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Systems of Circles analogous to Tucker Circles.*

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I.

SYSTEMS OF SIX-POINT CIRCLES CONNECTED WITH THE TRIANGLE.

GENERAL THEOREMS.†

1. If a circle meets the sides BC, CA, AB of a triangle (Fig. 6) in the pairs of points L and l, M and m, N and n respectively, it is obvious that the pairs of connectors Mn and mN, Nl and nL, Lm and lM are antiparallel with respect to the angles A, B, C respectively.

Conversely, by a modification of a theorem of Poncelet's, if the above-mentioned pairs are antiparallel with respect to A, B, C respectively, the six points L, l, M, m, N, n lie on a circle; for if not, M, n, N, m and N, l, L, n and L, m, M, l would lie on three distinct circles, and consequently BC, CA, AB, their radical axes, would be concurrent.

* Mr Third's Papers were received on 29th March. Sections I. and II. of the First Paper, in a different form, were read at the November and December Meetings.

† Since writing this paper I have found that a considerable number of the theorems contained in Part I. are given in two English memoirs, which I had somehow overlooked, viz., "The Relations of the Intersections of a Circle with a Triangle," by H. M. Taylor, *Proc. of the London Math. Soc.*, Vol. XV., pp. 122-139, in which the treatment is mainly analytical, and "Some Geometrical Proofs of Theorems connected with the Inscription of a Triangle of constant Form in a given Triangle," by M. Jenkins, *Quarterly Journal*, Vol. XXI., pp. 84-89. Among the theorems in Part I. which seem to be new, the most important, perhaps, are those referring to the connection between the S-point and the angles of a system, and to the relation of the systems to coaxal systems. The special cases of Arts. 19 and 22, Part II., are also referred to in the above-mentioned papers, but in the former of these cases the S-point is not determined.

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2. It follows at once from Pascal's theorem that in the figure of the preceding article (Fig. 6) the following are triads of collinear points

A, and the points of intersection of the pairs Lm, Nl and lM, nL;
B, " " " " " " " " Mn, Lm and mN, lM;
C, " " " " " " " " Nl, Mn and nL, mN;

and also that the three lines on which these triads of points lie are concurrent. Let the point of concurrence be denoted by S.

It is obvious that by joining the six points in which the circle meets the sides of the triangle in all possible ways by chords passing from side to side, we obtain, in general, three other points of concurrence similar to S.

3. If three lines be drawn parallel to Mn, Nl, Lm (Fig. 6) meeting CA, AB in M$'$, n'; AB, BC in N$'$, l'; BC, CA in L$'$, m' respectively, and intersecting two and two on SC, SA, SB, the six points L$'$, l', M$'$, m', N$'$, n' also lie on a circle.

Demonstration. Let Lm, Nl intersect in P, and L$'m'$, N$'l'$ in P$'$. Then $\text{AN}/\text{AN}' = \text{AP}/\text{AP}' = \text{A}m/\text{A}m'$. Therefore mN, m'N$'$ are parallel. Therefore M$'n'$, m'N$'$ are antiparallel with respect to the angle A. Similarly it may be shown that N$'l'$, n'L$'$ are antiparallel with respect to the angle B, and that L$'m'$, l'M$'$ are antiparallel with respect to the angle C. Therefore, by (1), L$'$, l', M$'$, m', N$'$, n' are concyclic.

By drawing a system of parallels to Mn, Nl, Lm as above, we obtain a system of six-point circles of which the original circle is one. Such a system may be called, in the meantime, an S-point system. The chords Mn, Nl, Lm of the circle which is regarded as the original circle of the system may be called the directive chords of the system.

It is obvious that the same S-point system would be obtained by drawing parallels (intersecting two and two on SC, SA, SB) to mN, nL, lM, *i.e.*, by drawing antiparallels to Mn, Nl, Lm with respect to the angles A, B, C respectively. In other words mN, nL, lM may equally well with Mn, Nl, Lm be regarded as directive chords of the system.

The following particular cases may be noticed.

(i) Two circles of the system are, in general, obtained by drawing (1) parallels and (2) antiparallels to the directive chords, so as to cointersect in S.

(ii) Two circles of the system are, in general, determined by drawing through any point P′ on SA (or SB or SC) (1) parallels and (2) antiparallels to the two directive chords which meet on that line; but in the case where P′ also lies on BC (or CA or AB), these two circles coalesce and touch BC (or CA or AB) at P′.

(iii) If from A, lines AX, AX′ be drawn parallel to the directive chords which intersect on SA, to meet the opposite side BC, and if from B and C analogous pairs of lines BY, BY′ and CZ, CZ′ be drawn to meet the opposite sides, the circumcircles of the triangles AXX′, BYY′ and CZZ′ belong to the system.

It is obvious that if we take in succession, as S-point, each of the three other points of concurrence referred to in Art. 2 as being obtainable in connection with the original circle, we obtain three other S-point systems. Thus, in general, every six-point circle connected with a triangle, belongs to four different S-point systems. It will be noticed that while each circle of a system has, in general, four S-points, the circles of the system have one and only one S-point in common.

4. If x, y, z be the trilinear coordinates of S, and α, β, γ respectively denote the angles AnM, BlN, CmL which the directive chords make with the sides of ABC, (Fig. 6) we have the following relations:

$$\begin{aligned}\text{(i)}\quad & x/\operatorname{cosec}\alpha\operatorname{cosec}(A+\alpha)\\ &= y/\operatorname{cosec}\beta\operatorname{cosec}(B+\beta)\\ &= z/\operatorname{cosec}\gamma\operatorname{cosec}(C+\gamma);\\ \text{and}\quad \text{(ii)}\quad & \alpha+\beta+\gamma=\pi.\end{aligned}$$

Demonstration. Let $M''n''$, $N''l''$, $L''m''$ be drawn through S parallel to Mn, Nl, Lm respectively, and meeting the sides of ABC as in the figure. Then, by Art. 3 (i), L'', l'', M'', m'', N'', n'' lie on a circle of the system. Therefore

$$M''S \,.\, Sn'' = N''S \,.\, Sl'' = L''S \,.\, Sm''.$$

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But $\quad M''S = y\operatorname{cosec} A\widehat{M''}S = y\operatorname{cosec}(A+\alpha)$,

and $\quad Sn'' = z\operatorname{cosec} A\widehat{n''}S = z\operatorname{cosec}\alpha$.

Therefore $\quad M''S \, . \, Sn'' = yz\operatorname{cosec}\alpha\operatorname{cosec}(A+\alpha)$.

Similarly it may be shown that

$$N''S \, . \, Sl'' = zx\operatorname{cosec}\beta\operatorname{cosec}(B+\beta),$$

and $\quad L''S \, . \, Sm'' = xy\operatorname{cosec}\gamma\operatorname{cosec}(C+\gamma)$.

Hence the first part of the proposition is established.

Again $\quad B\widehat{n}L = B\widehat{l}N = \beta$,

and $\quad M\widehat{n}L = C\widehat{m}L = \gamma$.

Therefore, the signs being suitably selected,

$$\alpha + \beta + \gamma = \pi.$$

The angles α, β, γ (or $\pi - A - \alpha$, $\pi - B - \beta$, $\pi - C - \gamma$) which the directive chords of an S-point system (or their antiparallels) make with the sides of ABC, may be called the angles of the system. The theorem established in the present article enables us to state that the angles of an S-point system and the position of S implicate each other, and that if either of these data be given, the system is determinate. The theorem is obviously calculated to be of great service in the investigation of particular cases of S-point systems connected with the triangle. The discussion of these, however, is reserved for Part II.

