Nagoya Math. J., **237** (2020), 98–126
DOI 10.1017/nmj.2018.12

MULTIPLICATION FORMULAS AND SEMISIMPLICITY FOR q-SCHUR SUPERALGEBRAS

JIE DU, HAIXIA GU* AND ZHONGGUO ZHOU

Abstract. We investigate products of certain double cosets for the symmetric group and use the findings to derive some multiplication formulas for the q-Schur superalgebras. This gives a combinatorialization of the relative norm approach developed in Du and Gu (*A realization of the quantum supergroup* $\mathbf{U}(\mathfrak{gl}_{m|n})$, J. Algebra **404** (2014), 60–99). We then give several applications of the multiplication formulas, including the matrix representation of the regular representation and a semisimplicity criterion for q-Schur superalgebras. We also construct infinitesimal and little q-Schur superalgebras directly from the multiplication formulas and develop their semisimplicity criteria.

§1. Introduction

The beautiful Beilinson–Lusztig–MacPherson construction [1] of quantum $\mathfrak{gl}_n$ has been generalized to the quantum affine $\mathfrak{gl}_n$ [4, 9], to the quantum super $\mathfrak{gl}_{m|n}$ [12], and partially to the other classical types [2, 20] and affine type C [19], in which certain coideal subalgebras of quantum $\mathfrak{gl}_n$ (or affine $\mathfrak{gl}_n$) are used to form various quantum symmetric pairs associated with Hecke algebras of type $B/C/D$ or affine type C. A key step of these works is the establishment of certain multiplication formulas in the relevant q-Schur algebras or Hecke endomorphism algebras. These formulas were originally derived by geometric methods. When the geometric approach is not available in the super case, a super version of the Curtis–Scott relative norm basis [8, 24], including a detailed analysis of the explicit action on the tensor space, is used in deriving such formulas; see [12, 14, 15]. However, it is natural to expect the existence of a direct Hecke algebra method involving only the combinatorics of symmetric groups.

Received July 19, 2017. Revised March 16, 2018. Accepted March 19, 2018.
2010 Mathematics subject classification. 20B30, 20G43, 17A70, 20C08.
The work was supported by a 2017 UNSW Science Goldstar Grant and the Natural Science Foundation of China (#11501197, #11671234). The third author would like to thank UNSW for its hospitality during his a year visit and thank the Jiangsu Provincial Department of Education for financial support.
*Corresponding author.

In this paper, we will develop such a method. The multiplication formulas require to compute certain structure constants associated with the double coset basis, a basis defined by the double cosets of a symmetric group. Since a double coset can be described by a certain matrix with nonnegative integer entries, our first step is to find formulas, in terms of the matrix entries, of decomposing products of certain double cosets into disjoint unions of double cosets. We then use the findings to derive the multiplication formulas in q-Schur superalgebras; see Theorem 4.1 and Corollary 4.2. This method simplifies the calculation in [12, Sections 2–3] using relative norms.

The multiplication formulas result in several applications. The first one is the matrix representation of the regular representations over any commutative ring R; see Theorem 4.5. When the ground ring R is a field, we establish a criterion for the semisimplicity of q-Schur superalgebras (see Theorem 5.4), generalizing a quantum result of Erdmann and Nakano to the super case and a classical super result of Marko and Zubkov [26] (cf. [6, 18]) to the quantum case. Finally, we introduce the infinitesimal and little q-Schur superalgebras directly from the multiplication formulas (Theorem 6.1, Corollary 6.3). We also determine semisimple infinitesimal q-Schur superalgebras and semisimple little q-Schur superalgebras (Theorem 6.4).

It should be interesting to point out that, unlike the traditional methods used in [7, 10], our definitions do not involve quantum enveloping algebras or quantum coordinate algebras and the semisimplicity proof is also independent of the representation theory of these ambient quantum groups or algebras. We expect that this combinatorial approach will give further applications to various q-Schur superalgebras of other types in the near future.

§2. q-Schur superalgebras

Let $W = \mathfrak{S}_{\{1,2,\dots,r\}}$ be the symmetric group on r letters and let $S = \{s_k \mid 1 \leqslant k < r\}$ be the set of basic transpositions $s_k = (k, k+1)$. Denote the length function with respect to S by $\ell : W \to \mathbb{N}$.

Let R be a commutative ring with 1 and let $q \in R^\times$. The Hecke algebra $\mathcal{H}_R = \mathcal{H}_R(W)$ is a free R-module with basis $\{T_w \mid w \in W\}$ and the multiplication defined by the rules: for $s \in S$,

$$T_w T_s = \begin{cases} T_{ws}, & \text{if } \ell(ws) > \ell(w); \\ (q-1)T_w + qT_{ws}, & \text{otherwise.} \end{cases} \tag{2.0.1}$$

The Hecke algebra over $R = \mathcal{Z} := \mathbb{Z}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$ and $q = \boldsymbol{v}^2$ is simply denoted by $\mathcal{H}$.

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

Let W_λ denote the parabolic subgroup of W associated with $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N) \in \Lambda(N, r)$ where

$$\Lambda(N, r) = \left\{ \lambda \in \mathbb{N}^N \middle| |\lambda| := \textstyle\sum_i \lambda_i = r \right\}.$$

Then W_λ consists of permutations that leave invariant the following sets of integers

$$\mathbb{N}_1^\lambda = \{1, 2, \dots, \lambda_1\}, \mathbb{N}_2^\lambda = \{\lambda_1 + 1, \lambda_1 + 2, \dots, \lambda_1 + \lambda_2\}, \dots.$$

Let $\mathcal{D}_\lambda := \mathcal{D}_{W_\lambda}$ be the set of all shortest coset representatives of the right cosets of W_λ in W. Let $\mathcal{D}_{\lambda\mu} = \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$ be the set of the shortest W_λ–W_μ double coset representatives.

For $\lambda, \mu \in \Lambda(N, r)$ and $d \in \mathcal{D}_{\lambda\mu}$, the subgroup $W_\lambda^d \cap W_\mu = d^{-1} W_\lambda d \cap W_\mu$ is a parabolic subgroup associated with a composition, which is denoted by $\lambda d \cap \mu$. In other words, we define

$$W_{\lambda d \cap \mu} = W_\lambda^d \cap W_\mu. \tag{2.0.2}$$

The composition $\lambda d \cap \mu$ can be easily described in terms of the following $N \times N$-matrix $A = (a_{i,j})$ with $a_{i,j} = |\mathbb{N}_i^\lambda \cap d(\mathbb{N}_j^\mu)|$: if $\nu^{(j)} = (a_{1,j}, a_{2,j}, \dots, a_{N,j})$ denotes the jth column of A, then

$$\lambda d \cap \mu = (\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(N)}). \tag{2.0.3}$$

Putting $\jmath(\lambda, d, \mu) = (|\mathbb{N}_i^\lambda \cap d(\mathbb{N}_j^\mu)|)_{i,j}$, we obtain a bijection

$$\jmath : \{(\lambda, d, \mu) \mid \lambda, \mu \in \Lambda(N, r), d \in \mathcal{D}_{\lambda\mu}\} \longrightarrow M(N, r), \tag{2.0.4}$$

where $M(N, r)$ is the set of all $N \times N$ matrices $A = (a_{i,j})$ over $\mathbb{N}$ whose entries sum to r, i.e., $|A| := \sum_{i,j} a_{i,j} = r$.

For $A \in M(N, r)$, if $\jmath^{-1}(A) = (\lambda, d, \mu)$, then $\lambda, \mu \in \Lambda(N, r)$ and

$$\begin{aligned} \lambda = \mathrm{ro}(A) &:= \left(\sum_{j=1}^N a_{1,j}, \dots, \sum_{j=1}^N a_{N,j} \right) \quad \text{and} \\ \mu = \mathrm{co}(A) &:= \left(\sum_{i=1}^N a_{i,1}, \dots, \sum_{i=1}^N a_{i,N} \right). \end{aligned} \tag{2.0.5}$$

For the definition of q-Schur superalgebra, we fix two nonnegative integers m, n and assume R has characteristic $\neq 2$. We also need the *parity function*

$$\widehat{h} = \begin{cases} 0 & \text{if } 1 \leqslant h \leqslant m; \\ 1 & \text{if } m+1 \leqslant h \leqslant m+n. \end{cases} \tag{2.0.6}$$

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

A composition λ of $m+n$ parts will be written

$$\lambda = (\lambda^{(0)}|\lambda^{(1)}) = (\lambda_1^{(0)}, \lambda_2^{(0)}, \dots, \lambda_m^{(0)}|\lambda_1^{(1)}, \lambda_2^{(1)}, \dots, \lambda_n^{(1)})$$

to indicate the "even" and "odd" parts of λ. Let

$$\Lambda(m|n,r) := \Lambda(m+n,r) = \bigcup_{r_1+r_2=r} (\Lambda(m,r_1) \times \Lambda(n,r_2)).$$

For $\lambda = (\lambda^{(0)} \mid \lambda^{(1)}) \in \Lambda(m|n,r)$, we also write

$$W_\lambda = W_{\lambda^{(0)}} W_{\lambda^{(1)}} \cong W_{\lambda^{(0)}} \times W_{\lambda^{(1)}}, \tag{2.0.7}$$

where $W_{\lambda^{(0)}} \leqslant \mathfrak{S}_{\{1,2,\dots,|\lambda^{(0)}|\}}$ and $W_{\lambda^{(1)}} \leqslant \mathfrak{S}_{\{|\lambda^{(0)}|+1,\dots,r\}}$ are the even and odd parts of W_λ, respectively.

Denote the Hecke algebra associated with the parabolic subgroup W_λ by $\mathcal{H}_\lambda$, which is spanned by T_w, $w \in W_\lambda$. The elements in $\mathcal{H}_\lambda$

$$[xy]_\lambda := x_{\lambda^{(0)}} y_{\lambda^{(1)}}, \quad [yx]_\lambda := y_{\lambda^{(0)}} x_{\lambda^{(1)}}, \tag{2.0.8}$$

where, for $i = 0, 1$,

$$x_{\lambda^{(i)}} = \sum_{w \in W_{\lambda^{(i)}}} T_w, \quad y_{\lambda^{(i)}} = \sum_{w \in W_{\lambda^{(i)}}} (-q)^{-\ell(w)} T_w$$

generate $\mathcal{H}_\lambda$-modules $R[xy]_\lambda$, $R[yx]_\lambda$. Define the "tensor space" (cf. [16, (8.3.4)])

$$\mathfrak{T}_R(m|n,r) = \bigoplus_{\lambda \in \Lambda(m|n,r)} [xy]_\lambda \mathcal{H}_R. \tag{2.0.9}$$

By the definition in [16], the endomorphism algebra

$$\mathcal{S}_R(m|n,r) = \mathrm{End}_{\mathcal{H}_R}(\mathfrak{T}_R(m|n,r))$$

is called a *q-Schur superalgebra* whose $\mathbb{Z}_2$-graded structure is given by

$$\mathcal{S}_R(m|n,r)_i = \bigoplus_{\substack{\lambda,\mu \in \Lambda(m|n,r) \\ |\lambda^{(1)}|+|\mu^{(1)}| \equiv i \pmod 2}} \mathrm{Hom}_{\mathcal{H}_R}([xy]_\lambda \mathcal{H}_R, [xy]_\mu \mathcal{H}_R) \quad (i = 0, 1).$$

We will use the notation $\mathcal{S}(m|n,r)$ to denote the υ^2-Schur algebra over $\mathcal{Z}$.

We now describe a characteristic-free basis for $\mathcal{S}_R(m|n,r)$.

For $\lambda, \mu \in \Lambda(m|n,r)$, let

$$\mathcal{D}^\circ_{\lambda\mu} = \{d \in \mathcal{D}_{\lambda\mu} \mid W^d_{\lambda^{(0)}} \cap W_{\mu^{(1)}} = 1, W^d_{\lambda^{(1)}} \cap W_{\mu^{(0)}} = 1\}. \tag{2.0.10}$$

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

This set is the super version of the usual $\mathcal{D}_{\lambda\mu}$. We need the following subsets of the $(m+n)\times(m+n)$ matrix ring $M_{m+n}(\mathbb{N})$ over $\mathbb{N}$:

$$M(m|n,r)=\{\jmath(\lambda,d,\mu)\mid \lambda,\mu\in\Lambda(m|n,r),d\in\mathcal{D}^{\circ}_{\lambda\mu}\},$$

$$M(m|n)=\bigcup_{r\geqslant 0}M(m|n,r)\subseteq M_{m+n}(\mathbb{N}). \tag{2.0.11}$$

Following [16, (5.3.2)], define, for $\lambda,\mu\in\Lambda(m|n,r)$ and $d\in\mathcal{D}^{\circ}_{\lambda\mu}$,

$$T_{W_\lambda dW_\mu}:=[xy]_\lambda T_d T_{\mathcal{D}_\nu\cap W_\mu}=T_{\mathcal{D}_{\nu'}\cap W_\lambda}T_d[xy]_\mu, \tag{2.0.12}$$

where $\nu=\lambda d\cap\mu$, $\nu'=\mu d^{-1}\cap\lambda$, and

$$T_{D\cap W_\eta}=\sum_{w_0\in D_0,w_1\in D_1}T_{w_0}(-q)^{-\ell(w_1)}T_{w_1}$$

for any $D\subseteq W$ ($\eta=\lambda$ or μ) with $D_i=D\cap W_{\eta^{(i)}}$ (cf. [16, (5.3.2)]).

The element $T_{W_\lambda dW_\mu}$ is used to define an $\mathcal{H}_R$-module homomorphism $\phi^d_{\lambda\mu}$ on $\mathfrak{T}_R(m|n,r)$:

$$\phi^d_{\lambda\mu}([xy]_\alpha h)=\delta_{\mu,\alpha}T_{W_\lambda dW_\mu}h,\ \forall\alpha\in\Lambda(m|n,r),h\in\mathcal{H}.$$

The first assertion of the following result is given in [16, 5.8], while the last assertion for the nonquantum case was observed in [23, Section 3.1]. Write $\phi_A:=\phi^d_{\lambda\mu}$ if $A=\jmath(\lambda,d,\mu)$.

