakly compact, or, equivalently, by

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Proposition 3.2, if $\mathrm{Ball}(B(\ell^2(X)_{\mathcal{U}}))$ is *wo*-compact for every ultrafilter $\mathcal{U}$. This motivates us to introduce the notion of ultrareflexivity in the next definition.

DEFINITION 4.2. A Banach space X is called ultrareflexive if $\mathrm{Ball}(B(X))(x_{\mathcal{U}})$ is weakly compact for every ultrafilter $\mathcal{U}$ and each $x_{\mathcal{U}} \in \ell^2(X)_{\mathcal{U}}$.

Clearly, every ultrareflexive space X is reflexive. Indeed, for each nonzero $x \in X$, $B(X)(x) = X$. Further, if X is superreflexive then X is ultrareflexive (see Corollary 4.5). Therefore ultrareflexivity lies between reflexivity and superreflexivity.

We are now ready to prove our main result characterising the Arens regularity of $B(X)$ in terms of the ultrareflexivity of X. The next technical lemma from [4] will be used in the proof of Theorem 4.4.

LEMMA 4.3 [4, Lemma 1]. *Let X be a Banach space and $W \subseteq X$ be a bounded, symmetric and convex subset. For each $n \in \mathbb{N}$ let the norm $\|\cdot\|_n$ denote the gauge of $U_n = 2^n W + 2^{-n}\mathrm{Ball}(X)$. Set $Y = \{x \in X : |||x||| = (\sum_{n=1}^{\infty} \|x\|_n^2)^{1/2} < \infty\}$. Then:*

(i) *$W \subseteq \mathrm{Ball}(Y)$;*
(ii) *$(Y, |||\cdot|||)$ is a Banach space and the identity embedding $j : Y \to X$ is bounded;*
(iii) *$j^{**} : Y^{**} \to X^{**}$ is one-to-one and $(j^{**})^{-1}(X) = Y$; and*
(iv) *Y is reflexive if and only if W is weakly relatively compact.*

THEOREM 4.4. *For a Banach space X the following assertions are equivalent:*

(a) *$B(X)$ is Arens regular;*
(b) *$f_{\mathcal{U}} \circ \mathrm{Ball}(B(X))$ is w^*-compact for every ultrafilter $\mathcal{U}$ and each $f_{\mathcal{U}} \in \ell^2(X)_{\mathcal{U}}{}^*$;*
(c) *$f_{\mathcal{U}} \circ \mathrm{Ball}(B(X))$ is w-compact for every ultrafilter $\mathcal{U}$ and each $f_{\mathcal{U}} \in \ell^2(X)_{\mathcal{U}}{}^*$;*
(d) *X is ultrareflexive.*

PROOF. (a) $\Rightarrow$ (b). Suppose $B(X)$ is Arens regular, $\mathcal{U}$ is an ultrafilter, $f_{\mathcal{U}} \in \ell^2(X)_{\mathcal{U}}{}^*$ and $x \in X$. These elements induce the functional $x \otimes f_{\mathcal{U}} \in B(X)^*$.

Suppose that $\{T_\alpha\}$ is a net in $\mathrm{Ball}(B(X))$. Since X is reflexive [3, Theorem 2.6.23], by Proposition 3.2, $\mathrm{Ball}(B(X))$ is *wo*-compact, so $\{T_\alpha\}$ has a subnet $\{T_{\alpha_\beta}\}$ such that $T_{\alpha_\beta} \xrightarrow{wo} T_0$, for some $T_0 \in \mathrm{Ball}(B(X))$. This implies that

$$(x \otimes f_{\mathcal{U}}) \cdot T_{\alpha_\beta} \xrightarrow{w^*} (x \otimes f_{\mathcal{U}}) \cdot T_0.$$

By Proposition 2.2, the Arens regularity of $B(X)$ also implies the weak compactness of the operator

$$T \mapsto (x \otimes f_{\mathcal{U}}) \cdot T : B(X) \to B(X)^*, \quad (T \in B(X)),$$

from which

$$(x \otimes f_{\mathcal{U}}) \cdot T_{\alpha_\beta} \xrightarrow{w} (x \otimes f_{\mathcal{U}}) \cdot T_0,$$

for some new subnet $\{T_{\alpha_\beta}\}$ of the previous one.