It may be noted that $LL'/L'L'' = ll'/l'l'' = MM'/M'M'' =$ etc., from which it could readily be proved that the centres of the circles of an S-point system are collinear; but this will be established by a shorter method in Art. 8.

5. If α, β, γ be the angles of an S-point system of circles connected with the triangle ABC, and if on the sides of ABC (Fig. 7) there be described externally three similar triangles aBC, AbC, ABc, such that

$$C\widehat{a}B = C\widehat{A}b = c\widehat{A}B = \alpha,$$

$$a\widehat{B}C = A\widehat{b}C = A\widehat{B}c = \beta,$$

and $\quad B\widehat{C}a = b\widehat{C}A = B\widehat{c}A = \gamma$,

then Aa, Bb, Cc cointersect in a point,* which I shall denote by Π,

* This construction is given by Mr Jenkins in the paper already cited.

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whose trilinear coordinates are

$$\sin\alpha\,\mathrm{cosec}(A+\alpha),\ \sin\beta\,\mathrm{cosec}(B+\beta),\ \sin\gamma\,\mathrm{cosec}(C+\gamma).$$

Also if other three similar triangles $a'BC$, $Ab'C$, ABc' be described externally on the sides of ABC, such that

$$C\widehat{a'}B = C\widehat{A}b' = c'\widehat{A}B = \pi - (A+\alpha),$$
$$a'\widehat{B}C = A\widehat{b'}C = A\widehat{B}c' = \pi - (B+\beta),$$

and
$$B\widehat{C}a' = b'\widehat{C}A = B\widehat{c'}A = \pi - (C+\gamma),$$

then Aa', Bb', Cc' cointersect in a point, which I shall denote by Π', whose trilinear coordinates are

$$\mathrm{cosec}\,\alpha\sin(A+\alpha),\ \mathrm{cosec}\,\beta\sin(B+\beta),\ \mathrm{cosec}\,\gamma\sin(C+\gamma).$$

The points Π and Π' are obviously isogonal conjugates with respect to the triangle ABC.

They will be referred to subsequently as the Π-points of the S-point system of circles whose angles are α, β, γ. It will be found that the Π-points of an S-point system play an analogous part to that played by the Brocard points, which are a particular case of them, in the theory of the Tucker circles.

Other methods of constructing the Π-points of a system might easily be deduced from the following considerations.

$$\Pi aBC,\ \Pi AbC,\ \Pi ABc,\ \Pi'a'BC,\ \Pi'Ab'C,\ \Pi'ABc'$$

are evidently quartets of concyclic points.

Hence $B\widehat{\Pi}C = \pi - \alpha$, etc., $\Pi\widehat{B}C + \Pi\widehat{C}B = \alpha$, etc.,

$$B\widehat{\Pi'}C = A + \alpha, \text{ etc., } \Pi'\widehat{B}C + \Pi'\widehat{C}B = \pi - (A+\alpha), \text{ etc.}$$

6. The pedal circle of the Π-points of an S-point system of circles connected with the triangle ABC, is one of the circles of the system.

Demonstration. Let l, m, n (Fig 8) be the feet of the perpendiculars from Π on BC, CA, AB respectively, and L, M, N the feet of the corresponding perpendiculars from Π'. Then, by a well-known theorem of Steiner's, since Π and Π' are isogonal conjugates L, l, M, m, N, n, lie on a circle. Hence, making the necessary joins and noticing that $BL\Pi'N$ and $CM\Pi'L$ are cyclic quadrilateral we have

$$A\widehat{n}M = N\widehat{L}M = \Pi'\widehat{B}A + \Pi'\widehat{C}A = \Pi\widehat{B}C + \Pi\widehat{C}B = \alpha.$$

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Similarly it may be shown that $B\widehat{l}N = \beta$, and $C\widehat{m}L = \gamma$. Therefore the circle $LlMmNn$ belongs to the S-point system whose angles are α, β, γ, *i.e.*, to the system whose Π-points are Π and Π′.

It will be noticed that the centre of the circle $LlMmNn$ is the mid-point of ΠΠ′.

It is evident that an S-point system connected with the triangle is completely determined, if its Π-points are given.

7. If the pedal triangles of Π and Π′ be rotated round Π and Π′ respectively, in opposite directions through equal angles, so as to remain inscribed in ABC and retain their species, they have a common circumcircle which belongs to the S-point system of which Π and Π′ are the Π-points.

Demonstration. Let Πl, Πm, Πn (Fig. 8) be rotated round Π in, say, the positive direction through an angle ϕ, and in their new positions let them meet the sides of ABC in l', m', n'. Similarly let $\Pi'L$, $\Pi'M$, $\Pi'N$ be rotated round Π′ in the negative direction through an equal angle ϕ, meeting in their new positions the sides of ABC in L', M', N'. Then, as might easily be proved, the triangles lmn and LMN remain constant in species during the above operation; and $L'l'M'm'$, $M'm'N'n'$, $N'n'L'l'$ are cyclic quartets, whence by Art. 1, L', l', M', m', N', n' are concyclic.

Again, making the necessary joins, and denoting the angles ΠAB, Π′AC by θ, we have

$$MM' = \Pi'M \tan\phi = AM \tan\theta \tan\phi,$$

and

$$nn' = \Pi n \tan\phi = An \tan\theta \tan\phi.$$

Therefore $M'n'$ is parallel to Mn. Hence $A\widehat{n'}M' = \alpha$.
Similarly it may be shown that $B\widehat{l'}N' = \beta$, and $C\widehat{m'}L' = \gamma$. Therefore the circle $L'l'M'm'N'n'$ belongs to the S-point system whose angles are α, β, γ, *i.e.* to the system whose Π-points are Π and Π′.

It will be noticed that

$$\widehat{nlm} = \Pi\widehat{BA} + \Pi\widehat{CA} = \Pi'\widehat{BC} + \Pi'\widehat{CB} = \pi - (A + \alpha), \text{ etc.},$$

and

$$\widehat{NLM} = \Pi'\widehat{BA} + \Pi'\widehat{CA} = \Pi\widehat{BC} + \Pi\widehat{CB} = \alpha, \text{ etc.}$$

Hence the angles of the variable triangle $(l'm'n')$ associated with Π, are equal to the supplements of $A + \alpha$, $B + \beta$, $C + \gamma$, while those of the variable triangle $(L'M'N')$ associated with Π′, are equal to α, β, γ.

An interesting particular case not amongst those discussed in Part II. may be referred to here. When Π is on the circumcircle of ABC, Π', its isogonal conjugate, is at infinity, and the pedal triangle of Π degenerates into the Simson line of Π. In this case the circles of the S-point system degenerate into a system of line-pairs, one line of each pair being the line at infinity and the other lines the Simson lines of Π (the lines joining the feet of isoclinals from Π to the sides of ABC being included in this designation).

8. The locus of the centres of an S-point system of circles is the straight line which bisects the connector of the Π-points perpendicularly.