LEMMA 2.1. *The set $\{\phi_A\mid A\in M(m|n,r)\}$ forms an R-basis for $\mathcal{S}_R(m|n,r)$. Hence, $\mathcal{S}_R(m|n,r)\cong\mathcal{S}(m|n,r)\otimes_{\mathcal{Z}}R$. Moreover, there is an R-algebra isomorphism*

$$\mathcal{S}_R(m|n,r)\cong\mathcal{S}_R(n|m,r).$$

Proof. We only need to prove the last assertion. The Hecke algebra $\mathcal{H}_R$ admits an R-algebra involutory automorphism φ sending T_s to $-qT_s^{-1}=(q-1)-T_s$ for all $s\in S$. Since $\varphi(x_\lambda)=q^{\ell(w_{0,\lambda})}y_\lambda$, where $w_{0,\lambda}$ is the longest element in W_λ (see, e.g., [5, (7.6.2)]), we have $\varphi([xy]_\lambda)=\varphi(x_{\lambda^{(0)}}y_{\lambda^{(1)}})=q^{\ell(w_{0,\lambda^{(0)}})-\ell(w_{0,\lambda^{(1)}})}y_{\lambda^{(0)}}x_{\lambda^{(1)}}$. If we denote by $([xy]_\lambda\mathcal{H}_R)^\varphi$ the module obtained by twisting the action on $[xy]_\lambda\mathcal{H}_R$ by φ, i.e., $([xy]_\lambda h)*h'=([xy]_\lambda h)\varphi(h')$ for all $h,h'\in\mathcal{H}_R$, then the map

$$\Phi_\lambda:([xy]_\lambda\mathcal{H}_R)^\varphi\to[yx]_\lambda\mathcal{H}_R,[xy]_\lambda h\mapsto\varphi([xy]_\lambda h)$$

is an $\mathcal{H}_R$ module isomorphism. These Φ_λ induce an $\mathcal{H}_R$ module isomorphism $\Phi:\mathfrak{T}_R(m|n,r)^\varphi\longrightarrow\mathfrak{T}_R(n|m,r)$. Now the required isomorphism follows.

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

§3. Decomposing products of double cosets

Throughout the section, let W be the symmetric group and let n, r be positive integers. We also fix the following notation in this section:

$$\text{(3.0.1)}\quad \begin{cases} M=(m_{ij})\in M(n,r) \quad \text{with } \jmath^{-1}(M)=(\lambda,d,\mu), \quad d_M:=d,\\ \nu_M:=\lambda d\cap\mu=(m_{1,1},m_{2,1},\dots,m_{n,1},\dots,m_{1,n},m_{2,n},\dots,m_{n,n}),\\ \sigma_{i,j}=\displaystyle\sum_{k=1}^{j-1}\sum_{h=1}^{n}m_{h,k}+\sum_{k\leqslant i,l\geqslant j}m_{k,l},\\ M^+_{h,k}=M+E_{h,k}-E_{h+1,k} \quad \text{if } m_{h+1,k}\geqslant 1,\\ M^-_{h,k}=M-E_{h,k}+E_{h+1,k} \quad \text{if } m_{h,k}\geqslant 1. \end{cases}$$

Moreover, to any sequence $(a_1,a_2,\dots,a_n)$, we associate its partial sum sequence $(\widetilde{a}_1,\widetilde{a}_2,\dots,\widetilde{a}_n)$ with $\widetilde{a}_i=a_1+\cdots+a_i$. Thus, $\widetilde{\lambda}_i=\lambda_1+\cdots+\lambda_i$ and $\widetilde{m}_{i,j}$ is the partial sum at the (i,j)-position of ν_M. We also note that $\sigma_{i,j}=\widetilde{\mu}_{j-1}+m^{\llcorner}_{i,j}$, where $m^{\llcorner}_{i,j}=\sum_{k\leqslant i,l\geqslant j}a_{k,l}$. In particular, $\sigma_{i,1}=m^{\llcorner}_{i,1}=\widetilde{\lambda}_i$.

The following result will be proved at the end of the section.

THEOREM 3.1. *Maintain the notation in (3.0.1) with $\lambda=(\lambda_1,\dots,\lambda_n)$ and, for $1\leqslant h\leqslant n$, let $\lambda^{[h^\pm]}:=\lambda\pm\mathbf{e}_h\mp\mathbf{e}_{h+1}=\mathrm{ro}(M^\pm_{h,k})$, where $\mathbf{e}_i=(\delta_{1,i},\dots,\delta_{n,i})$. Then*

$$(W_{\lambda^{[h^+]}}1W_\lambda)(W_\lambda d_M W_\mu)=\bigcup_{\substack{k\\ m_{h+1,k}\geqslant 1}}W_{\lambda^{[h^+]}}d_{M^+_{h,k}}W_\mu,$$

$$(W_{\lambda^{[h^-]}}1W_\lambda)(W_\lambda d_M W_\mu)=\bigcup_{\substack{k\\ m_{h,k}\geqslant 1}}W_{\lambda^{[h^-]}}d_{M^-_{h,k}}W_\mu.$$

We first describe some standard reduced expression for d_M.

If $m_{i,j}=0$, or $m_{i,j}>0$ but $\sigma_{i-1,j}=\widetilde{m}_{i-1,j}$ (i.e., $m^{\llcorner}_{i-1,j+1}=0$), set $w_{i,j}=1$; if $m_{i,j}>0$ and $\sigma_{i-1,j}>\widetilde{m}_{i-1,j}$, let

$$\begin{aligned} w_{i,j}=\ &(s_{\sigma_{i-1,j}}s_{\sigma_{i-1,j}-1}\cdots s_{\widetilde{m}_{i-1,j}+1})\\ &(s_{\sigma_{i-1,j}+1}s_{\sigma_{i-1,j}}\cdots s_{\widetilde{m}_{i-1,j}+2})\cdots\\ &(s_{\sigma_{i-1,j}+m_{i,j}-1}s_{\sigma_{i-1,j}+m_{i,j}-2}\cdots s_{\widetilde{m}_{i,j}}) \end{aligned}\quad\text{(3.1.1)}$$

and $w^+_{i,j}=s_{\sigma_{i-1,j}+1}s_{\sigma_{i-1,j}+2}\cdots s_{\sigma_{i-1,j}+m_{i,j}}w_{i,j}$ (and $w^+_{i,j}=1$ if $m_{i,j}=0$). Note that we may rewrite $w^+_{i,j}$ as

$$w^+_{i,j}=s_{\sigma_{i-1,j}+1}(s_{\sigma_{i-1,j}}s_{\sigma_{i-1,j}-1}\cdots s_{\widetilde{m}_{i-1,j}+1})$$

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

$$s_{\sigma_{i-1,j}+2}\big(s_{\sigma_{i-1,j}+1}s_{\sigma_{i-1,j}}\cdots s_{\widetilde{m}_{i-1,j}+2}\big)\cdots$$
$$s_{\sigma_{i-1,j}+m_{i,j}}\big(s_{\sigma_{i-1,j}+m_{i,j}-1}s_{\sigma_{i-1,j}+m_{i,j}-2}\cdots s_{\widetilde{m}_{i,j}}\big). \tag{3.1.2}$$

For example, if $M=\left(\begin{smallmatrix}1&3&2\\2&1&1\\1&0&2\end{smallmatrix}\right)$ then

$$(\sigma_{ij})=\begin{pmatrix}6&9&10\\10&11&11\\13&13&13\end{pmatrix},\qquad (\widetilde{m}_{ij})=\begin{pmatrix}1&7&10\\3&8&11\\4&8&13\end{pmatrix},$$

and

$$w_{2,1}=(s_6s_5\cdots s_2)(s_7s_6\cdots s_3)=\begin{pmatrix}2&3&4&5&6&7&8\\7&8&2&3&4&5&6\end{pmatrix},$$

$w_{3,1}=s_{10}s_9\cdots s_4=\left(\begin{smallmatrix}4&5&6&7&8&9&10&11\\11&4&5&6&7&8&9&10\end{smallmatrix}\right)$, and $w_{2,2}=s_9s_8=\left(\begin{smallmatrix}8&9&10\\10&8&9\end{smallmatrix}\right)$, $w_{3,2}=1$, then $w_{2,1}w_{3,1}w_{2,2}w_{3,2}=\left(\begin{smallmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13\\1&7&8&11&2&3&4&9&5&6&10&12&13\end{smallmatrix}\right)$, which is d_M.

LEMMA 3.2. [12, Algorithm 2.1] *Let M, d_M and $M^+_{h,k}$ be given as in (3.0.1). Then a reduced expression of d_M is of the form*

$$d_M=(w_{2,1}w_{3,1}\cdots w_{n,1})(w_{2,2}w_{3,2}\cdots w_{n,2})\cdots(w_{2,n-1}w_{3,n-1}\cdots w_{n,n-1}).$$

If $m_{h+1,k}\geqslant 1$, then

$$d_{M^+_{h,k}}=(w'_{2,1}w'_{3,1}\cdots w'_{n,1})(w'_{2,2}w'_{3,2}\cdots w'_{n,2})\cdots(w'_{2,n-1}w'_{3,n-1}\cdots w'_{n,n-1}),$$

where, for almost all i,j, $w'_{ij}=w_{ij}$, except $w'_{h+1,j}=w^+_{h+1,j}$ for $j<k$ and

$$w'_{h,k}=w^{\bullet}_{h,k}:=w_{h,k}(s_{\sigma_{h-1,k}+m_{h,k}}s_{\sigma_{h-1,k}+m_{h,k}-1}\cdots s_{\widetilde{m}_{h,k}+1}), \tag{3.2.1}$$
$$w'_{h+1,k}=w^{\circ}_{h+1,k}:=(s_{\sigma_{h,k}+1}s_{\sigma_{h,k}}\cdots s_{\widetilde{m}_{h,k}+2})(s_{\sigma_{h,k}+2}s_{\sigma_{h,k}+1}\cdots s_{\widetilde{m}_{h,k}+3})\cdots$$
$$(s_{\sigma_{h,k}+m_{h+1,k}-1}s_{\sigma_{h,k}+m_{h+1,k}-2}\cdots s_{\widetilde{m}_{h+1,k}}).$$

In particular, $\ell(d_{M^+_{h,k}})=\ell(d_M)+\sum_{j<k}m_{h+1,j}-\sum_{j>k}m_{h,j}$.

REMARK 3.3. (1) We display the factors $w_{i,j}$ of d_M through a matrix notation:

$$d_M=\begin{pmatrix}w_{2,1}&w_{2,2}&\cdots&w_{2,n-1}\\w_{3,1}&w_{3,2}&\cdots&w_{3,n-1}\\\vdots&\vdots&\cdots&\vdots\\w_{n,1}&w_{n,2}&\cdots&w_{n,n-1}\end{pmatrix}, \tag{3.3.1}$$

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

where d_M is simply a product of the entries down column 1, then down column 2, and so on. Note that $w_{i,j} = 1$ whenever $m_{i,j} = 0$ or $m^{\llcorner}_{i-1,j+1} = 0$.

(2) Note that a product of the form $s_{h-1}s_{h-2}\cdots s_k$ for $h > k$ is in fact the cycle permutation $h \to h-1 \to \cdots \to k+1 \to k \to h$. Thus, each $w_{i,j}$ is a product of cycle permutations. Note also that the largest number permuted (or moved) by the partial column product $w_{2,j}w_{3,j}\cdots w_{h,j}$ is $\sigma_{h-1,j} + m_{h,j}$.

LEMMA 3.4.

(1) *For any nonnegative integers k, i, h with $0 < k \leqslant i < h < r$,*

$$s_i(s_h s_{h-1}\cdots s_k) = (s_h s_{h-1}\cdots s_k)s_{i+1}.$$

Hence, for $0 < k \leqslant i < h_1 < h_2 < \cdots < h_l < r$,

$$\begin{aligned} &s_i(s_{h_1}s_{h_1-1}\cdots s_k)(s_{h_2}s_{h_2-1}\cdots s_{k+1})\cdots(s_{h_l}s_{h_l-1}\cdots s_{k+l-1}) \\ &\quad = (s_{h_1}s_{h_1-1}\cdots s_k)(s_{h_2}s_{h_2-1}\cdots s_{k+1})\cdots(s_{h_l}s_{h_l-1}\cdots s_{k+l-1})s_{i+l}. \end{aligned}$$

(2) *With the notation given in (3.0.1) and (3.1.1), if $\sigma_{h-1,j} + m_{h,j} < l < \sigma_{h,j}$ and $l \geqslant \widetilde{m}_{h,j} + 1$, then*

$$s_l(w_{2,j}w_{3,j}\cdots w_{n,j}) = (w_{2,j}w_{3,j}\cdots w_{n,j})s_{l+\sum_{i=h+1}^{n} m_{i,j}}.$$

(3) *For any $1 < k \leqslant n$, if $0 < x \leqslant m_{h,k}$ and assume $\sum_{j=1}^{k-1} m_{h,j} + x < \lambda_h$, then*

$$\begin{aligned} &s_{\sigma_{h-1,1}+\sum_{j=1}^{k-1} m_{h,j}+x}(w_{2,1}\cdots w_{n,1})\cdots(w_{2,k-1}\cdots w_{n,k-1}) \\ &\quad = (w_{2,1}\cdots w_{n,1})\cdots(w_{2,k-1}\cdots w_{n,k-1})s_{\sigma_{h-1,k}+x}. \end{aligned}$$

Proof. The proof for the first two assertions is straightforward. We now prove (3).

Consider the product $\prod_t$ of first t columns of d_M:

$$\begin{aligned} \Pi_t = &(w_{2,1}\cdots w_{h-1,1}w_{h,1}w_{h+1,1}\cdots w_{n,1}) \\ &\cdots(w_{2,t}\cdots w_{h-1,t}w_{h,t}w_{h+1,t}\cdots w_{n,t}). \end{aligned}$$

We claim for all $t < k$ that

$$s_{\sigma_{h-1,1}+\sum_{j=1}^{k-1} m_{h,j}+x}\cdot\Pi_t = \Pi_t\cdot s_{\sigma_{h-1,t+1}+\sum_{j=t+1}^{k-1} m_{h,j}+x}. \tag{3.4.1}$$

Thus, taking $t = k-1$ gives the assertion (3).