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Fix $x_{\mathcal{U}} \in \ell^2(X)_{\mathcal{U}}$ and choose $S_{\mathcal{U}} \in B(\ell^2(X)_{\mathcal{U}})$ so that $S_{\mathcal{U}}(x) = x_{\mathcal{U}}$. Define the operator $F_{S_{\mathcal{U}}} : (x \otimes f_{\mathcal{U}}) \cdot B(X) \to \mathbb{C}$ by

$$F_{S_{\mathcal{U}}}((x \otimes f_{\mathcal{U}}) \cdot T) = f_{\mathcal{U}}(T(S(x))), \quad (T \in B(X)).$$

Then F_{S_U} is linear and bounded. Indeed

$$\begin{aligned} \|F_{S_{\mathcal{U}}}\| &= \sup_{\|(x\otimes f_{\mathcal{U}})\cdot T\|\le 1} |F_{S_{\mathcal{U}}}((x \otimes f_{\mathcal{U}}) \cdot T)| = \sup_{\|f_{\mathcal{U}}\circ T\|\le 1} |f_{\mathcal{U}}(T \circ S_{\mathcal{U}}(x))| \\ &\le \sup_{\|f_{\mathcal{U}}\circ T\|\le 1} \|f_{\mathcal{U}} \circ T\| \, \|x\| \, \|S_{\mathcal{U}}\| \le \|S_{\mathcal{U}}\|. \end{aligned}$$

So we can extend $F_{S_{\mathcal{U}}}$ to an element $F_{S_U} \in B(X)^{**}$ with the same norm. This gives

$$\begin{aligned} (f_{\mathcal{U}} \circ T_{\alpha_\beta})(x_{\mathcal{U}}) &= f_{\mathcal{U}}(T_{\alpha_\beta}(S_{\mathcal{U}}(x))) = F_{S_{\mathcal{U}}}((x \otimes f_{\mathcal{U}}) \cdot T_{\alpha_\beta}) \\ &\to F_{S_{\mathcal{U}}}((x \otimes f_{\mathcal{U}}) \cdot T_0) = (f_{\mathcal{U}} \circ T_0)(x_{\mathcal{U}}), \end{aligned}$$

which implies that $(f_{\mathcal{U}} \circ T_{\alpha_\beta}) \xrightarrow{w^*} (f_{\mathcal{U}} \circ T_0)$, so $f_{\mathcal{U}} \circ \mathrm{Ball}(B(X))$ is w^*-compact in ${\ell^2(X)_{\mathcal{U}}}^*$.

(b) $\Rightarrow$ (c). Fix an ultrafilter $\mathcal{U}$ and $f_{\mathcal{U}} \in {\ell^2(X)_{\mathcal{U}}}^*$. By Proposition 2.1, there exists an embedding ${\ell^2(X)_{\mathcal{U}}}^{**} \hookrightarrow \ell^2(X)_{\mathcal{U}\times\mathcal{V}}$, for some ultrafilter $\mathcal{V}$. We also consider the identification $f_{\mathcal{U}} \mapsto (f_{\mathcal{U}})_{\mathcal{V}} : {\ell^2(X)_{\mathcal{U}}}^* \hookrightarrow {\ell^2(X)_{\mathcal{U}\times\mathcal{V}}}^*$. By (b), $(f_{\mathcal{U}})_{\mathcal{V}} \circ \mathrm{Ball}(B(X))$ is w^*-compact in ${\ell^2(X)_{\mathcal{U}\times\mathcal{V}}}^*$. From the above identifications, $f_{\mathcal{U}} \circ \mathrm{Ball}(B(X))$ is w-compact in ${\ell^2(X)_{\mathcal{U}}}^*$.

(c) $\Rightarrow$ (d). First note that, since $B(X)$ is isometrically $*$-antiisomorphic to $B(X^*)$, the Arens regularity of $B(X)$ implies that of $B(X^*)$. Now using (c) for $x_{\mathcal{U}} \circ \mathrm{Ball}(B(X^*))$, the identification $\mathrm{Ball}(B(X))(x_{\mathcal{U}}) \cong x_{\mathcal{U}} \circ \mathrm{Ball}(B(X^*))$ implies that $\mathrm{Ball}(B(X))(x_{\mathcal{U}})$ is weakly compact in $\ell^2(X)_{\mathcal{U}}$ for each $x_{\mathcal{U}} \in \ell^2(X)_{\mathcal{U}}$.

(d) $\Rightarrow$ (a). Suppose that X is ultrareflexive. Then X must be reflexive, so, by Lemma 4.1, each element $\lambda \in B(X)^*$ has the tensorial form $\lambda = x_{\mathcal{U}} \otimes f_{\mathcal{U}}$ for some $x_{\mathcal{U}}$ in $\ell^2(X)_{\mathcal{U}}, f_{\mathcal{U}} \in {\ell^2(X)_{\mathcal{U}}}^*$. To prove that $B(X)$ is Arens regular, we use Proposition 2.2. For this we first apply Lemma 4.3 to the subset $W = \mathrm{Ball}(B(X))(x_{\mathcal{U}})$ of $\ell^2(X)_{\mathcal{U}}$. It induces a reflexive subspace $Y_{x_{\mathcal{U}}}$ of $\ell^2(X)_{\mathcal{U}}$ such that $W \subseteq \mathrm{Ball}(Y_{x_{\mathcal{U}}})$ and the identity embedding $j : Y_{x_{\mathcal{U}}} \longrightarrow \ell^2(X)_{\mathcal{U}}$ is bounded. Now we define $\phi : B(X) \to Y_{x_{\mathcal{U}}}$ and $\psi : B(X) \to {Y_{x_{\mathcal{U}}}}^*$ by $\phi(T) = T(x_{\mathcal{U}})$ and $\psi(T) = f_{\mathcal{U}} \circ T \circ j$, respectively. A direct verification reveals that ϕ, ψ are bounded linear mappings satisfying