Demonstration. Let K, K′ (Fig. 8) be the centres of the circles $LlMmNn$ and $L'l'M'm'N'n'$ respectively, and join KK′, K′Π, K′Π'. Then K is the mid-point of $\Pi\Pi'$, and the angles $K\Pi K'$, $K\Pi' K'$ are obviously each equal to the angle of rotation ϕ. Therefore KK′ bisects $\Pi\Pi'$ perpendicularly.

When Lm, $L'm'$ etc., are equally inclined to BC, CA and Nl, $N'l'$, etc., to AB, BC, it is obvious that the perpendicular bisectors of Lm, $L'm'$, etc., coalesce, as also those of Nl, $N'l'$, etc.; so that in this case the line of centres reduces to a point.

9. The radii of the circles of an S-point system bear a constant ratio to the distances of their centres from either of the Π-points of the system.

Demonstration. If Kn, $K'n'$ (Fig. 8) be joined, it is obvious that the triangles ΠKn, $\Pi K'n'$ are similar. Hence $K'n'/Kn = K'\Pi/K\Pi$, which proves the proposition.

From this property it is evident that if we construct a system of coaxal circles having Π and Π' as common points, the circles of the S-point system are simply those of the coaxal system with their radii increased or diminished in a constant ratio, the centres remaining fixed. I propose to call a system of circles arrived at in this manner, by the elongation or shortening in a constant ratio of the radii of a coaxal system whether of the common-point or limiting-point species, a coaxaloid system of circles. The ratio in which the radii of the kernel coaxal system are increased or diminished will be referred to as the ratio of the resulting coaxaloid system. It is evident that coaxaloid systems of circles could be studied altogether apart from their connection with the triangle, and in fact many of their properties can be so obtained, of which the theorems of Arts. 14 to 17 are examples. It is evident also that while every S-point system as previously defined, is a coaxaloid system, it does not follow that every coaxaloid system is an S-point system of a triangle with which its Π-points are connected as isogonal conjugates. In fact for every pair of real Π-points, connected with a triangle as isogonal conjugates, we have an infinite number of coaxaloid systems with different ratios, but only one of these systems with a definite ratio depending on the position of the Π-points, is a system of six-point circles of the triangle, possessing the property of parallel chords. When the Π-points are imaginary, *i.e.* when the coaxaloid systems are derived from a limiting-point instead of a common-point coaxal system, none of the coaxaloid systems are S-point systems. In future an S-point system of circles will be called a coaxaloid system connected with the triangle.

10. Π and Π′ being the Π-points of a coaxaloid system of circles connected with the triangle, we will now consider the connection between the magnitude of the ratio of the system, denoted by r, and the position of Π and Π′. Let d denote the diameter of the pedal circle of Π and Π′. Then $d/\Pi\Pi' = r$.

(i) When Π and Π′ both lie within the triangle, $d > \Pi\Pi'$ and $r > 1$.

(ii) When Π and Π′ lie on the perimeter of the triangle, in which case one of them coincides with a vertex and the other lies on the opposite side, $d = \Pi\Pi'$, and $r = 1$. In this case the coaxaloid system connected with the triangle is an ordinary common-point coaxal system.

(iii) When one of the Π-points lies outside the triangle but within the circumcircle, in which case the other lies outside the circumcircle, $d < \Pi\Pi'$, and $r < 1$.

(iv) When one of the Π-points lies on the circumcircle, in which case the other is at infinity (see Art. 8), $d = \Pi\Pi' = \infty$, and $r = 1$.

(v) When Π and Π′ both lie outside the circumcircle, $d > \Pi\Pi'$ and $r > 1$.

(vi) When Π and Π' coincide, as they do in the incentre and excentres, $\Pi\Pi' = 0$ and $r = \infty$.

(vii) If Kl (Fig. 8) be joined it may easily be proved that $4Kl^2 = \Pi\Pi'^2 + 4\Pi l . \Pi' L$. But Kl is a radius of the pedal circle of Π and Π'. Hence $r^2 = 1 + 4\Pi l . \Pi' L / \Pi\Pi'^2$. This result supplies us with a ready means of calculating the ratio in certain cases.

11. The envelope of a coaxaloid system of circles connected with a triangle ABC is the conic inscribed in (or escribed to) the triangle, and having the Π-points of the system as foci.

Demonstration. Let K be the mid-point of $\Pi\Pi'$ (Fig. 9), and TT′ the diameter through Π and Π' of the pedal circle of Π and Π'. Let K′ be the centre of any other circle of the system, and let this circle be cut by the circumcircle of $\Pi\Pi' K'$ at J. Join $J\Pi$, $J\Pi'$, JK′, $\Pi K'$ and $\Pi' K'$.

(i) Suppose $r > 1$, and consequently $KT > K\Pi$. Then by Ptolemy's theorem

$$J\Pi . K'\Pi' + J\Pi' . K'\Pi = \Pi\Pi' . K'J.$$

But $K'\Pi' = K'\Pi$.

Therefore
$$\begin{aligned} J\Pi + J\Pi' &= \Pi\Pi' . K'J / K'\Pi \\ &= \Pi\Pi' . KT / K\Pi = TT'. \end{aligned}$$

Therefore J lies on an ellipse having Π and Π' as foci, and the pedal circle of Π and Π' as auxiliary circle.

Again, since $K'\Pi' = K'\Pi$, JK′ bisects the angle $\Pi J \Pi'$, and is, consequently, the normal to the ellipse at J. Therefore the circle K′ touches the ellipse at J. Similarly it might be proved to touch the ellipse at the image of J with respect to the line of centres of the coaxaloid system.

Again, since the feet of the perpendiculars from Π and Π' on the sides of ABC, lie on the pedal circle of Π and Π', *i.e.* on the auxiliary circle of the ellipse, the sides of ABC are tangents to the ellipse.

It may be noted that when Π and Π' coincide, the ellipse becomes the incircle or one of the excircles.

(ii) When $r < 1$, it may be proved in precisely the same way that the envelope is a hyperbola touching the sides of ABC and having Π, Π' as foci and their pedal circle as auxiliary circle.

(iii) In the particular case when one of the Π-points lies on the circumcircle, the coaxaloid system degenerates, as we have seen, into the Simson lines of the point, and the other Π-point lies at infinity. In this case, therefore, the envelope becomes a parabola. We thus obtain the well-known theorem that the envelope of the Simson lines of a point is a parabola escribed to the triangle and having the point as focus.

It is obvious that the foregoing proof establishes the more general theorem that the envelope of a coaxaloid system of circles, without restriction as to connection with the triangle, is a conic having the Π-points of the system as foci, and the minimum circle of the system as auxiliary circle.

12. The envelope of a coaxaloid system of circles connected with the triangle, touches the sides of the triangle at the points where these are met by the lines joining the S-point to the opposite vertices.

Demonstration. Let ABC (Fig. 10) be the triangle, and S any point in its plane. Let AS, BS, CS meet BC, CA, AB in D, E, F respectively. Then by Art 3 (ii) one of the circles of the coaxaloid system connected with the triangle and having S as S-point, touches BC at D. This is sufficient to prove that the envelope of the system also touches BC at D. Similarly it might be shown that it touches CA and AB at E and F respectively.

We are now in a position to find a geometrical solution of the following problem: Given the S-point of a coaxaloid system of circles connected with the triangle ABC, to find the Π-points, and thus determine the system generally.