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

We prove (3.4.1) by induction on t. If $t=1$, then $x>0$ implies

$$l=\sigma_{h-1,1}+\sum_{j=1}^{k-1}m_{h,j}+x>\sigma_{h-1,1}+m_{h,1}.$$

As the largest number permuted by $w_{2,1}\cdots w_{h,1}$ is $\sigma_{h-1,1}+m_{h,1}$, we have

$$s_l(w_{2,1}\cdots w_{h,1})=(w_{2,1}\cdots w_{h,1})s_l. \tag{3.4.2}$$

Now we consider $s_l(w_{h+1,1}\cdots w_{n,1})$. Assume $w_{h+1,1}\neq 1$ (and so $m_{h+1,1}>0$). Since $k>1$ and $\widetilde{m}_{h,1}+1\leqslant l=\sigma_{h-1,1}+\sum_{j=1}^{k-1}m_{h,j}+x<\sigma_{h-1,1}+\lambda_h=\sigma_{h,1}$, by (2), $s_l w_{h+1,1}=w_{h+1,1}s_{l+m_{h+1,1}}$ and, by an inductive argument as above,

$$s_l w_{h+1,1}w_{h+2,1}\cdots w_{n,1}=w_{h+1,1}w_{h+2,1}\cdots w_{n,1}s_{l+\sum_{i=h+1}^{n}m_{i,1}}.$$

But $l+\sum_{i=h+1}^{n}m_{i,1}=\sigma_{h-1,2}+\sum_{j=2}^{k-1}m_{h,j}+x$. This proves (3.4.1) for $t=1$.

Suppose now $t>1$ and (3.4.1) is true for $t-1$. That is, assume

$$\begin{aligned}&s_{\sigma_{h-1,1}+\sum_{j=1}^{k-1}m_{h,j}+x}(w_{2,1}\cdots w_{n,1})\cdots(w_{2,t-1}\cdots w_{n,t-1})\\&\quad=(w_{2,1}\cdots w_{n,1})\cdots(w_{2,t-1}\cdots w_{n,t-1})s_{\sigma_{h-1,t}+\sum_{j=t}^{k-1}m_{h,j}+x}.\end{aligned}$$

Since $\sigma_{h-1,t}+\sum_{j=t}^{k-1}m_{h,j}+x>\sigma_{h-1,t}+m_{h,t}$ and

$$\sigma_{h,t}=\sigma_{h-1,t}+\sum_{j=t}^{n}m_{h,j}>\sigma_{h-1,t}+\sum_{j=t}^{k-1}m_{h,j}+x\geqslant\widetilde{m}_{h,t}+1,$$

applying (2) with $l=\sigma_{h-1,t}+\sum_{j=t}^{k-1}m_{h,j}+x$ gives

$$\begin{aligned}s_l(w_{2,t}\cdots w_{h,t}w_{h+1,t}\cdots w_{n,t})&=(w_{2,t}\cdots w_{h,t})s_l(w_{h+1,t}\cdots w_{n,t})\\&=(w_{2,t}\cdots w_{h,t}w_{h+1,t}\cdots w_{n,t})s_{l+\sum_{i=h+1}^{n}m_{i,t}},\end{aligned}$$

where

$$\begin{aligned}l+\sum_{i=h+1}^{n}m_{i,t}&=\sigma_{h-1,t}+\sum_{j=t}^{k-1}m_{h,j}+x+\sum_{i=h+1}^{n}m_{i,t}\\&=\sigma_{h-1,t+1}+\sum_{j=t+1}^{k-1}m_{h,j}+x.\end{aligned}$$

This proves (3.4.1) for t and, hence, (3).

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

COROLLARY 3.5. *For* $0 < x \leqslant m_{h,k}$, $l = \sigma_{h-1,1} + \sum_{j=1}^{k-1} m_{h,j}$ *with* $l + x < \sigma_{h,1}$, *we have*

$$s_{l+x} d_M = \begin{pmatrix} w_{2,1} & \cdots & w_{2,k-1} & w_{2,k} & w_{2,k+1} & \cdots & w_{2,n-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ w_{h-1,1} & \cdots & w_{h-1,k-1} & w_{h-1,k} & w_{h-1,k+1} & \cdots & w_{h-1,n-1} \\ w_{h,1} & \cdots & w_{h,k-1} & w^*_{h,k} & w_{h,k+1} & \cdots & w_{h,n-1} \\ w_{h+1,1} & \cdots & w_{h+1,k-1} & w_{h+1,k} & w_{h+1,k+1} & \cdots & w_{h+1,n-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ w_{n,1} & \cdots & w_{n,k-1} & w_{n,k} & w_{h,k+1} & \cdots & w_{n,n-1} \end{pmatrix},$$

where $w^*_{h,k} = s_{\sigma_{h-1,k}+x} w_{h,k}$. *In particular,* $s_{l+1} s_{l+2} \cdots s_{l+m_{h,k}} d_M$ *can be expressed by the same matrix with* $w^*_{h,k} = w^+_{h,k}$, *the element defined in (3.1.2).*

The next result is the key to establish the decomposition in Theorem 3.1 and the multiplication formulas in Theorem 4.1.

PROPOSITION 3.6. *Maintain the notation as given in (3.0.1) and Theorem 3.1, and let* $a = \sum_{j=1}^{k-1} m_{h+1,j}$, *and* $b = \sum_{j=k+1}^{n} m_{h,j}$.

(1) *If* $m_{h+1,k} \geqslant 1$ *then, for* $\lambda^+ = \lambda^{[h^+]} = \lambda + \mathbf{e}_h - \mathbf{e}_{h+1}$ *and* $0 \leqslant p < m_{h+1,k}$,

$$\begin{aligned} & s_{\widetilde{\lambda}_h+1} s_{\widetilde{\lambda}_h+2} \cdots s_{\widetilde{\lambda}_h+a+p} d_M \\ &= s_{\widetilde{\lambda}^+_h-1} s_{\widetilde{\lambda}^+_h-2} \cdots s_{\widetilde{\lambda}^+_h-b} d_{M^+_{h,k}} (s_{\widetilde{m}_{h,k}+1} \cdots s_{\widetilde{m}_{h,k}+p}) \\ &= s_{\widetilde{\lambda}_h} s_{\widetilde{\lambda}_h-1} \cdots s_{\widetilde{\lambda}_h-b+1} d_{M^+_{h,k}} (s_{\widetilde{m}_{h,k}+1} \cdots s_{\widetilde{m}_{h,k}+p}). \end{aligned}$$

(2) *If* $m_{h,k} \geqslant 1$ *then, for* $\lambda^- = \lambda^{[h^-]} = \lambda - \mathbf{e}_h + \mathbf{e}_{h+1}$ *and* $q = m_{h,k} - p$ *with* $0 < p \leqslant m_{h,k}$ *(so* $0 \leqslant q < m_{h,k}$*),*

$$\begin{aligned} & s_{\widetilde{\lambda}_h-1} s_{\widetilde{\lambda}_h-2} \cdots s_{\widetilde{\lambda}_h-b-q} d_M \\ &= s_{\widetilde{\lambda}^-_h+1} s_{\widetilde{\lambda}^-_h+2} \cdots s_{\widetilde{\lambda}^-_h+a} d_{M^-_{h,k}} (s_{\widetilde{m}_{h,k}-1} s_{\widetilde{m}_{h,k}-2} \cdots s_{\widetilde{m}_{h,k}-q}) \\ &= s_{\widetilde{\lambda}_h} s_{\widetilde{\lambda}_h+1} \cdots s_{\widetilde{\lambda}_h+a-1} d_{M^-_{h,k}} (s_{\widetilde{m}_{h,k}-1} s_{\widetilde{m}_{h,k}-2} \cdots s_{\widetilde{m}_{h,k}-q}). \end{aligned}$$

Here every product of the s_i*'s is regarded as 1 if its "length" is 0.*

Proof. We only prove (1), (2) follows from (1) with a similar argument. We first assume that $p = 0$. In this case, we want to prove

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

$$s_{\widetilde{\lambda}_h+1}s_{\widetilde{\lambda}_h+2}\cdots s_{\widetilde{\lambda}_h+a}d_M = s_{\widetilde{\lambda}_h^+-1}s_{\widetilde{\lambda}_h^+-2}\cdots s_{\widetilde{\lambda}_h^+-b}d_{M_{h,k}^+}. \tag{3.6.1}$$

Since $a = m_{h+1,1}+\cdots+m_{h+1,k-1}$, repeatedly applying Corollary 3.5 (with h replaced by $h+1$, noting $m_{h+1,k}>0$) yields

$$s_{\widetilde{\lambda}_h+1}s_{\widetilde{\lambda}_h+2}\cdots s_{\widetilde{\lambda}_h+a}d_M = \begin{pmatrix} w_{2,1} & \cdots & w_{2,k-1} & w_{2,k} & \cdots & w_{2,n-1} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ w_{h,1} & \cdots & w_{h,k-1} & w_{h,k} & \cdots & w_{h,n-1} \\ w^+_{h+1,1} & \cdots & w^+_{h+1,k-1} & w_{h+1,k} & \cdots & w_{h+1,n-1} \\ w_{h+2,1} & \cdots & w_{h+2,k-1} & w_{h+2,k} & \cdots & w_{h+2,n-1} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ w_{n,1} & \cdots & w_{n,k-1} & w_{n,k} & \cdots & w_{n,n-1} \end{pmatrix}.$$

(Note that, if $k=1$, then $a=0$ and so left-hand side (LHS) of (3.6.1) $= d_M$. Note also that $w^+_{h+1,j}=1$ if $m_{h+1,j}=0$.) By comparing this with the "matrix" of $d_{M_{h,k}^+}$, we now show that multiplying $d_{M_{h,k}^+}$ by $s_{\widetilde{\lambda}_h^+-1}s_{\widetilde{\lambda}_h^+-2}\cdots s_{\widetilde{\lambda}_h^+-b}$ on the left will turn the product $w^\bullet_{h,k}w^\circ_{h+1,k}$ into $w_{h,k}w_{h+1,k}$.

If $b=0$, then $\sigma_{h,k}=\sigma_{h-1,k}+m_{h,k}$ and so $w^\bullet_{h,k}w^\circ_{h+1,k}=w_{h,k}w_{h+1,k}$ (cf. Lemma 3.2). This proves (3.6.1) in this case. Assume now $b>0$. Observe that, for $\lambda^+=\mathrm{ro}(M_{h,k}^+)$, $\widetilde{\lambda}_h^+-\sum_{j>k}m_{h,j}=\widetilde{\lambda}_{h-1}+\sum_{j=1}^k m_{h,j}+1$. Let $l=\widetilde{\lambda}_{h-1}+\sum_{j=1}^{k-1}m_{h,j}$ and $1\leqslant x\leqslant m_{h,k}$. Then $l+x<l+x+m_{h+1,k}\leqslant\lambda_{h+1}$. By Lemma 3.4(3) (cf. (3.4.1)),

$$s_{l+x}\Pi^+_{k-1} = \Pi^+_{k-1}s_{\sigma_{h-1,k}+x}, \tag{3.6.2}$$

where Π^+_{k-1} is the product of the first $k-1$ columns of $d_{M_{h,k}^+}$. By (3.3.1) for $M_{h,k}^+$ and noting (3.2.1),

$$\begin{aligned} &s_{\widetilde{\lambda}_h^+-1}s_{\widetilde{\lambda}_h^+-2}\cdots s_{\widetilde{\lambda}_h^+-\sum_{j>k}m_{h,j}}d_{M_{h,k}^+} \\ &\quad=\Pi^+_{k-1}\cdot s_{\sigma_{h,k}}s_{\sigma_{h,k}-1}\cdots s_{\sigma_{h-1,k}+m_{h,k}+1} \\ &\qquad(w_{2,k}\cdots w_{h-1,k}w^\bullet_{h,k}w^\circ_{h+1,k}w_{h+2,k}\cdots w_{n,k}) \\ &\qquad\cdots \\ &\qquad(w_{2,n-1}\cdots w_{h,n-1}w_{h+1,n-1}w_{h+2,n-1}\cdots w_{n,n-1}). \end{aligned} \tag{3.6.3}$$

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

Since the smallest number permuted by $s_{\sigma_{h,k}}s_{\sigma_{h,k}-1}\cdots s_{\sigma_{h-1,k}+m_{h,k}+1}$ is $\sigma_{h-1,k}+m_{h,k}+1$, while the largest number permuted by $w_{2,1}\cdots w_{h-1,k}w_{h,k}$ is $\sigma_{h-1,k}+m_{h,k}$, it follows that $s_{\sigma_{h,k}}s_{\sigma_{h,k}-1}\cdots s_{\sigma_{h-1,k}+m_{h,k}+1}$ commutes with $w_{2,k}\cdots w_{h-1,k}$ and $w_{h,k}$. Thus,

$$\begin{aligned}
&s_{\sigma_{h,k}}s_{\sigma_{h,k}-1}\cdots s_{\sigma_{h-1,k}+m_{h,k}+1}w^{\bullet}_{h,k}w^{\circ}_{h+1,k}\\
&\quad= w_{h,k}(s_{\sigma_{h,k}}\cdots s_{\sigma_{h-1,k}+m_{h,k}+1})s_{\sigma_{h-1,k}+m_{h,k}}s_{\sigma_{h-1,k}+m_{h,k}-1}\\
&\qquad\cdots s_{\widetilde{m}_{h,k}+1}w^{\circ}_{h+1,k}\\
&\quad= w_{h,k}(s_{\sigma_{h,k}}s_{\sigma_{h,k}-1}\cdots s_{\widetilde{m}_{h,k}+1})w^{\circ}_{h+1,k}\\
&\quad= w_{h,k}w_{h+1,k}.
\end{aligned}$$

Hence, $s_{\widetilde{\lambda}^+_h-1}s_{\widetilde{\lambda}^+_h-2}\cdots s_{\widetilde{\lambda}^+_h-\sum_{j>k}m_{h,j}}d_{M^+_{h,k}}=\text{LHS}$, proving the $p=0$ case.