$$\lambda(ST) = (x_{\mathcal{U}} \otimes f_{\mathcal{U}})(ST) = \langle \psi(S), \phi(T) \rangle, \quad (S, T \in B(X)).$$

So $B(X)$ is Arens regular, as required. □

For every superreflexive space X, the algebra $B(X)$ is Arens regular (see [5, Theorem 1]). This remark gives the next corollary.

Corollary 4.5. *Every superreflexive space is ultrareflexive.*

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We do not know of an ultrareflexive space which is not superreflexive and it would be highly desirable to find such an example. An example of a reflexive space which is not ultrareflexive can be found in [5, Corollary 2].

5. On Daws' conjecture

Daws [5, Theorem 1] showed that, if X is superreflexive, then $B(X)$ is Arens regular and he conjectured the validity of the converse statement. The following incomplete idea may suggest a way of resolving Daws' conjecture.

Let $B(X)$ be Arens regular and let $\mathcal{U}$ be an arbitrary ultrafilter. For $x_{\mathcal{U}} \in X_{\mathcal{U}}$ choose $f_{x_{\mathcal{U}}} \in X_{\mathcal{U}}{}^*$ so that $\|f_{x_{\mathcal{U}}}\| = 1$ and $f_{x_{\mathcal{U}}}(x_{\mathcal{U}}) = \|x_{\mathcal{U}}\|$. This induces a functional $\lambda_{x_{\mathcal{U}}} : B(X) \to \mathbb{C}$ defined by $\langle \lambda_{x_{\mathcal{U}}}, T\rangle = \langle f_{\mathcal{U}}, T(x_{\mathcal{U}})\rangle$. By Proposition 2.2, there exist a reflexive space $Z_{x_{\mathcal{U}}}$ and operators $\phi_{x_{\mathcal{U}}} : B(X) \to Z_{x_{\mathcal{U}}}$ and $\psi_{x_{\mathcal{U}}} : B(X) \to Z_{x_{\mathcal{U}}}{}^*$ such that $\|\phi_{x_{\mathcal{U}}}\| \le \|\lambda_{x_{\mathcal{U}}}\|$ and $\langle \lambda_{x_{\mathcal{U}}}, ST\rangle = \langle \psi_{x_{\mathcal{U}}}(S), \phi_{x_{\mathcal{U}}}(T)\rangle$, for all $S, T \in B(X)$. Define $Z = \bigoplus^{\ell^2}_{x_{\mathcal{U}} \in X_{\mathcal{U}}} Z_{x_{\mathcal{U}}}$. Trivially, Z is reflexive. If one could establish an (isometric) embedding from $X_{\mathcal{U}}$ into Z, then the reflexivity of Z implies that X is superreflexive. However, we do not know how to find such an embedding.

For such an embedding, one may consider the map $\theta : X_{\mathcal{U}} \to Z$, which is defined by $\theta(x_{\mathcal{U}}) = \phi_{x_{\mathcal{U}}}(I)$. Then θ preserves the norm. Indeed,

$$\|\theta(x_{\mathcal{U}})\| = \|\phi_{x_{\mathcal{U}}}(I)\| \le \|\phi_{x_{\mathcal{U}}}\| \le \|\lambda_{x_{\mathcal{U}}}\| \le \|x_{\mathcal{U}}\|$$

and

$$\|\theta(x_{\mathcal{U}})\| = \|\phi_{x_{\mathcal{U}}}(I)\| = \sup_{\|z^*\|\le 1, z^* \in Z^*} |\langle z^*, \phi_{x_{\mathcal{U}}}(I)\rangle| \ge |\langle \psi_{x_{\mathcal{U}}}(I), \phi_{x_{\mathcal{U}}}(I)\rangle| = |\langle \lambda_{x_{\mathcal{U}}}, I\rangle| = \|x_{\mathcal{U}}\|.$$

However, we know nothing about the linearity of θ.

Acknowledgement

We would like to thank Professor Matthew Daws for his very valuable comments which improved an earlier version of this paper.

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