Solution. Let X, Y, Z (Fig. 10) be the mid-points of the connectors EF, FD, DE respectively. Then since ABC is circumscribed to the envelope of the system and DEF is inscribed in it, the connectors AX, BY, CZ meet in the centre of the envelope. Let this point be denoted by K. It may be noted that S and K coincide when S is the centroid of ABC. If now two points J and J′ be taken on AK, on opposite sides of K, such that $KJ^2 = KJ'^2 = KX \cdot KA$, JJ′ is obviously the diameter of the envelope conjugate to the chord EF. Being given a chord and the conjugate diameter, we can find the foci of the envelope by any one of a number of well-known constructions, *i.e.*, we can find the Π-points of the system. Then by drawing perpendiculars Πl, Πm, Πn, $\Pi'L$, $\Pi'M$, $\Pi'N$ from these points to the sides of ABC as in Fig. 8, we determine one circle of the system, and by drawing parallels (or antiparallels) to the connectors Mn, Nl, Lm so as to intersect two and two on CS, AS, BS we determine the other circles of the system.

The following particular case of the problem just solved may be noticed: Given a point S in the plane of a triangle, to draw through S three lines terminated by the sides of the triangle, two and two, such that their extremities shall lie on a circle.

Combining the theorem of the present article with the proposition that every six-point circle connected with the triangle belongs, in general, to four coaxaloid systems connected with the triangle, we obtain the following interesting result. Every six-point circle connected with the triangle has, in general, double contact (real or imaginary) with four conics which touch the sides of the triangle at the points, taken three and three, where the connectors of the vertices with the four S-points meet the opposite sides.

13. Every coaxaloid system of circles having real Π-points, is an S-point system of an infinite number of triangles, viz., of all the triangles circumscribed to the envelope of the system.

Demonstration. Let Π and Π′ be the Π-points of a coaxaloid system, and let any triangle be circumscribed to its envelope. The minimum circle of the system is the auxiliary circle of the envelope, and therefore, by a well-known property of tangents to a conic, is the pedal circle of Π and Π′ with respect to the triangle. Again, by another well-known property of tangents to a conic, Π and Π′ are isogonal conjugates with respect to the triangle, and hence are the Π-points of an S-point system connected with the triangle. Of this system the pedal circle of Π and Π′ with respect to the triangle is one of the circles. Hence the ratio of both systems—the coaxaloid and the S-point system—is the diameter of the pedal circle of Π and Π′ over the distance ΠΠ′. Therefore, since both systems have the same Π-points and the same ratio, they are identical, which proves the proposition.

The theorem is important as affording a starting-point for the discussion of coaxaloid systems of circles connected with the quadrilateral and general polygon.

14. The tangent to the minimum circle of a coaxaloid system (with real Π-points) at either extremity of the diameter through Π and Π′ meets every other circle of the system in two points which connect with the centre of the latter through Π and Π′.

Demonstration. Let K (Fig. 9) be the mid-point of ΠΠ′, and, therefore, the centre of the minimum circle, and K′ the centre of any other circle of the system. Let the tangent to the minimum circle at T, one of the extremities of the diameter through Π and Π′, meet the connectors ΠK′, Π′K′ in U and U′ respectively. Then by *Euc.* VI., 2, $K'U/K'\Pi = KT/K\Pi$. Therefore, by Art. 10, U lies on the circle K′. Similarly $K'U'/K'\Pi' = KT/K\Pi'$. Therefore U′ also lies on the circle K′.

15. The circles of the coaxal system from which a coaxaloid system is derived, are the circles of similitude of the circles of the coaxaloid system taken in pairs.

Demonstration. Let K′, K″ (Fig. 9) be the centres of two circles of the coaxaloid system, E their external and I their internal centre of similitude. Then $EK'/EK'' = K'I/IK''$

$$= \text{radius of circle } K'/\text{radius of circle } K''$$
$$= \Pi K'/\Pi K''.$$

Therefore EI subtends a right angle at Π, which proves the proposition.

In the proof Π and Π′ are real points, but the same result holds good when they are imaginary.

16. The locus of the extremities of a system of parallel diamaters of a coaxaloid system of circles is a hyperbola.

Demonstration. Through K, the mid-point of ΠΠ′, (Fig. 9) draw any line KV, and from K′, the centre of any circle of the coaxaloid system, draw K′W a radius, parallel to KV. Then if x and y are the coordinates of W with respect to the axes KV, KK′, we have

$$\begin{aligned} x^2 = K'W^2 &= K'\Pi^2 . KT^2/K\Pi^2 \\ &= (K'K^2 + K\,\Pi^2)KT^2/K\Pi^2 \\ &= (y^2 + K\Pi^2)KT^2/K\Pi^2. \end{aligned}$$

Hence the equation of the locus of W is $x^2/KT^2 - y^2/K\Pi^2 = 1$, which proves the proposition

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By varying the direction of the parallel diameters we obtain a system of hyperbolas associated with the coaxaloid system. These are obviously concentric, with K as centre, and have a common diameter equal to ΠΠ′ coinciding with the line of centres of the coaxaloid system. Obviously also their diameters conjugate to this common diameter are diameters of the minimum circle of the coaxaloid system; hence the tangents to the hyperbolas at the points where the latter are cut by the minimum circle are perpendicular to ΠΠ′. When the coaxaloid system is coaxal, $KT = K\Pi$, and hence the associated hyperbolas are all equilateral. When the diameters of the circles are drawn parallel to ΠΠ′, the hyperbola on which their extremities lie has TT′, the diameter through Π and Π′ of the minimum circle, as transverse axis, and the common diameter of all the hyperbolas, as conjugate axis. In this case the minimum circle of the coaxaloid system is the auxiliary circle of the hyperbola.

The theorem has been proved for the case when Π and Π′ are real, but might easily be extended to the case when they are imaginary.* In the former case the extremities of the parallel diameters of the circles lie on different branches of the hyperbolas, and the common diameter of the latter is ideal; in the latter case the extremities of the parallel diameters of the circles lie on the same branches of the hyperbolas and the common diameter of the latter is real.

The converse theorem may be stated: Circles described on parallel chords of a hyperbola as diameters form a coaxaloid system, with real Π-points if the extremities of the chords lie on different branches of the curve, with imaginary Π-points if they lie on the same branch. In the particular case when the hyperbola is a line-pair, the coaxaloid circles reduce to circles having a common centre of similitude, and the other associated hyperbolas are also line-pairs.

17. The hyperbolas associated with a coaxaloid system have the same envelope as the system.

Demonstration. If KV, KK′ (Fig. 9) be taken as axes, the equation of the hyperbola obtained by drawing diameters of the system parallel to KV is $x^2/a^2 - y^2/c^2 = 1$, where $a = KT$, and $c = K\Pi$.

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* See note at end of paper.

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Transforming to the axes KT, KK′, we have

$$x^2\sec^2\phi/a^2 - (y - x\tan\phi)^2/c^2 = 1,$$

where ϕ denotes the angle VKT′. Rewriting we have

$$x^2(a^2 - c^2)\tan^2\phi - 2a^2xy\tan\phi - c^2x^2 + a^2y^2 + a^2c^2 = 0.$$

Hence the equation of the envelope of the hyperbolas is

$$(a^2 - c^2)(a^2y^2 - c^2x^2 + a^2c^2) = a^4y^2,$$

$$i.e. \quad x^2/a^2 + y^2/(a^2 - c^2) = 1,$$

which represents the conic having Π, Π′ as foci, and TT′ as transverse axis.