Assume now $p>0$. Then one can easily prove by Corollary 3.5 that

$$s_{l+1}\cdots s_{l+p}d_M = d_M s_{\widetilde{m}_{h,k}+1}s_{\widetilde{m}_{h,k}+2}\cdots s_{\widetilde{m}_{h,k}+p}.$$

Now the required formula follows from (3.6.1).

Proof of Theorem 3.1. Set $D^+_h=\operatorname{diag}(\lambda-\mathbf{e}_{h+1})+E_{h,h+1}$. Then $\operatorname{ro}(D^+_h)=\lambda^{[h^+]}$, $\operatorname{co}(D^+_h)=\lambda$, and

$$\nu' := \nu_{D^+_h}=(\lambda_1,\lambda_2,\dots,\lambda_h,1,\lambda_{h+1}-1,\lambda_{h+2},\dots,\lambda_n).$$

Note that in this case $d_{D^+_h}=1$. Observe that

(3.6.4)
$$\mathcal{D}_{\nu'}\cap W_\lambda=\{1,s_{\widetilde{\lambda}_h+1},s_{\widetilde{\lambda}_h+1}s_{\widetilde{\lambda}_h+2},\dots,s_{\widetilde{\lambda}_h+1}s_{\widetilde{\lambda}_h+2}\cdots s_{\widetilde{\lambda}_h+\lambda_{h+1}-1}\}.$$

Putting $d_i=s_{\widetilde{\lambda}_h+1}s_{\widetilde{\lambda}_h+2}\cdots s_{\widetilde{\lambda}_h+i}$ for $0\leqslant i\leqslant\lambda_{h+1}-1$, the LHS becomes $\bigcup_i W_{\lambda^{[h^+]}}d_i d_M W_\mu$. Since $\lambda_{h+1}=\sum_{k;m_{h+1,k}\geqslant 1}m_{h+1,k}$, the first decomposition follows from Proposition 3.6(1). The second decomposition can be proved similarly.

§4. Regular representation of the q-Schur superalgebra

We now use Proposition 3.6 to derive certain multiplication formulas in $\mathcal{S}(m|n,r)$ and the matrix representation of the regular representation. For any integers $0\leqslant t\leqslant s$, define Gaussian polynomials in $\mathcal{Z}=\mathbb{Z}[\upsilon,\upsilon^{-1}]$ by

$$\begin{bmatrix}s\\t\end{bmatrix}=\begin{bmatrix}s\\t\end{bmatrix}_{\mathbf{q}}=\frac{[\![s]\!]^!}{[\![t]\!]^![\![s-t]\!]^!},$$

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

where $[\![r]\!]^! := [\![1]\!][\![2]\!]\cdots[\![r]\!]$ with $[\![i]\!] = 1 + \mathbf{q} + \cdots + \mathbf{q}^{i-1}$ $(\mathbf{q} = \boldsymbol{\upsilon}^2)$. Define $[r]^!$ similarly with $[i] = (\boldsymbol{\upsilon}^i - \boldsymbol{\upsilon}^{-i})/(\boldsymbol{\upsilon} - \boldsymbol{\upsilon}^{-1})$.

For $\lambda \in \Lambda(m|n, r)$, denote $\mathcal{P}_{W_\lambda}$ to be the *super* Poincaré polynomial

$$\mathcal{P}_{W_\lambda} = \sum_{w_0 \in W_{\lambda^{(0)}}, w_1 \in W_{\lambda^{(1)}}} (\mathbf{q})^{\ell(w_0)} (\mathbf{q}^{-1})^{\ell(w_1)}. \tag{4.0.5}$$

For $1 \leqslant h \leqslant m+n$, define $\dot{\mathbf{q}}_h, \ddot{\mathbf{q}}_h, \boldsymbol{\upsilon}_h$ by

$$\begin{cases} \dot{\mathbf{q}}_h = 1, & \ddot{\mathbf{q}}_h = \mathbf{q}, \quad \boldsymbol{\upsilon}_h = \boldsymbol{\upsilon}, \quad \text{if } 1 \leqslant h \leqslant m; \\ \dot{\mathbf{q}}_h = -\mathbf{q}^{-1}, & \ddot{\mathbf{q}}_h = -1, \quad \boldsymbol{\upsilon}_h = \boldsymbol{\upsilon}^{-1}, \text{ if } m < h \leqslant m+n, \end{cases}$$

and let $\mathbf{q}_h = \boldsymbol{\upsilon}_h^2$. Recall the basis $\{\phi_A\}_{A \in M(m|n,r)}$ given in Lemma 2.1.

THEOREM 4.1. *For any $A = (a_{i,j}) \in M(m|n, r)$ and $1 \leqslant h < m+n$, let D_h^+, D_h^- be the matrices defined by the conditions that $D_h^+ - E_{h,h+1}, D_h^- - E_{h+1,h}$ are diagonal and $\mathrm{co}(D_h^+) = \mathrm{co}(D_h^-) = \mathrm{ro}(A)$, and assume $D_h^+, D_h^- \in M(m|n, r)$. Then the following multiplication formulas hold in $\mathcal{S}(m|n, r)$:*

$$(1) \quad \phi_{D_h^+}\phi_A = \sum_{\substack{k \in [1,m+n] \\ a_{h+1,k} \geqslant 1}} \dot{\mathbf{q}}_{h+1}^{\sum_{j<k} a_{h+1,j}} \ddot{\mathbf{q}}_h^{\sum_{j>k} a_{h,j}} [\![a_{h,k}+1]\!]_{\mathbf{q}_h} \phi_{A_{h,k}^+};$$

$$(2) \quad \phi_{D_h^-}\phi_A = \sum_{\substack{k \in [1,m+n] \\ a_{h,k} \geqslant 1}} \dot{\mathbf{q}}_h^{\sum_{j>k} a_{h,j}} \ddot{\mathbf{q}}_{h+1}^{\sum_{j<k} a_{h+1,j}} [\![a_{h+1,k}+1]\!]_{\mathbf{q}_{h+1}} \phi_{A_{h,k}^-}.$$

(Here $[1, m+n] = \{1, 2, \ldots, m+n\}$.)

Proof. We only prove (1). The proof of (2) is symmetric.

Let $\lambda = \mathrm{ro}(A)$, $\mu = \mathrm{co}(A)$, $d = d_A$ and $W_\nu = W_\lambda^d \cap W_\mu = W_{\nu^{(0)}} \times W_{\nu^{(1)}}$ (cf. (3.0.1)), where $W_{\nu^{(i)}} = W_{\lambda^{(i)}}^d \cap W_{\mu^{(i)}}$ for $i = 0, 1$. Then $\lambda = \mathrm{co}(D_h^+)$, $\lambda^{[h^+]} = \mathrm{ro}(D_h^+) = \lambda + \mathbf{e}_h - \mathbf{e}_{h+1}$, and $\jmath(\lambda^{[h^+]}, 1, \lambda) = D_h^+$.

Putting $W_{\nu'(h)} = W_{\lambda^{[h^+]}} \cap W_\lambda$, we see from (3.6.4),

$$\mathcal{D}_{\nu'(h)} \cap W_\lambda = \{1, s_{\widetilde{\lambda}_h+1}, s_{\widetilde{\lambda}_h+1} s_{\widetilde{\lambda}_h+2}, \ldots, s_{\widetilde{\lambda}_h+1} \cdots s_{\widetilde{\lambda}_h+\lambda_{h+1}-1}\}.$$

Since $\mathcal{D}_{\nu'(h)} \cap W_\lambda \subseteq W_{\lambda^{(1)}}$ whenever $h \geqslant m$, the element $T_{\mathcal{D}_{\nu'(h)} \cap W_\lambda}$ used in (2.0.12) can be written as $T_{\mathcal{D}_{\nu'(h)} \cap W_\lambda} = \sum_{w \in \mathcal{D}_{\nu'(h)} \cap W_\lambda} (\dot{\mathbf{q}}_{h+1})^{\ell(w)} T_w$.

By definition, to compute $\phi_{D_h^+}\phi_A$, it suffices to write $\phi_{D_h^+}\phi_A([xy]_\mu)$ as a linear combination of some $T_{W_\xi d' W_\mu}$, where $\xi = \lambda^{[h^+]}$. We compute this

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

within $\mathcal{S}_{\mathbb{Q}(\upsilon)}(m|n,r)$:

$$\begin{aligned}\phi_{D_h^+}\phi_A([xy]_\mu) &= \phi^1_{\xi,\lambda}\phi^d_{\lambda,\mu}([xy]_\mu) = \phi^1_{\xi,\lambda}(T_{W_\lambda d W_\mu})\\ &= \phi^1_{\xi,\lambda}([xy]_\lambda T_d T_{\mathcal{D}_\nu\cap W_\mu}) \text{ (by (2.0.12))}\\ &= T_{W_\xi W_\lambda} T_d T_{\mathcal{D}_\nu\cap W_\mu} = (\mathcal{P}_{W_\nu})^{-1} T_{W_\xi W_\lambda} T_d [xy]_\mu\\ &= (\mathcal{P}_{W_\nu})^{-1}[xy]_\xi T_{\mathcal{D}_{\nu'(h)}\cap W_\lambda} T_d [xy]_\mu\\ &= (\mathcal{P}_{W_\nu})^{-1} \sum_{w\in\mathcal{D}_{\nu'(h)}\cap W_\lambda} [xy]_\xi(\dot{\mathbf{q}}_{h+1}^{\ell(w)} T_w) T_d [xy]_\mu. \qquad (4.1.1)\end{aligned}$$

Note that $d = d_A \in \mathcal{D}_{\lambda\mu}$. If $a_{h+1,k} > 0$ and

$$w_p := s_{\widetilde{\lambda}_h+1} s_{\widetilde{\lambda}_h+2} \cdots s_{\widetilde{\lambda}_h+\sum_{j=1}^{k-1} a_{h+1,j}+p}$$

for some $0 \leqslant p < a_{h+1,k}$, then by Proposition 3.6(1), we have

$$w_p d = s_{\widetilde{\lambda}_h} s_{\widetilde{\lambda}_h-1} \cdots s_{\widetilde{\lambda}_h-\sum_{j=k+1}^{m+n} a_{h,j}+1} d^+ (s_{\widetilde{a}_{h,k}+1} \cdots s_{\widetilde{a}_{h,k}+p}),$$

where $d^+ = d_{A^+_{h,k}}$. Clearly, $\sum_{j<k} a_{h+1,j} = \ell(w_p) - p$. If we put $Q_{h+1,k} = \dot{\mathbf{q}}_{h+1}^{\sum_{j<k} a_{h+1,j}}$, then

$$\begin{aligned}\sum_{p=0}^{a_{h+1,k}-1} \dot{\mathbf{q}}_{h+1}^{\ell(w_p)} T_{w_p} T_d &= Q_{h+1,k} T_{\widetilde{\lambda}_h} T_{\widetilde{\lambda}_h-1} \cdots T_{\widetilde{\lambda}_h-\sum_{j>k} a_{h,j}+1} T_{d^+}\\ &\cdot (1 + \dot{\mathbf{q}}_{h+1} T_{\widetilde{a}_{h,k}+1} + \cdots + \dot{\mathbf{q}}_{h+1}^{a_{h+1,k}-1} T_{\widetilde{a}_{h,k}+1} \cdots T_{\widetilde{a}_{h,k}+a_{h+1,k}-1}).\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{w\in\mathcal{D}_{\nu'}\cap W_\lambda} [xy]_\xi(\dot{\mathbf{q}}_{h+1}^{\ell(w)} T_w T_d)[xy]_\mu\\ &= \sum_{\substack{k\in[1,m+n]\\ a_{h+1,k}\geqslant 1}} Q_{h+1,k}[xy]_\xi T_{\widetilde{\lambda}_h} T_{\widetilde{\lambda}_h-1} \cdots T_{\widetilde{\lambda}_h-\sum_{j>k} a_{h,j}+1} T_{d^+}\\ &\quad \cdot (1 + (\dot{\mathbf{q}}_{h+1}) T_{\widetilde{a}_{h,k}+1} + \cdots + (\dot{\mathbf{q}}_{h+1})^{a_{h+1,k-1}} T_{\widetilde{a}_{h,k}+1} \cdots T_{\widetilde{a}_{h,k}+a_{h+1,k}-1})\\ &\quad \cdot [xy]_\mu.\end{aligned}$$