II.

Particular Cases of Coaxaloid Systems of Circles connected with the Triangle.

Some of the more obvious particular cases of coaxaloid systems connected with the triangle ABC will now be noticed.

18. *The Tucker Circles.* When the angles α, β, γ (Art. 4) equal B, C, A (or C, A, B) respectively, the trilinear coordinates of S are

$$\sin A, \quad \sin B, \quad \sin C;$$

i.e., S is the symmedian point.

By Art. 5, the trilinear coordinates of the Π-points are

$$c/b, \quad a/c, \quad b/a \quad \text{and} \quad b/c, \quad c/a, \quad a/b$$

i.e., the Π-points are the Brocard points.

This is the well-known case of the Tucker circles.

The ratio of this system may be calculated in terms of the angles of ABC as follows: If Ω, Ω' are the Brocard points, and ΩL, Ω'L′ be drawn perpendicular to BC, we have, by Art. 11 (vii)

$$r^2 = 1 + 4\Omega L \,.\, \Omega' L'/\Omega\Omega'^2.$$

Hence, using the well-known results,

$$\Omega L \,.\, \Omega' L' = 4R^2\sin^4\omega,$$

$$\Omega\Omega'^2 = 4R^2\sin^2\omega(1 - 4\sin^2\omega),$$

and $$\operatorname{cosec}^2\omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C,$$

where R is the radius of the circumcircle, and ω the Brocard angle, we have

$$r^2 = (1 - 3\sin^2\omega)/(1 - 4\sin^2\omega)$$

$$= \Sigma\cot^2 A/(\Sigma\cot^2 A - 1).$$

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19. *A System derived from the Nine-Point Circle.*

If A′, B′, C′ are the mid-points of BC, CA, AB, and D, E, F the feet of the altitudes on these sides respectively, then the connectors FB′, DC′, EA′ are the directive chords of one of the four systems derivable from the nine-point circle.

Since $\qquad A\widehat{FB}' = A,\ B\widehat{DC}' = B$, and $C\widehat{EA}' = C$,

the angles of the system are A, B, C (or $\pi - 2A$, $\pi - 2B$, $\pi - 2C$). Hence, by Art. 4, the trilinear coordinates of S are

$$\sec A/a^2,\ \sec B/b^2,\ \sec C/c^2;$$

i.e., S is the isotomic conjugate of the circumcentre. Thus the other circles of the system are obtained by drawing transversals Mn, Nl, Lm, terminated by the sides as in Fig. 6, so as to intersect two and two on CS, AS, BS, where S is the isotomic conjugate of the circumcentre, and so as to make with the sides of ABC three isosceles triangles whose vertices are M, N, L (or m, n, l) respectively.

By Art. 5 the trilinear coordinates of the Π-points are

$$\cos A,\ \cos B,\ \cos C \text{ and } \sec A,\ \sec B,\ \sec C;$$

i.e., the Π-points are the circumcentre, O, and the orthocentre, H. Thus the line of centres of the system is the perpendicular bisector of OH, and the nine-point circle, being the pedal circle of O and H, is the minimum circle of the system.

By Art. 10 (viii), and the well-known relations,

$$OA' = R\cos A,\ HD = 2R\cos B\cos C,$$

and $\qquad OH^2 = R^2(1 - 8\cos A\cos B\cos C)$,

we have $\qquad r^2 = 1/(1 - 8\cos A\cos B\cos C)$.

It may be noticed that the following theorem is involved in the foregoing: The connector of A with the isotomic conjugate of O contains the points of intersection of the pairs DC′, EA′ and FA′, DB′; with similar theorems for the vertices B and C.

20. *Other Systems derived from the Nine-Point Circle.*

Take B′F, C′A′, DE (notation of preceding article) as directive chords. Since $A\widehat{FB}' = A$, $B\widehat{A'C}' = C$, and $C\widehat{ED} = B$, the angles of the resulting system are A, C, B (or $\pi - 2A$, A, A). Hence by Art. 4 the trilinear coordinates of S are

$$\operatorname{cosec} 2A,\ \operatorname{cosec} C,\ \operatorname{cosec} B,$$

by which it at once appears that AS is the symmedian through A, and the following construction, independent of the nine-point circle, is suggested. On CA, AB describe isosceles triangles VCA, WAB having their base angles equal to A. Then BV, CW and the symmedian through A cointersect in S.

By Art. 5 the trilinear coordinates of Π are

$$\sin A \operatorname{cosec} 2A, \ \sin C \operatorname{cosec} A, \ \sin B \operatorname{cosec} A,$$

from which it is evident that Π lies on the median through A, and that, consequently, Π' lies on the corresponding symmedian, and, therefore on AS. Since $\widehat{B\Pi C} = \pi - A$ (Art. 5), and $\widehat{B\Pi' D} = 2A$, the median through A meets the circumcircle of BHC in Π, and the symmedian through A meets the circumcircle of BOC in Π'. This suggests a construction for finding Π and Π'. The line of centres, which, it may be noted, passes through the nine-point centre, may be found by drawing the perpendicular bisector of the distance between Π and Π', or it may be found by the following construction. Let K be the symmedian point of ABC, and at L where AK meets BC draw a perpendicular to BC meeting OK in T. Then the connector of T with the nine-point centre is the line of centres. This may be proved as follows. Through L draw LM, Lm to meet CA, and LN, Ln to meet AB, so that Lm shall be parallel to AB and LM antiparallel to it with respect to C, and that LN shall be parallel to CA, and Ln antiparallel to it with respect to B. Then by Art. 3 (ii) since $\widehat{BLN} = C$, and $\widehat{CML} = B$, and Ln, Lm are the antiparallels of LN, LM respectively, the points L, M, m, N, n lie on a circle of the system which touches BC at L, and whose centre, therefore lies on LT. But since AL is a symmedian, and $\widehat{BLN} = C$, and $\widehat{CmL} = A$, and Ln, LM are the antiparallels of LN and Lm respectively, this circle is also a Tucker circle, and, therefore, by a well-known theorem relating to Tucker circles, its centre lies on OK. Therefore T is the centre of the circle, and since the nine-point circle is another circle of the system, the statement is proved.

Two other analogous systems may be derived from the nine-point circle by taking

(i) EF, C′D, A′B′ and (ii) B′C′, FD, A′E

as directive chords.

The following theorem is easily deducible from the foregoing:

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The point of intersection of $C'A'$ and DE, and also the point of intersection of the antiparallel pair DF and $A'B'$ lie on the symmedian AK; with similar statements for BK and CK.

21. *The Three Rectangular Systems.*

The circumcircles of all the rectangles inscribed in ABC so as to have one of their sides lying on a side of the triangle, form a coaxaloid system connected with the triangle.

Demonstration. Let $lLmN$ (Fig. 11) be any rectangle inscribed in ABC, having lL lying on BC, m on CA, and N on AB; and let the circumcircle of the rectangle meet CA again in M, and AB again in n. Then the connector Mn is antiparallel to mN, and therefore to BC, with respect to A. Therefore the chords Mn, Nl, Lm are constant in direction, which proves the proposition.

The system thus determined may be called the A rectangular system. Two other analogous systems are obtained by taking the rectangles so as to have a side coinciding (i) with CA and (ii) with AB. These may be called the B and C rectangular systems respectively.

Since, as is well-known, the six points in which the Cosine circle meets the sides of the triangle can be joined so as to form three rectangles having each a side coincident with a side of the triangle, the Cosine circle belongs to each of the rectangular systems. Hence the three lines of centres of the system cointersect in the centre of the Cosine circle, *i.e.*, in the symmedian point.

The angles of the A system are obviously

$$C, \frac{\pi}{2}, \frac{\pi}{2} - C \left(\text{or } B, \frac{\pi}{2} - B, \frac{\pi}{2}\right).$$

Hence, by Art. 4, the trilinear coordinates of the S-point of this system are

$$\operatorname{cosec}B \operatorname{cosec}C, \ \sec B, \ \sec C,$$

by which it is evident that S lies on the altitude AD, and the following construction is suggested: From D, the foot of the altitude AD, draw perpendiculars to CA and AB meeting the parallel to BC through A in V and W respectively; then AD, BV, CW cointersect in S.

The line of centres can be readily determined without first

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finding the Π-points. The side BC and the altitude AD are degenerate forms of the rectangles which give rise to the A system, and, therefore, the line of centres of the system is the line joining the mid-points of BC and AD.

To solve the problem of finding the Π-points of the A system, let $lLmN$ (Fig. 11) be the rectangle inscribed in the minimum circle of the system. From l draw a perpendicular to CA meeting CA in M and Lm in Π′; and from L draw a perpendicular to AB meeting AB in n and Nl in Π. Then it is obvious that M and n are points on the circumcircle of $lLmN$, and that Π and Π′, being the points of concurrence of perpendiculars to the sides at the points where these are cut by the minimum circle, are the Π-points. Now $\Pi l/lL = \cot B$, and $\Pi'L/lL = \cot C$. But since Π′N and Ln are perpendicular to AB, $\Pi'L = \Pi N$.

Therefore $$Nl/lL = \cot B + \cot C = BC/AD.$$

This result determines the species of the rectangle $lLmN$, which can, consequently, be easily constructed. The Π-points are then obtained by drawing lM, Ln perpendicular to CA, AB respectively as above.

To determine the ratio of the A system we have

$$\Pi l \, . \, \Pi'L = lL^2 \cot B \cot C,$$

and $$\Pi\Pi'^2 = lL^2 + (Nl - 2\Pi'L)^2$$
$$= lL^2\{1 + (\cot B - \cot C)^2\}.$$

Hence, by Art. 11 (vii) we obtain

$$r^2 = \frac{1 + (\cot B + \cot C)^2}{1 + (\cot B - \cot C)^2}$$

The following is given as an example of the numerous theorems dealing with collinearity and concurrency, which arise in connection with these systems. If $lLMn$, $mMNl$, $nNLm$ are the three inscribed rectangles of the Cosine circle, lL, mM, nN coinciding with BC, CA, AB respectively, and AD, BE, CF the altitudes, the following are triads of concurrent lines,

AD, NL, lm; BE, LM, mn; CF, MN, nl.

This follows at once from the fact that pairs of directive chords of the rectangular systems intersect on the altitudes.

22. *The Four Concentric Systems.*

When to α, β, γ are assigned the values

$$\tfrac{1}{2}(B+C),\quad \tfrac{1}{2}(C+A),\quad \tfrac{1}{2}(A+B)$$

respectively, *i.e.*, when the directive chords Mn, Nl, Lm (Fig. 12) make with the sides of ABC three isosceles triangles whose vertical angles are the interior angles A, B, C respectively, the trilinear coordinates of S are found by Art. 5 (i) to be

$$\sec^2\frac{A}{2},\quad \sec^2\frac{B}{2},\quad \sec^2\frac{C}{2};$$

i.e., S is the point of concurrence of the lines joining the vertices of ABC to the points where the incircle touches the opposite sides (Gergonne point).

Since the directive chords in this case are isoclinal to the sides of the triangle taken in pairs, it follows from the last paragraph of Article 8 that the circles of the system are concentric.

The incircle evidently belongs to this system, for if X, Y, Z (Fig. 12) are the points where this circle touches BC, CA, AB respectively, the chords YZ, ZX, XY are obviously parallel to the directive chords of the system. It follows that the incentre is the common centre of the system.

By Art. 5, or otherwise, the Π-points of the system are found to coincide in the incentre. Hence the incircle besides being the minimum circle is the (ideal) envelope of the system.

It may be noticed also that α and $\pi-(A+\alpha)$ are in this case equal, and so for the other pairs, and that, consequently, the two variable inscribed triangles of constant species which by their rotation (round the incentre) determine the system, are similar, their angles opposite the vertices A, B, C being equal to $\tfrac{1}{2}(B+C)$, $\tfrac{1}{2}(C+A)$, $\tfrac{1}{2}(A+B)$ respectively.

By Art. 10 (vii), $\qquad r=\infty.$

The following particulars may be noted with respect to the figure of the present article (Fig. 12)

(i) mN, nL, lM are both parallel and antiparallel (with respect to A, B, C) to Mn, Nl, Lm respectively.

(ii) $Ll=Mm=Nn$, and X, Y, Z are their mid-points.

(iii) If P, Q, R are the points on AS, BS, CS respectively,

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where Mn, Nl, Lm intersect, and p, q, r the points on AS, BS, CS where mN, nL, lM intersect,

$$\widehat{RPQ} = \widehat{rpq} = \tfrac{1}{2}(B + C),$$
$$\widehat{PQR} = \widehat{pqr} = \tfrac{1}{2}(C + A),$$
$$\widehat{QRP} = \widehat{qrp} = \tfrac{1}{2}(A + B).$$

Hence Ll, Mm, Nn are antiparallel to QR, RP, PQ with respect to the angles RPQ, PQR, QRP respectively, and to qr, rp, pq with respect to the angles rpq, pqr, qrp respectively. They are also antiparallel to YZ, ZX, XY with respect to the angles ZXY, XYZ, YZX respectively.

(iv) S is the symmedian point of the triangles PQR, pqr, XYZ.

(v) The circle LlMmNn is a Tucker circle of the triangles PQR, pqr, XYZ.

(vi) Hence, if I be the incentre of ABC, J the circumcentre of PQR and J′ the circumcentre of pqr, then I, S, J, J′ are collinear, and I is the mid-point of JJ′.

By taking S as the point of concurrence of the lines joining A, B, C to the points where the excircle opposite to A touches the opposite sides, and drawing Mn, Nl, Lm so as to make with the sides of ABC three isosceles triangles whose vertical angles are the interior angle at A and the exterior angles at B and C, we obtain an analogous system concentric with the excircle and including it. An analogous system exists in connection with each of the other excircles.

23. *Systems connected with the Isogonic Centres.*

When $\alpha = \beta = \gamma = 60°$ (or $\alpha = 120° - A$, $\beta = 120° - B$, $\gamma = 120° - C$), the trilinear coordinates of S are found by Art. 4 to be

$$\operatorname{cosec}(60° + A),\ \operatorname{cosec}(60° + B),\ \operatorname{cosec}(60° + C).$$

Hence S is the point of concurrence of the lines joining the vertices of ABC to the vertices of equilateral triangles described externally on the opposite sides, *i.e.*, it is one of the isogonic centres of ABC.