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

Since

$$\begin{aligned}&(1+(\dot{\mathbf{q}}_{h+1})T_{\widetilde{a}_{h,k}+1}+\cdots+(\dot{\mathbf{q}}_{h+1})^{a_{h+1,k-1}}T_{\widetilde{a}_{h,k}+1}\cdots T_{\widetilde{a}_{h,k}+a_{h+1,k-1}})[xy]_\mu\\&\quad=(1+\dot{\mathbf{q}}_{h+1}\ddot{\mathbf{q}}_k+\cdots+(\dot{\mathbf{q}}_{h+1}\ddot{\mathbf{q}}_k)^{a_{h+1,k}-1})[xy]_\mu\\&\quad=[\![a_{h+1,k}]\!]_{\dot{\mathbf{q}}_{h+1}\ddot{\mathbf{q}}_k}[xy]_\mu\end{aligned}$$

and

$$[xy]_\xi T_{\widetilde{\lambda}_h}T_{\widetilde{\lambda}_h-1}\cdots T_{\widetilde{\lambda}_h-\sum_{j>k}a_{h,j}+1}=\ddot{\mathbf{q}}_h^{\sum_{j>k}a_{h,j}}[xy]_\xi,$$

it follows that

$$\begin{aligned}\phi_{D_h^+}\phi_A([xy]_\mu)&=\mathcal{P}_{W_\nu}^{-1}\sum_{a_{h+1,k}\geqslant 1}Q_{h+1,k}\ddot{\mathbf{q}}_h^{\sum_{j>k}a_{h,j}}[\![a_{h+1,k}]\!]_{\dot{\mathbf{q}}_{h+1}\ddot{\mathbf{q}}_k}[xy]_\xi T_{d^+}[xy]_\mu\\&=\sum_{a_{h+1,k}\geqslant 1}\frac{\mathcal{P}_{W_{\nu''}}}{\mathcal{P}_{W_\nu}}Q_{h+1,k}\ddot{\mathbf{q}}_h^{\sum_{j>k}a_{h,j}}[\![a_{h+1,k}]\!]_{\dot{\mathbf{q}}_{h+1}\ddot{\mathbf{q}}_k}T_{W_\xi d^+W_\mu}\\&=\sum_{a_{h+1,k}\geqslant 1}\frac{\mathcal{P}_{W_{\nu''}}}{\mathcal{P}_{W_\nu}}Q_{h+1,k}\ddot{\mathbf{q}}_h^{\sum_{j>k}a_{h,j}}[\![a_{h+1,k}]\!]_{\dot{\mathbf{q}}_{h+1}\ddot{\mathbf{q}}_k}\phi_{A_{h,k}^+}([xy]_\mu),\end{aligned}$$

where $\nu''=\nu_M$ with $M=A_{h,k}^+$ or $W_{\nu''}=W_\xi^{d^+}\cap W_\mu$. Hence, noting

$$\frac{\mathcal{P}_{W_{\nu''}}}{\mathcal{P}_{W_\nu}}=\frac{[\![a_{h,k}+1]\!]_{\mathbf{q}_k}^!\,[\![a_{h+1,k}-1]\!]_{\mathbf{q}_k}^!}{[\![a_{h,k}]\!]_{\mathbf{q}_k}^!\,[\![a_{h+1,k}]\!]_{\mathbf{q}_k}^!}=\frac{[\![a_{h,k}+1]\!]_{\mathbf{q}_k}}{[\![a_{h+1,k}]\!]_{\mathbf{q}_k}},$$

we obtain

(4.1.2)

$$\phi_{D_h^+}\phi_A=\sum_{\substack{k\\ a_{h+1,k}\geqslant 1}}\dot{\mathbf{q}}_{h+1}^{\sum_{j<k}a_{h+1,j}}\ddot{\mathbf{q}}_h^{\sum_{j>k}a_{h,j}}\frac{[\![a_{h,k}+1]\!]_{\mathbf{q}_k}[\![a_{h+1,k}]\!]_{\dot{\mathbf{q}}_{h+1}\ddot{\mathbf{q}}_k}}{[\![a_{h+1,k}]\!]_{\mathbf{q}_k}}\phi_{A_{h,k}^+}.$$

It remains to prove that

$$\frac{[\![a_{h,k}+1]\!]_{\mathbf{q}_k}[\![a_{h+1,k}]\!]_{\dot{\mathbf{q}}_{h+1}\ddot{\mathbf{q}}_k}}{[\![a_{h+1,k}]\!]_{\mathbf{q}_k}}=[\![a_{h,k}+1]\!]_{\mathbf{q}_h}.\tag{4.1.3}$$

This can be seen in cases. For example, if $h<m$ and $k\leqslant m$ (resp., $h>m$ and $k>m$), then $\dot{\mathbf{q}}_{h+1}=1$, $\ddot{\mathbf{q}}_k=\mathbf{q}$ (resp., $\dot{\mathbf{q}}_{h+1}=-\mathbf{q}^{-1}$, $\ddot{\mathbf{q}}_k=-1$), and so $\mathbf{q}_k=\mathbf{q}_h$ (resp., $\dot{\mathbf{q}}_{h+1}\ddot{\mathbf{q}}_k=\mathbf{q}_h$). Hence,

$$\frac{[\![a_{h,k}+1]\!]_{\mathbf{q}_k}[\![a_{h+1,k}]\!]_{\dot{\mathbf{q}}_{h+1}\ddot{\mathbf{q}}_k}}{[\![a_{h+1,k}]\!]_{\mathbf{q}_k}}=[\![a_{h,k}+1]\!]_{\mathbf{q}_h}.$$

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

When $h \leqslant m$ and $k > m$, or $h > m$ and $k \leqslant m$, we must have $a_{h,k}+1 = a_{h+1,k} = 1$. Thus, $[\![a_{h,k}+1]\!]_{\mathbf{q}_k} = [\![a_{h,k}+1]\!]_{\dot{\mathbf{q}}_{h+1}\ddot{\mathbf{q}}_k} = [\![a_{h+1,k}]\!]_{\mathbf{q}_k} = 1 = [\![a_{h,k}+1]\!]_{\mathbf{q}_h}$. Finally, when $h = m$ and $k \leqslant m$, we have $\mathbf{q}_h = \mathbf{q}_k$ and $\dot{\mathbf{q}}_{h+1}\ddot{\mathbf{q}}_k = -\mathbf{q}^{-1}\mathbf{q} = -1$. But $a_{h+1,k} = a_{m+1,k} = 1$, forcing $[\![a_{h+1,k}]\!]_{\dot{\mathbf{q}}_{h+1}\ddot{\mathbf{q}}_k} = [\![a_{h+1,k}]\!]_{\mathbf{q}_k} = 1$. Hence,

$$\frac{[\![a_{h,k}+1]\!]_{\mathbf{q}_k}[\![a_{h+1,k}]\!]_{\dot{\mathbf{q}}_{h+1}\ddot{\mathbf{q}}_k}}{[\![a_{h+1,k}]\!]_{\mathbf{q}_k}} = [\![a_{h,k}+1]\!]_{\mathbf{q}_h},$$

proving (4.1.3) and, hence, formula (1).

If $n = 0$, then $\mathcal{S}(m|0,r)$ is the usual $\mathbf{q}$-Schur algebra, which is defined in [1] as a convolution algebra of the m-step flags of an r-dimensional space. Similar multiplication formulas are obtained in [1, Lemma 3.4] by counting intersections of certain orbits. Observe that, for $h < m$,

$$\dot{\mathbf{q}}_{h+1}^{\sum_{j<k} a_{h+1,j}} \ddot{\mathbf{q}}_h^{\sum_{j>k} a_{h,j}} = \mathbf{q}^{\sum_{j>k} a_{h,j}}, \quad \dot{\mathbf{q}}_h^{\sum_{j>k} a_{h,j}} \ddot{\mathbf{q}}_{h+1}^{\sum_{j<k} a_{h+1,j}} = \mathbf{q}^{\sum_{j<k} a_{h+1,j}}.$$

Corollary 4.2. *The multiplication formulas in Theorem 4.1 for $\mathcal{S}(m|0,r)$ coincide with the ones in [1, Lemma 3.4].*

We now make a comparison of these new formulas with ones given in [12, Lemma 3.1], derived through the relative norm method.

The $\mathcal{H}$-module $\mathfrak{T}(m|n,r)$ is isomorphic to the tensor superspace $V(m|n)^{\otimes r}$ (over $\mathcal{Z}!$) with an $\mathcal{H}$-action defined in [12, (1.0.10)]; see [16, Proposition 8.3]. In fact, the endomorphism algebra of $V(m|n)^{\otimes r}$ has a relative norm basis $\{N_A\}_{A\in M(m|n,r)}$ acting on the right. Matrix transposing may turn the right action to a left action and result in a basis denoted by $\{\zeta_A\}_{A\in M(m|n,r)}$. The $\mathcal{H}$-module isomorphism induces an algebra isomorphism (cf. [16, Corollary 8.4] and [13, Lemma 2.3])

$$\mathrm{End}_{\mathcal{H}}(V(m|n)^{\otimes r})^{\mathrm{op}} \longrightarrow \mathcal{S}(m|n,r), \zeta_A \longmapsto (-1)^{\widehat{A}}\phi_A,$$

where $\widehat{A} = \sum_{m<k<i\leqslant m+n, 1\leqslant j<l\leqslant m+n} a_{i,j}a_{k,l}$.

Corollary 4.3. *Let*

$$f^+_{h,k}(\mathbf{q},A) = \dot{\mathbf{q}}_{h+1}^{\sum_{j<k} a_{h+1,j}} \ddot{\mathbf{q}}_h^{\sum_{j>k} a_{h,j}}, \quad f^-_{h,k}(\mathbf{q},A) = \dot{\mathbf{q}}_h^{\sum_{j>k} a_{h,j}} \ddot{\mathbf{q}}_{h+1}^{\sum_{j<k} a_{h+1,j}}.$$

Then

$$(-1)^{\widehat{D}^+_h+\widehat{A}+\widehat{A}^+_{h,k}} f^+_{h,k}(\mathbf{q},A) = f_k(\mathbf{q},A,h) \quad \textit{and}$$

$$(-1)^{\widehat{D}^-_h+\widehat{A}+\widehat{A}^-_{h,k}} f^-_{h,k}(\mathbf{q},A) = g_k(\mathbf{q},A,h),$$

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

where $f_k(\mathbf{q}, A, h)$ and $g_k(\mathbf{q}, A, h)$ are defined in [12, (3.0.1-2)]. In particular, rewriting the multiplication formulas in Theorem 4.1 in terms of the ζ-basis results in the formulas in [12, Lemma 3.1].

Proof. We have

(4.3.1)
$$f^+_{h,k}(\mathbf{q}, A) = \begin{cases} \mathbf{q}^{\sum_{j>k} a_{h,j}}, & \text{if } h < m; \\ (-1)^{\sum_{j<k} a_{m+1,j}} \mathbf{q}^{-\sum_{j<k} a_{m+1,j} + \sum_{j>k} a_{m,j}}, & \text{if } h = m; \\ (-1)^{\sum_{j<k} a_{h+1,j} + \sum_{j>k} a_{h,j}} \mathbf{q}^{-\sum_{j<k} a_{h+1,j}}, & \text{if } h > m. \end{cases}$$

On the other hand (cf. [12, Lemma 5.1]), for the choice of $+$ or $-$,

$$\widehat{D}^{\pm}_h + \widehat{A} + \widehat{A}^{\pm}_{h,k} = \begin{cases} 2\widehat{A} & \text{if } h < m; \\ \mp \sum_{i>m+1, j<k} a_{i,j} + 2\widehat{A} & \text{if } h = m; \\ \mp \sum_{j>k} a_{h,j} \pm \sum_{j<k} a_{h+1,j} + 2\widehat{A} & \text{if } h > m. \end{cases}$$

Adjusting the right-hand side of (4.3.1) by the corresponding sign for the "+" case gives $f_k(\mathbf{q}, A, h)$. The "−" case is similar.

Theorem 4.1 and Corollary 4.3 give a new method to derive the key fundamental multiplication formulas given in [12, Lemma 3.1].

By introducing the normalized basis $\{[A]\}_{A \in M(m|n,r)}$, where[1]

$$[A] = (-1)^{\widehat{A}} \boldsymbol{v}^{-d(A)} \phi_A \quad \text{with } d(A) = \sum_{i>k, j<l} a_{i,j} a_{k,l} + \sum_{j<l} (-1)^{\widehat{i}} a_{i,j} a_{i,l},$$

we may modify the formulas given in Theorem 4.1 to obtain further multiplication formulas for the [ ]-basis; cf. (the $p = 1$ case of) [12, Propositions 4.4 and 4.5].

COROLLARY 4.4. *Maintain the notation above and let $\epsilon_{h,k} = 0$ for $h \neq m$, and $\epsilon_{m,k} = \sum_{i>m, j<k} a_{i,j}$. The following multiplication formulas hold in $\mathcal{S}_R(m|n, r)$:*

(1) $$[D^+_h][A] = \sum_{\substack{k \in [1, m+n] \\ a_{h+1,k} \geqslant 1}} (-1)^{\epsilon_{h,k}} \boldsymbol{v}_h^{f^+_{h,k}} \overline{[\![a_{h,k} + 1]\!]}_{\boldsymbol{v}_h^2} [A^+_{h,k}],$$

where $f^+_{h,k} = \sum_{j \geqslant k} a_{h,j} - (-1)^{\widehat{h} + \widehat{h+1}} \sum_{j>k} a_{h+1,j}$;

[1] The element $[A]$ is denoted by ξ_A in [12, (4.2.1)].

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

(2) $$[D_h^-][A] = \sum_{\substack{k\in[1,m+n]\\ a_{h,k}\geqslant 1}} (-1)^{\epsilon_{h,k}} \boldsymbol{\upsilon}_{h+1}^{f_{h,k}^-} \overline{[\![a_{h+1,k}+1]\!]}_{\boldsymbol{\upsilon}_{h+1}^2} [A_{h,k}^-],$$

where $f_{h,k}^- = \sum_{j\leqslant k} a_{h+1,j} - (-1)^{\widehat{h}+\widehat{h+1}} \sum_{j<k} a_{h,j}$.

The first important application of the multiplication formulas above is a new realization of the quantum supergroup $\mathbf{U}_{\boldsymbol{\upsilon}}(\mathfrak{gl}_{m|n})$; see the argument from [12, Section 5] onwards and, in particular, see [12, Definition 6.1, Theorem 8.4].

We now seek further applications of these multiplication formulas.

We will show below that the formulas provide enough information for the regular representation of the integral q-Schur superalgebra $\mathcal{S}_R(m|n,r)$. We then use such a representation to determine the semisimplicity of q-Schur superalgebras and to construct infinitesimal and little ones without involving the quantum supergroup or quantum coordinate superalgebra.

We return to the general setting for $\mathcal{S}_R(m|n,r)$ defined relative to a commutative ring R and an invertible parameter $\upsilon\in R$ or $q=\upsilon^2$. Base change via $\mathcal{Z}\to R, \boldsymbol{\upsilon}\mapsto\upsilon$, we may turn the multiplication formulas in $\mathcal{S}(m|n,r)$ into similar formulas in $\mathcal{S}_R(m|n,r)$. In fact, these formulas can be interpreted as the matrix representation of certain generators for $\mathcal{S}_R(m|n,r)$ relative to the basis $\{[A]\}_{A\in M(m|n,r)}$.

Let

$$M(m|n)^{\pm} = \{A=(a_{i,j})\in M(m|n) \mid a_{i,i}=0, 1\leqslant i\leqslant m+n\}.$$

For $A\in M(m|n)^{\pm}$ and $\mathbf{j}=(j_1,j_2,\dots,j_{m+n})\in\mathbb{Z}^{m+n}$, define

$$A(\mathbf{j},r) = \begin{cases} \displaystyle\sum_{\lambda\in\Lambda(m|n,r-|A|)} (-1)^{\overline{A+\lambda}} \upsilon^{\lambda*\mathbf{j}}[A+\lambda], & \text{if } |A|\leqslant r;\\ 0, & \text{otherwise,}\end{cases} \tag{4.4.1}$$

where $\lambda*\mathbf{j}=\sum_{i=1}^{m+n}(-1)^{\widehat{i}}\lambda_i j_i$ is the super (or signed) "dot product", $A+\lambda=A+\mathrm{diag}(\lambda)$ and $\overline{M}=\sum_{\substack{m+n\geqslant i>m\geqslant k\geqslant 1\\ m<j<l\leqslant m+n}} m_{i,j}m_{k,l}$ for a matrix M. We also let $1_\lambda=[\mathrm{diag}(\lambda)]$ for all $\lambda\in\Lambda(m|n,r)$, the identity map on $[xy]_\lambda\mathcal{H}_R$. Then $1_\lambda[A]=\delta_{\lambda,\mathrm{ro}(A)}[A]$. For the zero matrix O, $\mathbf{e}_i\in\Lambda(m|n,1)$ and $p\geqslant 1$, set

$$\mathrm{K}_i = O(\mathbf{e}_i,r), \quad \mathrm{E}_h^{(p)} = (pE_{h,h+1})(\mathbf{0},r), \quad \mathrm{F}_h^{(p)} = (pE_{h+1,h})(\mathbf{0},r).$$

Note that $\mathrm{K}_i=\sum_{\lambda\in\Lambda(m|n,r)} \upsilon^{(-1)^{\widehat{i}}\lambda_i}1_\lambda$ and $\mathrm{E}_m^2=0=\mathrm{F}_m^2$.

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

Let $\mathcal{S}_R^-$, $\mathcal{S}_R^+$ be the subsuperalgebra of $\mathcal{S}_R(m|n,r)$ generated respectively by $\mathrm{F}_h^{(p)}$, $\mathrm{E}_h^{(p)}$ for all $1 \leqslant h < m+n$, $p \geqslant 1$, and $\mathcal{S}_R^0$ the subsuperalgebra spanned by all 1_λ.

The first assertion of the following is [12, Corollary 8.5].

THEOREM 4.5. *The q-Schur superalgebra $\mathcal{S}_R = \mathcal{S}_R(m|n,r)$ is generated by K_i, 1_λ, $\mathrm{E}_h^{(p)}, \mathrm{F}_h^{(p)}$ for all $1 \leqslant h, i \leqslant m+n$, $h \neq m+n$, $\lambda \in \Lambda(m|n,r)$, $1 \leqslant p \leqslant r$, and $\mathcal{S}_R = \mathcal{S}_R^+ \mathcal{S}_R^0 \mathcal{S}_R^-$. These generators have the following matrix representations relative to the basis $\{[A]\}_{A \in M(m|n,r)}$:*

(0) $\mathrm{K}_i[A] = \upsilon^{(-1)^{\widehat{i}}\mathrm{ro}(A)_i}[A]$, $1_\lambda[A] = \delta_{\lambda,\mathrm{ro}(A)}[A]$;

(1) $$\mathrm{E}_h^{(p)}[A] = \sum_{\substack{\nu \in \Lambda(m|n,p) \\ \nu \leqslant \mathrm{row}_{h+1}(A)}} \upsilon_h^{f_h^+(\nu,A)} \prod_{k=1}^{m+n} \overline{\left[\!\left[\begin{matrix} a_{h,k}+\nu_k \\ \nu_k \end{matrix} \right]\!\right]}_{\upsilon_h^2} [A + \sum_l \nu_l (E_{h,l} - E_{h+1,l})],$$
where $h \neq m$, $f_h^+(\nu, A) = \sum_{j \geqslant t} a_{h,j}\nu_t - \sum_{j>t} a_{h+1,j}\nu_t + \sum_{t<t'} \nu_t \nu_{t'}$ *and* $\nu \leqslant \nu'$ *means that* $\nu_i \leqslant \nu_i'$ *for all* i;

(2) $$\mathrm{F}_h^{(p)}[A] = \sum_{\substack{\nu \in \Lambda(m|n,p) \\ \nu \leqslant \mathrm{row}_h(A)}} \upsilon_{h+1}^{f_h^-(\nu,A)} \prod_{k=1}^{m+n} \overline{\left[\!\left[\begin{matrix} a_{h+1,k}+\nu_k \\ \nu_k \end{matrix} \right]\!\right]}_{\upsilon_{h+1}^2} [A - \sum_l \nu_l (E_{h,l} - E_{h+1,l})],$$
where $h \neq m$ *and* $f_h^-(\nu, A) = \sum_{j \leqslant t} a_{h+1,j}\nu_t - \sum_{j<t} a_{h,j}\nu_t + \sum_{t<t'} \nu_t \nu_{t'}$;

(3) $$\mathrm{E}_m[A] = \sum_{\substack{k \\ a_{m+1,k} \geqslant 1}} (-1)^{\sum_{i>m, j<k} a_{i,j}} \upsilon_m^{f_{m,k}^+(A)} \overline{[\![a_{m,k}+1]\!]}_{\upsilon_m^2} [A_{m,k}^+],$$
where $f_{m,k}^+(A) = \sum_{j \geqslant k} a_{m,j} + \sum_{j>k} a_{m+1,j}$;

(4) $$\mathrm{F}_m[A] = \sum_{\substack{k \\ a_{m,k} \geqslant 1}} (-1)^{\sum_{i>m, j<k} a_{i,j}} \upsilon_{m+1}^{f_{m,k}^-(A)} \overline{[\![a_{m+1,k}+1]\!]}_{\upsilon_{m+1}^2} [A_{m,k}^-],$$
where $f_{m,k}^-(A) = \sum_{j \leqslant k} a_{m+1,j} + \sum_{j<k} a_{m,j}$.

Proof. The first assertion follows from [12, Corollary 8.5] (cf. [12, Theorem 6.3]). Now the relations in (0) are clear. Since $\mathrm{E}_h^{(p)}[A] = \mathrm{E}_h^{(p)} 1_{\mathrm{ro}(A)}[A]$, $\mathrm{F}_h^{(p)}[A] = \mathrm{F}_h^{(p)} 1_{\mathrm{ro}(A)}[A]$, and $\mathrm{E}_h^{(p)} 1_{\mathrm{ro}(A)} = (-1)^{\overline{D_{h,p}^+}}[D_{h,p}^+]$, $\mathrm{F}_h^{(p)} 1_{\mathrm{ro}(A)} = (-1)^{\overline{D_{h,p}^-}}[D_{h,p}^-]$, where the matrices $D_{h,p}^\pm \in M(m|n,r)$ are defined by the conditions that $\mathrm{co}(D_{h,p}^\pm) = \mathrm{ro}(A)$ and $D_{h,p}^+ - pE_{h,h+1}$,

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

$D^-_{h,p} - pE_{h+1,h}$ are diagonal, (1) and (2) follow from [12, Proposition 4.4][2] and [12, Lemma 5.1(1)] which tells $\overline{D^{\pm}_{h,p}} = 0$. The remaining (3) and (4) follow from the $h = m$ case of Corollary 4.4; see [12, Proposition 4.5].

Note that we have in $\mathcal{S}_F(m|n, r)$

$$\mathrm{E}_h\mathrm{F}_k - (-1)^{\widehat{h}\widehat{k}}\mathrm{F}_k\mathrm{E}_h = \delta_{h,k}\frac{\mathrm{K}_h\mathrm{K}_{h+1}^{-1} - \mathrm{K}_h^{-1}\mathrm{K}_{h+1}}{\upsilon_h - \upsilon_h^{-1}}. \tag{4.5.1}$$

§5. Semisimple q-Schur superalgebras

The most fabulous application of the multiplication formulas is the realizations of quantum $\mathfrak{gl}_n$ [1] and quantum super $\mathfrak{gl}_{m|n}$ [12]. We now use these formulas to construct certain modules from which we obtain a semisimplicity criterion of q-Schur superalgebras. *From now on, let F be a field of characteristic $\neq 2$ and assume that $\upsilon \in F^\times$ and $q = \upsilon^2 \neq 1$.* Since every simple $\mathcal{S}_F(m|n, r)$-supermodule is also a simple $\mathcal{S}_F(m|n, r)$-module (see e.g., [15, Proposition 4.1]), we will drop the prefix "super" in the sequel for simplicity.

We first determine the semisimplicity for $\mathcal{S}_F(1|1, r)$ (see [25] for the $q = 1$ case).

LEMMA 5.1. *Assume that $q \neq 1$ is a primitive l-th root of unity.*

(1) *If $l \nmid r$ then $\mathcal{S}_F(1|1, r)$ is semisimple and has exact r nonisomorphic irreducible modules which are all two dimensional.*
(2) *If $l \mid r$ then $\mathcal{S}_F(1|1, r)$ is not semisimple and has exact $r + 1$ nonisomorphic irreducible modules which are all one dimensional.*

Proof. Let $\mathcal{S}_F = \mathcal{S}_F(1|1, r)$. We first observe that

$$M(1|1, r) = \{A_a, A_b^+, A_c^-, A_d^{\pm} \mid a \in [0, r], b, c \in [0, r-1], d \in [0, r-2]\},$$

where $A_a, A_b^+, A_c^-, A_d^{\pm}$ denote respectively the following matrices

$$\begin{pmatrix} a & 0 \\ 0 & r-a \end{pmatrix}, \quad \begin{pmatrix} b & 1 \\ 0 & r-b-1 \end{pmatrix}, \quad \begin{pmatrix} c & 0 \\ 1 & r-c-1 \end{pmatrix}, \quad \begin{pmatrix} d & 1 \\ 1 & r-d-2 \end{pmatrix}.$$

Note that $1_a := 1_{(a,r-a)} = [A_a]$ and $\sum_{a=0}^r 1_a$ is the identity element. So

$$\mathcal{S}_F = \bigoplus_{a=0}^{r} \mathcal{S}_F 1_a \quad \text{and} \quad \dim \mathcal{S}_F = 4r.$$

[2] $D^+_{h,p}$, $D^-_{h,p}$ are denoted there by U_p, L_p.

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

Since $\mathcal{S}_F 1_a$ is spanned by $[A]$ with $\mathrm{co}(A) = (a, r-a)$, it follows that

$$\mathcal{S}_F 1_0 = \mathrm{span}\{1_0, [A_0^+]\}, \quad \mathcal{S}_F 1_r = \mathrm{span}\{1_r, [A_{r-1}^-]\},$$
$$\mathcal{S}_F 1_a = \mathrm{span}\{1_a, [A_a^+], [A_{a-1}^-], [A_{a-1}^\pm]\}, \quad \forall a \in [1, r-1].$$

By Theorem 4.5(3) and (4), we have

$$\mathrm{E}_1[A_0^+] = 0, \quad \mathrm{F}_1[A_0^+] = \upsilon^{-(r-1)}[\![r]\!]_q 1_0, \quad \mathrm{E}_1 1_0 = [A_0^+], \quad \mathrm{F}_1 1_0 = 0,$$
$$\mathrm{F}_1[A_{r-1}^-] = 0, \quad \mathrm{E}_1[A_{r-1}^-] = \upsilon^{r-1}[\![r]\!]_{q^{-1}} 1_r, \quad \mathrm{E}_1 1_r = 0, \quad \mathrm{F}_1 1_r = [A_{r-1}^-].$$

If $l \nmid r$, then $\upsilon^{-(r-1)}[\![r]\!]_q = \upsilon^{r-1}[\![r]\!]_{q^{-1}} \neq 0$ in F, and we see easily that $L(1) := \mathcal{S}_F 1_0$ is irreducible. Similarly, $L(r) := \mathcal{S}_F 1_r$ is irreducible if $l \nmid r$.

If $l \mid r$, then $L(1)$ is indecomposable and $[A_0^+]$ spans a submodule $\overline{L}(1)$ of $L(1)$. Let $\overline{L}(0) = L(1)/\overline{L}(1)$. Similarly, $[A_{r-1}^-]$ spans a submodule $\overline{L}(r-1)$. Let $\overline{L}(r) = L(r)/\overline{L}(r-1)$.

For $a \in [1, r-1]$, applying Theorem 4.5 again yields

$$\begin{aligned}
&(1) \quad \mathrm{E}_1[A_a^+] = 0, \quad \mathrm{F}_1[A_a^+] = \upsilon^{-(r-1)}[\![r-a]\!]_q 1_a + [A_{a-1}^\pm],\\
&(2) \quad \mathrm{F}_1[A_{a-1}^-] = 0, \quad \mathrm{E}_1[A_{a-1}^-] = \upsilon^{r-1}[\![a]\!]_{q^{-1}} 1_a - [A_{a-1}^\pm],\\
&(3) \quad \mathrm{E}_1[A_{a-1}^\pm] = \upsilon^{r-1}[\![a]\!]_{q^{-1}}[A_a^+], \quad \mathrm{E}_1 1_a = [A_a^+],\\
&(4) \quad \mathrm{F}_1[A_{a-1}^\pm] = -\upsilon^{-(r-1)}[\![r-a]\!]_q[A_{a-1}^-], \quad \mathrm{F}_1 1_a = [A_{a-1}^-].
\end{aligned} \tag{5.1.1}$$

Let

$$L(a+1) = \mathrm{span}\{[A_a^+], \mathrm{F}_1[A_a^+]\} \quad \text{and} \quad L(a) = \mathrm{span}\{[A_{a-1}^-], \mathrm{E}_1[A_{a-1}^-]\}.$$

If $l \nmid r$, we claim that $\mathcal{S}_F 1_a = L(a+1) \oplus L(a)$ is a direct sum of irreducible submodules. Indeed, $[\![a]\!]_{q^{-1}}$ and $[\![r-a]\!]_q$ cannot be both zero in this case. So $L(a+1) \cap L(a) = 0$, forcing $\mathcal{S}_F 1_a = L(a+1) \oplus L(a)$ as vector spaces. Since, by (4.5.1),

$$\mathrm{E}_1\mathrm{F}_1[A_a^+] = (\mathrm{E}_1\mathrm{F}_1 + \mathrm{F}_1\mathrm{E}_1)[A_a^+] = \frac{\mathrm{K}_1\mathrm{K}_2^{-1} - \mathrm{K}_1^{-1}\mathrm{K}_2}{\upsilon - \upsilon^{-1}}[A_a^+] = \frac{\upsilon^r - \upsilon^{-r}}{\upsilon - \upsilon^{-1}}[A_a^+], \tag{5.1.2}$$

and $(\upsilon^r - \upsilon^{-r})/(\upsilon - \upsilon^{-1}) \neq 0$, every nonzero element in $L(a+1)$ generates $L(a+1)$. Hence, $L(a+1)$ is an irreducible submodule. Likewise, $L(a)$ is a submodule. This proves that $\mathcal{S}_F 1_a$ is semisimple for all $a \in [1, r-1]$. Hence, $\mathcal{S}_F$ is semisimple.