Using Art. 5 we find that, in the case of this system, Π coincides with the isogonic centre. Hence the line of centres of the system is the perpendicular bisector of the distance between the isogonic centre and its isogonal conjugate.

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It follows from Art. 7 that the circles of this system are the circumcircles of a variable equilateral triangle inscribed in ABC.

When two of the angles α, β, γ are taken each equal to $120°$, and the remaining angle to $-60°$, an analogous system is obtained for which the trilinear coordinates of both S and Π are found to be

$$\operatorname{cosec}(60° - A),\ \operatorname{cosec}(60° - B),\ \operatorname{cosec}(60° - C).$$

Thus S and Π coincide in the point of concurrence of the lines joining the vertices of ABC to the vertices of equilateral triangles described internally on the opposite sides, *i.e.*, in the other isogonic centre of ABC. In this case the directive chords are inclined at angles of $60°$ to the corresponding chords in the previous case.

24. *Coaxal Systems.*

It has been remarked in Art. 10 (ii) that a coaxaloid system connected with the triangle reduces to a coaxal system, when the Π-points lie on the perimeter of the triangle. In illustration the cases which arise in connection with the circle described on a side as diameter may be referred to.

Let AD, BE, CF be the three altitudes of ABC, and consider the circle described on BC as diameter. The four systems to which this circle belongs may be obtained by taking the following sets of directive chords,

(i) CB, FC, BE, which give $\alpha = B$, $\beta = \frac{\pi}{2} - B$, $\gamma = \frac{\pi}{2}$,

(ii) CF, BC, BE, which give $\alpha = \frac{\pi}{2}$, $\beta = 0$, $\gamma = \frac{\pi}{2}$,

(iii) EB, FC, CB, which give $\alpha = \frac{\pi}{2} - A$, $\beta = \frac{\pi}{2} - B$, $\gamma = \pi - C$,

(iv) EF, BC, CB which give $\alpha = C$, $\beta = 0$, $\gamma = \pi - C$.

In the first case we have the A-rectangular system (Art. 21), and in the second, third, and fourth cases we have the coaxal systems which have BE, CF and BC, respectively, as common chords.

III.

Additional Theorems.

25. The envelope of the polars of a point with respect to the circles of a coaxaloid system is a parabola.

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Demonstration. The equation of any circle of the system referred to $\Pi\Pi'$ and KK' as axes (Fig. 9) is

$$x^2 + (y - c\tan\phi)^2 = a^2\sec^2\phi,$$

where $KT' = a$, $K\Pi' = c$, and ϕ denotes the angle subtended at Π' by the distance of the centre of the circle from K.

The equation of the polar of a point x', y' with respect to the circle is

$$(c^2 - a^2)\tan^2\phi - c(y + y')\tan\phi + xx' + yy' - a^2 = 0.$$

The equation of the envelope of this, when ϕ varies, is

$$4(c^2 - a^2)(xx' + yy' - a^2) = c^2(y + y')^2,$$

which represents a parabola.

When the system is coaxal, $c = a$, and the parabola, as is well known, reduces to a point.

26. The locus of the poles of a line with respect to the circles of a coaxaloid system is, in general, a hyperbola, but when the line is parallel to the line of centres it is a parabola.

Demonstration. Let the line meet the line of centres at O (Fig. 9) and make with it an angle a. Let $K'M$ be the perpendicular to the line from K', the centre of a circle of the system, and let P be the pole of the line with respect to the circle K'.

Let $\quad K\Pi = c$, $OK = d$, and $r =$ the ratio of the system.

OK is taken as the axis of x, and the line through O parallel to the fixed direction $K'M$ as the axis of y.

$$\text{Then } K'P.K'M = (\text{radius of circle } K')^2 = r^2.K'\Pi^2 = r^2(c^2 + K'K^2).$$
$$= r^2\left(c^2 + \overline{K'O - d}^2\right).$$

And $K'M = K'O . \sin a$.

Hence we obtain as the equation of the locus of P,

$$yx\sin a = r^2\left(c^2 + \overline{x - d}^2\right),$$

$$i.e., \quad r^2x^2 - xy\sin a - 2r^2dx + r^2(c^2 + d^2) = 0,$$

which represents a hyperbola, except when $\sin a = 0$, when it represents a parabola.

This theorem holds also when the circles are coaxal.

27. The envelope of the radical axes of any circle and the circles of a coaxaloid system is a parabola.

Demonstration. As in Art. 25 the equation of any circle of the coaxaloid system referred to ΠΠ′ and KK′ as axes (Fig. 9) is

$$x^2+(y-c\tan\phi)^2=a^2\sec^2\phi \quad - \quad - \quad - \quad (1).$$

Let the equation of a circle not belonging to the system be

$$(x-p)^2+(y-q)^2=r^2 \quad - \quad - \quad - \quad (2).$$

Then the equation of the radical axis of (1) and (2) is

$$(c^2-a^2)\tan^2\phi-2cy\tan\phi$$
$$+2px+2qy+r^2-p^2-q^2-a^2=0.$$

Therefore the equation of the envelope is

$$c^2y^2=(c^2-a^2)(2px+2qy+r^2-p^2-q^2-a^2),$$

which represents a parabola.

When the system is coaxal, $c=a$, and the parabola, as is well known, reduces to a point.

28. Suppose that CA in Fig. 8 is made to move, in contact with the envelope of the system, until it becomes parallel to BC.

Then, in this case, the angles L′Π′M′ and l'Πm' which are supplementary to the angle C, become each equal to two right angles, so that L′M′, $l'm'$ pass through Π′ and Π respectively.

Again, since in the general case, Π′L′, Πl' are equally inclined to BC, L′M′, $l'm'$ are equally inclined to BC and CA, and therefore are equal. Hence the segments, known to be of constant form, cut off from the variable circle L′l'M′m' by L′M′ and $l'm'$ are, in this case, equal for each position of the circle.

Also when the circle L′l'M′m' is the auxiliary circle of the envelope, L′M′, $l'm'$ are perpendicular to BC and CA.

Therefore L′m' = l'M′ = diameter of the auxiliary circle

= focal axis of the envelope.

But since BC, CA are parallel, L′m', l'M′, already known to be fixed in direction, are also constant in length for all positions of the circle L′l'M′m', and are therefore each equal to the focal axis of the envelope. From this it follows that they are equally inclined to BC and CA. By making L′ and l' coincide at D, the point where BC touches the envelope, we see that the fixed directions of L′m', l'M′ are parallel to ΠD, Π′D respectively.

The foregoing may be summed up in the following theorem, of which that of Art. 15 is a particular case.

If two fixed parallel tangents to the envelope of a coaxaloid system whose Π-points are Π and Π′, meet a variable circle of the system in the points L, l and M, m, then

(i) Two of the connectors of these four points, say LM and lm, pass through Π′ and Π respectively, and LM, lm are equally inclined to Ll and Mm, and cut off two pairs of equal segments of constant form from the circle; and

(ii) Lm, lM are constant in length being each equal to the focal axis of the envelope, and are fixed in direction being parallel to ΠD, Π′E and Π′D, ΠE respectively, where D and E are the points of contact of Ll and Mm with the envelope.