Assume now $l \mid r$. Then, by (5.1.2), $\mathrm{E}_1(\mathrm{F}_1[A_a^+]) = 0$. On the other hand, $\mathrm{F}_1^2 = 0$ implies $\mathrm{F}_1(\mathrm{F}_1[A_a^+]) = 0$. Thus, $\mathrm{F}_1[A_a^+]$ spans a submodule $\overline{L}(a)$ of

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

$L(a+1)$. Similarly, $\mathrm{E}_1[A^-_{a-1}]$ spans a submodule $\overline{L}(a)'(\cong \overline{L}(a))$ of $L(a)$. Moreover, (cf. [25, Theorem 1])

$$\overline{L}(a+1) \cong L(a+1)/\overline{L}(a), \quad \overline{L}(a-1) \cong L(a)/\overline{L}(a)'.$$

Hence, $\overline{L}(a), 0 \leqslant a \leqslant r$, form a complete set of all irreducible $\mathcal{S}_F$-modules.

REMARK 5.2. The classification of irreducible modules for $\mathcal{S}_k(1|1,r)$ in the semisimple case is consistent with a classification given in [16, Theorem 7.5].

LEMMA 5.3. *With the same assumption on l as in Lemma 5.1, the superalgebras $\mathcal{S}_F(2|1,r)$ and $\mathcal{S}_F(1|2,r)$ are not semisimple for all $r \geqslant l$.*

Proof. By Lemma 2.1, it suffices to consider $\mathcal{S}_F = \mathcal{S}_F(2|1,r)$. Let $e = 1_{(r,0,0)}$. Then, for $P = \mathcal{S}_F e$, $\mathrm{End}_{\mathcal{S}_F}(P) \cong F$ and so P is an indecomposable $\mathcal{S}_F$-module. We now show the existence of a proper submodule of P if $r \geqslant l$. Observe that P is spanned by all $[A]$ with $\mathrm{co}(A) = (r,0,0)$. Such A will be written as $A_{a,b,c}$ where $(a,b,c)^t$ is the first column of A. We have two cases to consider.

Case 1. If $r = al + b$ with $0 \leqslant b \leqslant l-2$ (i.e., $l \nmid r+1$), then $b+1 < l$ and $\mathrm{F}_1^{(b+1)} e = [A_{al-1,b+1,0}] \in P$. We now claim that $[A_{al-1,b+1,0}]$ is a maximal vector in the sense that $\mathrm{E}_h^{(p)}[A_{al-1,b+1,0}] = 0$ for all $h = 1, 2$ and $p \geqslant 1$. This is clear if $h = 2$ since all $a_{h+1,k} = a_{3,k} = 0$. Also, by Theorem 4.5(1), we have $\mathrm{E}_1^{(p)}[A_{al-1,b+1,0}] = 0$ for $p > b+1$ and, for $p \leqslant b+1 < l$,

$$\mathrm{E}_1^{(p)}[A_{al-1,b+1,0}] = \frac{\mathrm{E}_1^{p-1}}{[p]^!_\upsilon}\mathrm{E}_1[A_{al-1,b+1,0}] = \frac{\upsilon^{al-1}[\![al]\!]_{q^{-1}}}{[p]^!_\upsilon}\mathrm{E}_1^{p-1}[A_{al,b,0}] = 0.$$

By the claim, we see that $P' := \mathcal{S}_F[A_{al-1,b+1,0}] = \mathcal{S}_F^-[A_{al-1,b+1,0}]$ is a proper submodule of P since $e \notin P'$.

Case 2. If $r = al - 1$ (and so $a \geqslant 2$), then by Theorem 4.5, $\mathrm{F}_2(\mathrm{F}_1^{(l)} e) = \mathrm{F}_2[A_{r-l,l,0}] = [A_{r-l,l-1,1}] \in P$. Now, since $r - l + 1 = (a-1)l$, we have $\mathrm{E}_1[A_{r-l,l-1,1}] = \upsilon^{r-l}[\![r-l+1]\!]_{q^{-1}}[A_{r-l+1,l-2,1}] = 0$ and $\mathrm{E}_2[A_{r-l,l-1,1}] = \upsilon^{l-1}[\![l]\!]_{q^{-1}}[A_{r-l,l,0}] = 0$. Hence, $\mathrm{E}_h^{(p)}[A_{r-l,l-1,1}] = 0$ for all $h = 1, 2$ and $p < l$. Similarly, by Theorem 4.5(1), $\mathrm{E}_h^{(p)}[A_{r-l,l-1,1}] = 0$ for $h = 1, 2$ and $p \geqslant l$. This proves that $\mathcal{S}_F[A_{r-l,l-1,1}] = \mathcal{S}_F^-[A_{r-l,l-1,1}]$ is a proper submodule of P.

Combining the two cases, we conclude that $\mathcal{S}_F$ is not semisimple whenever $r \geqslant l$.

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

The following result is the quantum analogue of a result of Marko and Zubkov [26], which is stated in the abstract.

THEOREM 5.4. *Let F be a field containing elements $q \neq 0, 1$. Then the q-Schur superalgebra $\mathcal{S}_F(m|n, r)$ with $m, n \geqslant 1$ is semisimple if and only if one of the following holds:*

(1) *q is not a root of unity;*
(2) *q is a primitive lth root of unity and $r < l$;*
(3) *$m = n = 1$ and q is an lth root of unity with $l \nmid r$.*

Proof. The first two conditions imply that $\mathcal{H}_F$ is semisimple and so is $\mathcal{S}_F$. The semisimplicity under (3) follows from Lemma 5.1. We now show that, if all three conditions fail, then $\mathcal{S}_F$ is not semisimple. By Lemmas 2.1 and 5.1, it suffices to look at the case for $m \geqslant 2$ and $n \geqslant 1$ and $l \leqslant r$.

Consider the subset

$$\begin{aligned}\Lambda(m|n, r)' = \{\lambda \in \Lambda(m|n, r) \mid \lambda^{(0)} = (\lambda_1, \lambda_2, 0, \dots, 0), \lambda^{(1)} \\ = (\lambda_{m+1}, 0, \dots, 0)\}\end{aligned}$$

and let $f = \sum_{\lambda \in \Lambda(m|n,r)'} 1_\lambda$ and $e = 1_{(r,0,\dots,0)}$. Then $ef = e = fe$ and it is clear that there is an algebra isomorphism $\mathcal{S}_F(2|1, r) \cong f\mathcal{S}_F(m|n, r)f$. By identifying the two algebras under this isomorphism, we see that there is an $f\mathcal{S}_F(m|n, r)f$-module isomorphism $\mathcal{S}_F(2|1, r)1_{(r,0,0)} \cong f\mathcal{S}_F(m|n, r)e$. This $f\mathcal{S}_F(m|n, r)f$-module is indecomposable, but not irreducible, by Lemma 5.3. Since $\mathcal{S}_F(m|n, r)e$ is indecomposable and its image $f\mathcal{S}_F(m|n, r)e$ under the "Schur functor" is indecomposable, but not irreducible, we conclude that $\mathcal{S}_F(m|n, r)e$ is not irreducible (see [22, (6.2g)]). Hence, $\mathcal{S}_F(m|n, r)$ is not semisimple.

REMARK 5.5. Semisimple q-Schur algebras have been classified by K. Erdmann and D. Nakano [18, Section 1.3, Theorem (A)]. By Corollary 4.2, we may also use this new approach to get their result; see appendix A.

§6. Infinitesimal and little q-Schur superalgebras

We now give another application of the multiplication formulas. We first construct certain subsuperalgebras of the q-Schur superalgebra $\mathcal{S}_R(m|n, r)$ over the commutative ring R in which $q = v^2 \neq 1$ is a primitive l-th root of unity. (So $l \geqslant 2$.)

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

Let $\mathfrak{s}_R(m|n,r)$ be the R-submodule spanned by all $[A]$ with $A \in M(m|n,r)_l$, where

$$M(m|n,r)_l = \{(a_{i,j}) \in M(m|n,r) \mid a_{i,j} < l \ \forall i \neq j\}.$$

We have the following super analogue of the infinitesimal q-Schur algebras (cf. [3]).

THEOREM 6.1. *The R-submodule $\mathfrak{s}_R(m|n,r)$ is a subsuperalgebra generated by $\mathrm{E}_h, \mathrm{F}_h, 1_\lambda$ for all $1 \leqslant h < m+n$, $\lambda \in \Lambda(m|n,r)$.*

Proof. Let $\mathfrak{s}'_R(m|n,r)$ be the subalgebra generated by $[aE_{h,h+1}+D]$ and $[bE_{h+1,h}+D']$, where D, D' are diagonal matrices with $aE_{h,h+1}+D, bE_{h+1,h}+D' \in M(m|n,r)_l$ and $0 \leqslant a, b < l$. Observe from the multiplication formulas in Theorem 4.5 that if $A \in M(m|n,r)_l$ then $\mathrm{E}_h^{(a)}[A] = [aE_{h,h+1}+D][A]$ and $\mathrm{F}_h^{(b)}[A] = [bE_{h+1,h}+D'][A]$, for some D, D', are linear combinations of $[B]$ with $B \in M(m|n,r)_l$. This implies that $\mathfrak{s}'_R(m|n,r) \subseteq \mathfrak{s}_R(m|n,r)$. Now, by the triangular relation [12, Theorem 7.4]:

$$\prod_{i \leqslant h < j}^{(\leqslant_2)} [a_{j,i}E_{h+1,h} + D_{i,h,j}] \prod_{i \leqslant h < j}^{(\leqslant_1)} [a_{i,j}E_{h,h+1} + D_{i,h,j}]$$

$$= (-1)^{\overline{A}}[A] + \text{lower terms}, \tag{6.1.1}$$

an inductive argument on the Bruhat order on $M(m|n,r)$ shows that every $[A]$ with $A \in M(m|n,r)_l$ belongs to $\mathfrak{s}'_R(m|n,r)$. Hence, $\mathfrak{s}_R(m|n,r) = \mathfrak{s}'_R(m|n,r)$ is a subalgebra and, hence, a subsuperalgebra. From the argument above, we see easily that $\mathrm{E}_h, \mathrm{F}_h, 1_\lambda$ can be generators.

Remarks 6.2. By [14, Corollary 8.4], $\mathfrak{s}_R(m|n,r)$ is isomorphic to the infinitesimal q-Schur superalgebra defined in [3, Section 3] by using quantum coordinate superalgebra.

We now construct a subsuperalgebra $\mathfrak{u}_R(m|n,r)$. Let $\mathbb{Z}_l := \mathbb{Z}/l\mathbb{Z}$ and let $^- : \mathbb{Z} \to \mathbb{Z}_l$ be the quotient map. Extend this map to $M(m|n,r)$, $\Lambda(m|n,r)$ by baring on the entries. Thus, we may identify the image $\overline{M(m|n,r)}$ with the following set:

$$\overline{M(m|n,r)} = \{A^{\pm} + \mathrm{diag}(\overline{\partial}_A) \mid A \in M(m|n,r)_l\} = \overline{M(m|n,r)_l},$$

where $A^{\pm}$ is obtained by replacing the diagonal of A with 0's and $\partial_A \in \mathbb{Z}^{m+n}$ is the diagonal of A (i.e., $A = A^{\pm} + \mathrm{diag}(\partial_A)$). For $A = A^{\pm} + \mathrm{diag}(\overline{\partial}_A) \in$

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

$\overline{M(m|n,r)}$, define

$$\overline{\xi}_A = \sum_{\substack{\lambda\in\Lambda(m|n,r-|A^{\pm}|)\\ \overline{\lambda}=\overline{\partial}_A}} [A^{\pm}+\mathrm{diag}(\lambda)] = \sum_{\substack{\lambda\in\Lambda(m|n,r-|A^{\pm}|)\\ \overline{\lambda}=\overline{\partial}_A}} \xi_{A^{\pm}+\mathrm{diag}(\lambda)},$$

and let $\overline{1}_\lambda = \overline{\xi}_{\mathrm{diag}(\lambda)}$. Note that every $\overline{\xi}_A$ is a homogeneous element with respect the super structure on $\mathcal{S}_R(m|n,r)$.

We now have the super analogue of the *little* q-Schur algebra introduced in [10].

COROLLARY 6.3. *The subsuperspace* $\mathfrak{u}_R(m|n,r)$ *of* $\mathfrak{s}_R(m|n,r)$ *spanned by* $\overline{\xi}_A$ *for all* $A\in\overline{M(m|n,r)}$ *is a subsuperalgebra with identity* $\sum_{x\in\overline{\Lambda(m|n,r)}}\overline{1}_{\mathrm{diag}(x)}$ *and generated by* $\mathrm{E}_h, \mathrm{F}_h, \overline{1}_\lambda$ *for all* $1\leqslant h<m+n, \lambda\in\overline{\Lambda(m|n,r)}$.

Proof. In this case, with a proof similar to that for Theorem 6.1, we see that $\mathfrak{u}_R(m|n,r)$ is the subalgebra generated by $\overline{\xi}_{aE_{h,h+1}+D}$ and $\overline{\xi}_{bE_{h+1,h}+D'}$, where D, D' are diagonal matrices with $aE_{h,h+1}+D, bE_{h+1,h}+D'\in\overline{M(m|n,r)}$. Note that by taking the sum of the triangular relations (6.1.1) for every $A^{\pm}+\mathrm{diag}(\lambda)$ with $\overline{\lambda}=\overline{\partial}_A$, we obtain the required triangular relation for $\overline{\xi}_A$'s (cf. the proof of [12, Theorem 8.1]). The last assertion is clear as every $\overline{\xi}_{aE_{h,h+1}+D}$ or $\overline{\xi}_{bE_{h+1,h}+D'}$ has the form $\mathrm{E}_h^{(a)}\overline{1}_\lambda$ or $\mathrm{F}_h^{(b)}\overline{1}_\lambda$.

We end the paper with the following semisimplicity criteria for the infinitesimal/little q-Schur superalgebras; compare the nonsuper case [11, Section 7] and [21].

THEOREM 6.4. *The superalgebra* $\mathfrak{s}_F(m|n,r)$ *or* $\mathfrak{u}_F(m|n,r)$ *with* $m,n\geqslant 1$ *is semisimple if and only if one of the following holds:*

(1) $r<l$;
(2) $m=n=1, l\nmid r$.

Proof. We first look at the "infinitesimal" case. We observe that, if $r<l$ or $m=n=1$, then $\mathfrak{s}_F(m|n,r)=\mathcal{S}_F(m|n,r)$. The "if" part is clear. Conversely, suppose $\mathfrak{s}_F(m|n,r)$ is semisimple. Since $\mathfrak{s}_F(1|1,r)=\mathcal{S}_F(1|1,r)$, its semisimplicity forces $l\nmid r$. Assume $m\geqslant 2, n\geqslant 1$ and $l\leqslant r$. By the proof of Lemma 5.3, we see that $\mathfrak{s}_F(2|1,r)e$ $(e=1_{(r,0,0)})$ is indecomposable and contains the proper submodule $\mathfrak{s}_F(2|1,r)[A_{al,b,0}]$ if $l\nmid r+1$, or $\mathfrak{s}_F(2|1,r)[A_{r-l,l-1,1}]$ if $l\mid r+1$. Hence, we can use the Schur functor argument to conclude $\mathfrak{s}_F(m|n,r)$ is not semisimple unless $r<l$.

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

We now look at the "little" case. If $r < l$, then $\mathfrak{u}_F(m|n,r) = \mathcal{S}_F(m|n,r)$ is semisimple. If $m = n = 1$ and $l \nmid r$, then the simple module $L(a)$ constructed in the proof of Lemma 5.1 remains irreducible when restricted to $\mathfrak{u}_F(m|n,r)$. This is seen from the last assertion of Corollary 6.3. Thus, $\mathfrak{s}_F(m|n,r)$ as an $\mathfrak{u}_F(m|n,r)$-module is semisimple. As a $\mathfrak{u}_F(m|n,r)$-submodule of $\mathfrak{s}_F(m|n,r)$, $\mathfrak{u}_F(m|n,r)$ is semisimple. Conversely, if conditions (1) and (2) both fail. Then $r \geqslant l$. If one of the m and n is great than 1, then $\mathfrak{u}_F(m|n,r)$ is not semisimple. To see this, it is enough to show that $M = \mathfrak{s}_F(2|1,r)e$ as an $\mathfrak{u}_F(2|1,r)$-module is indecomposable. Indeed, suppose $M = M_1 \oplus M_2$ where M_i are nonzero $\mathfrak{u}_F(2|1,r)$-submodules. Then, for any $\lambda \in \Lambda(m|n,r)$, $1_\lambda M_1$ and $1_\lambda M_2$ cannot be both nonzero since $\dim 1_\lambda M = 1$. This shows that M_i is a direct sum of some $1_\lambda M$. Hence, M_i is an $\mathfrak{s}_F(2|1,r)$-module, contrary to the fact that M is an indecomposable $\mathfrak{s}_F(2|1,r)$-module. If $m = n = 1$, then $l \mid r$. In this case, $\mathfrak{u}_F(1|1,r)$ is clearly non-semsimple as $\mathfrak{u}_F(1|1,r)\overline{1}_0$ is indecomposable, but not irreducible.

Acknowledgments. We thank the referee for several helpful comments.

Appendix A. A Theorem of Erdmann–Nakano

THEOREM A.1. [18, Section 1.3, Theorem (A)] *Let F be a field of characteristic $p \geqslant 0$ containing elements $q \neq 0, 1$. Then the q-Schur algebra $\mathcal{S}_F(m,r)$ is semisimple if and only if one of the following holds:*

(1) *q is not a root of unity;*
(2) *q is a primitive lth root of unity and $r < l$;*
(3) *$m = 2, p = 0, l = 2$ and r is odd;*
(4) *$m = 2, p \geqslant 3, l = 2$ and r is odd with $r < 2p + 1$.*

Proof. If q satisfies (1) or (2), then $\mathcal{S}_F(m,r)$ is clearly semisimple. Suppose now that q is a primitive lth root of unity and $r \geqslant l > 1$. By Corollary 4.2, an argument similar to those given in the proofs of Lemma 5.3 and Theorem 5.4 shows that both $\mathcal{S}_F(m,r)1_{(r,0,\dots,0)}$, $m \geqslant 3$, and $\mathcal{S}_F(2,r)1_{(r,0)}$, $l \nmid r+1$, are indecomposable but not irreducible. In particular, both $\mathcal{S}_F(2,l)$ and $\mathcal{S}_F(2,l+1)$ are not semisimple if $l \geqslant 3$. Since tensoring an $\mathcal{S}_F(2,r)$-module with the determinant representation gives an $\mathcal{S}_F(2,r+2)$-module, we see that $\mathcal{S}_F(2,r)$ is not semisimple for all $r \geqslant l \geqslant 3$. Hence, a semisimple $\mathcal{S}_F(m,r)$ forces $m = 2, l = 2$ and $2|r+1$. It remains to determine the semisimplicity of $\mathcal{S}_F(2,r)$ when $r \geqslant l = 2$ and r odd (and so $2|r+1$). We claim that, for $r \geqslant l = 2$ with r odd, $\mathcal{S}_F(2,r)$ is semisimple if and

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

only if either $p=0$ or $p\geqslant 3$ but $r<2p+1$. Indeed, $\mathcal{S}_F(2,r)$ is semisimple if and only if all q-Weyl modules $\Delta(\lambda)$, $\lambda\in\Lambda^+(2,r)$, are irreducible. For $\lambda=(\lambda_1,\lambda_2)\in\Lambda^+(2,r)$, if $x_\lambda\in\Delta(\lambda)$ is a highest weight vector, then $\Delta(\lambda)$ has a basis $x_\lambda, \mathrm{F}_1x_\lambda, \mathrm{F}_1^{(2)}x_\lambda,\dots,\mathrm{F}_1^{(\lambda_1-\lambda_2)}x_\lambda$ and, for $1\leqslant a\leqslant\lambda_1-\lambda_2$, we have

$$\mathrm{E}_1^{(a)}\mathrm{F}_1^{(a)}x_\lambda=\sum_{s=0}^{a}\mathrm{F}_1^{(a-s)}\begin{bmatrix}\lambda_1-\lambda_2;2s-2a\\ s\end{bmatrix}_\upsilon\mathrm{E}_1^{(a-s)}x_\lambda=\begin{bmatrix}\lambda_1-\lambda_2\\ a\end{bmatrix}_\upsilon x_\lambda,$$

where $\upsilon=\sqrt{q}$. Thus, the irreducibility of $\Delta(\lambda)$ is equivalent to

$$\prod_{0\leqslant a\leqslant\lambda_1-\lambda_2}\begin{bmatrix}\lambda_1-\lambda_2\\ a\end{bmatrix}_\upsilon\neq 0.$$

Since $r=\lambda_1+\lambda_2$ is odd and $l=2$, we see that $\lambda_1-\lambda_2$ is also odd and

$$\begin{bmatrix}\lambda_1-\lambda_2\\ a\end{bmatrix}_\upsilon=\binom{\frac{\lambda_1-\lambda_2-1}{2}}{a_1}\begin{bmatrix}1\\ a_0\end{bmatrix}_\upsilon,$$

where $a=2a_1+a_0$ with $a_0=0,1$. Obviously, $\begin{bmatrix}1\\ a_0\end{bmatrix}_\upsilon=1$. Thus, if $p=0$ or $p\geqslant 3$ but $r<2p+1$ then $\binom{\frac{\lambda_1-\lambda_2-1}{2}}{a_1}\neq 0$ for all $(\lambda_1,\lambda_2)\in\Lambda^+(2,r)$ and $1\leqslant a\leqslant\lambda_1-\lambda_2$. Hence, $\mathcal{S}_F(2,r)$ is semisimple in this case. Conversely, if $r\geqslant 2p+1$, choose λ so that $\lambda_1-\lambda_2=2p+1$ and $a=3$. Then

$$\begin{bmatrix}\lambda_1-\lambda_2\\ 3\end{bmatrix}_\upsilon=\binom{\frac{\lambda_1-\lambda_2-1}{2}}{1}=\binom{p}{1}=0.$$

Hence, $\Delta(\lambda)$ is not simple in this case and so $\mathcal{S}_F(2,r)$ is not semisimple.

References

[1] A. A. Beilinson, G. Lusztig and R. MacPherson, *A geometric setting for the quantum deformation of GL_n*, Duke Math. J. **61** (1990), 655–677.

[2] H. Bao, J. Kujawa, Y. Li and W. Wang, *Geometric Schur duality of classical type*, Transf. Groups, to appear.

[3] X. Chen, H. Gu and J. Wang, *Infinitesimal and little q-Schur superalgebras*, Comm. Algebra, to appear.

[4] B. Deng, J. Du and Q. Fu, *A Double Hall Algebra Approach to Affine Quantum Schur–Weyl Theory*, LMS Lecture Note Series **401**, Cambridge University Press, 2012.

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

[5] B. Deng, J. Du, B. Parshall and J. P. Wang, *Finite Dimensional Alegebras and Quantum Groups*, Mathematical Surveys and Monographs **150**, American Mathematical Society, Providence, RI, 2008.
[6] S. Doty and D. Nakano, *Semisimple Schur algebras*, Math. Proc. Cambridge Philos. Soc. **124** (1998), 15–20.
[7] S. Doty, D. Nakano and K. Peters, *Infinitesimal Schur algebras*, Proc. Lond. Math. Soc. **72** (1996), 588–612.
[8] J. Du, *The modular representation theory of q-Schur algebras*, Trans. Amer. Math. Soc. **329** (1992), 253–271.
[9] J. Du and Q. Fu, *Quantum affine $\mathfrak{gl}_n$ via Hecke algebras*, Adv. Math. **282** (2015), 23–46.
[10] J. Du, Q. Fu and J. Wang, *Infinitesimal quantum $\mathfrak{gl}_n$ and little q-Schur algebras*, J. Algebra **287** (2005), 199–233.
[11] J. Du, Q. Fu and J. Wang, *Representations of little q-Schur algebras*, Pacific J. Math. **257** (2012), 343–378.
[12] J. Du and H. Gu, *A realization of the quantum supergroup $\mathbf{U}(\mathfrak{gl}_{m|n})$*, J. Algebra **404** (2014), 60–99.
[13] J. Du and H. Gu, *Canonical bases for the quantum supergroup $\mathbf{U}(\mathfrak{gl}_{m|n})$*, Math. Z. **281** (2015), 631–660.
[14] J. Du, H. Gu and J. Wang, *Irreducible representations of q-Schur superalgebra at a root of unity*, J. Pure Appl. Algebra **218** (2014), 2012–2059.
[15] J. Du, H. Gu and J. Wang, *Representations of q-Schur superalgebras in positive characteristics*, J. Algebra **481** (2017), 393–419.
[16] J. Du and H. Rui, *Quantum Schur superalgebras and Kazhdan–Lusztig combinatorics*, J. Pure Appl. Algebra **215** (2011), 2715–2737.
[17] H. El Turkey and J. Kujawa, *Presenting Schur superalgebras*, Pacific J. Math. **262** (2013), 285–316.
[18] K. Erdmann and D. K. Nakano, *Representaiton type of q-Schur algebras*, Trans. Amer. Math. Soc. **353** (2001), 4729–4756.
[19] Z. Fan, C. Lai, Y. Li, L. Luo and W. Wang, *Affine flag varieties and quantum symmetric pairs*, Mem. Amer. Math. Soc., to appear.
[20] Z. Fan and Y. Li, *Geometric Schur duality of classical type, II*, Trans. Amer. Math. Soc., Ser. B **2** (2015), 51–92.
[21] Q. Fu, *Semisimple infinitesimal q-Schur algebras*, Arch. Math. **90** (2008), 295–303.
[22] J. A. Green, *Polynomial Representations of GL_n*, Second edition, Lecture Notes in Mathematics **830**, Springer, Berlin, 2007, with an appendix on Schensted correspondence and Littelmann paths by K. Erdmann, J. Green and M. Schocker.
[23] D. J. Hemmer, J. Kujawa and D. Nakano, *Representation type of Schur algebras*, J. Group Theory **9** (2006), 283–306.
[24] L. Jones, *Centers of generic Hecke algebras*, Trans. Amer. Math. Soc. **317** (1990), 361–392.
[25] F. Marko and A. N. Zubkov, *Schur superalgebras in characteristic p*, Algebr. Represent. Theory **9** (2006), 1–12.
[26] F. Marko and A. N. Zubkov, *Schur superalgebras in characteristic p, II*, Bull. Lond. Math. Soc. **38** (2006), 99–112.

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press

Jie Du
School of Mathematics and Statistics
University of New South Wales
Sydney
NSW 2052
Australia
j.du@unsw.edu.au

Haixia Gu
School of Science
Huzhou University
Huzhou 313000
China
ghx@zjhu.edu.cn

Zhongguo Zhou
College of Science
Hohai University
Nanjing 210098
China
zhgzhou@hhu.edu.cn

https://doi.org/10.1017/nmj.2018.12 Published online by Cambridge University Press