It might be easily proved also that

$$\mathrm{LD}/\mathrm{D}l = l\Pi/\Pi m = \mathrm{ME}/\mathrm{E}m = \mathrm{M}\Pi'/\Pi'\mathrm{L} = \Pi'\mathrm{D}/\mathrm{D}\Pi = \Pi\mathrm{E}/\mathrm{E}\Pi'.$$

The following theorem is deducible from the foregoing as a particular case.

If a circle have double contact with a conic whose foci are Π and Π′, and lines be drawn from L, one of the points of contact, through Π and Π′ meeting the circle in M and m respectively, then Mm is the tangent to the conic parallel to the tangent at L, and LM = Lm = the focal axis of the conic.

Another interesting particular case might be obtained by supposing that Ll and Mm coincide on an asymptote to the envelope.

29. If LMN (Fig. 13) be a variable triangle of constant species inscribed in the triangle ABC, not only the circumcircle of LMN, but any circle invariably connected with it (such as the incircle, Taylor circle, Brocard circle, etc.) forms, in its various positions, a coaxaloid system.

Demonstration. Let Π be the centre of similitude of the triangle LMN in its various positions. Draw through Π three lines making constant angles with MN, NL, LM respectively, and meeting the circle in l, m, n and the sides BC, CA, AB in l', m', n' respectively.

Then the ratios ΠL/Πl, etc., are constant, and also the ratios ΠL/Πl', etc. Therefore the ratios Πl/Πl', etc., are constant. Hence lmn is a triangle of constant species, and l, m, n move on straight lines parallel to BC, CA, AB respectively. That is the circle is the circumcircle of a variable triangle of constant species inscribed in a fixed triangle, and, therefore, by Art. 7, forms a coaxaloid system.

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If the circle passes through Π the system is coaxal; if LMN is similar to the orthocentric triangle of ABC, and the circle the incircle of LMN, the system is concentric.

30. If a system of directly similar triangles have the same centre of similitude when considered in pairs, the circle passing through the points where the sides of a variable triangle of the system meet the homologous sides of a fixed triangle of the system, forms, in its various positions, a coaxaloid system having the centre of similitude as one of its Π-points.

Demonstration. In the figure (Fig. 14) Π is the common centre of similitude, ABC the fixed triangle, A′B′C′ the variable triangle similar to it, L, M, N the points where BC, CA, AB are intersected by the homologous sides of A′B′C′ respectively, and l, m, n the feet of the perpendiculars from Π on BC, CA, AB.

Then, since ΠAB, ΠA′B′ are equal angles, Π, A, A′, N are concyclic, and since MAN, MA′N are equal angles, M, A, A′, N are concyclic. Therefore Π, M, A, N are concyclic. Therefore MΠN, $m\Pi n$ are equal angles. Similarly it may be shown that NΠL, $n\Pi l$ are equal angles, and also LΠM, $l\Pi m$. Therefore LΠl, MΠm, NΠn are equal angles. Hence it might easily be shown that the triangle LMN is similar to the triangle lmn. Therefore, by Art. 8, its circumcircle forms, in its various positions, a coaxaloid system having Π as one of the Π-points.

31. If a number of coaxaloid systems have the same ratio r, and have besides a common Π-point, Π, while the other Π-points lie on any locus, say a straight line, they form a web of circles possessing the following obvious properties.

(i) If a transversal be drawn in any direction across the web, the circles whose centres lie on it form a coaxaloid system one of whose Π-points is Π, and whose ratio is r; for if the transversal pass through K_1, K_2, K_3 ... centres of circles of the web, and ρK denote the radius of the circle whose centre is K, the ratios $\rho K_1/K_1\Pi$, $\rho K_2/K_2\Pi$, $\rho K_3/K_3\Pi$ ... are each equal to r.

(ii) When the transversal is drawn through Π, the circles whose centres lie on it have Π as a common centre of similitude; for if M_1, M_2, M_3 ... be centres lying on the transversal,

$$\rho M_1/M_1\Pi = \rho M_2/M_2\Pi = \ldots = r.$$

(iii) The centres of equal circles of the web lie on a circle whose centre is Π; for if C_1, C_2, $C_3 \ldots$ be centres of equal circles,

$$\rho C_1/C_1\Pi = \rho C_2/C_2\Pi = \rho C_3/C_3\Pi = \ldots = r,$$

and therefore since $\rho C_1 = \rho C_2 = \rho C_3 =$ etc.,

$$C_1\Pi = C_2\Pi = C_3\Pi = \text{ etc.}$$

A web such as has been described may be called a coaxaloid web of the first kind. It may be generated in a great number of different ways, of which the two following may be mentioned.

(1) If a variable triangle of constant species have a fixed point Π invariably connected with it, a circle invariably connected with it, forms, in its different positions, a coaxaloid web of the first kind; for, obviously, the radius of the circle bears a constant ratio to the distance of its centre from Π.

(2) If in the system of triangles referred to in the last article, different triangles be taken in succession as the fixed triangle, the coaxaloid systems so obtained form a web of the first kind.

Demonstration. The centre of similitude, Π, of the triangles is invariably connected with them. Therefore its pedal triangle with respect to each of them is of constant species. Therefore the triangle determined by the intersections of homologous sides of every pair of them is also of constant species, and has Π invariably connected with it. Therefore by (1) above, its circumcircle forms a web of the first kind.

32. If a variable line terminated at Π and Π' move parallel to itself so that Π and Π' always lie on two fixed straight lines intersecting in S, a coaxaloid system of circles having Π and Π' as Π-points and a constant ratio r, generates, as $\Pi\Pi'$ varies, a web of circles which may be called a coaxaloid web of the second kind, and which possesses the following properties.

(i) Those circles of the web whose centres lie on any transversal through S, have S as a common centre of similitude.

(ii) Generally the circles whose centres lie on any transversal are coaxaloid.

(iii) The coaxaloid systems determined by parallel transversals have the same ratio, and have their Π-points lying on two straight lines through S.

(iv) Systems of equal circles belonging to the web have their centres lying on a series of similar and similarly placed ellipses with S as centre.

Demonstration. Let $\Pi_1\Pi'_1$ and $\Pi_2\Pi'_2$ (Fig. 15) be two positions of $\Pi\Pi'$, and let O_1, O_2 be their respective mid-points. Let the perpendiculars at O_1, O_2 to $\Pi_1\Pi'_1$ and $\Pi_2\Pi'_2$ (the lines of centres of the coaxaloid systems whose Π-points are Π_1, Π'_1 and Π_2, Π'_2 respectively), meet a transversal through S in C_1, C_2, and a parallel transversal in K_1, K_2 respectively. Through S draw a parallel to O_1K_1, meeting K_1K_2 in K. It will be noticed that SK is the line of centres of that one of the original coaxaloid systems whose Π-points coincide in S. Let ρC denote the radius of the circle (belonging to the web) whose centre is C.

(i) $C_1S/C_2S = O_1S/O_2S = \Pi_1S/\Pi_2S$.

Therefore $C_1\Pi_1$ and $C_2\Pi_2$ are parallel.

Therefore $C_1\Pi_1/C_1S = C_2\Pi_2/C_2S$.

But $\rho C_1/C_1\Pi_1 = \rho C_2/C_2\Pi_2 = r$.

Therefore $\rho C_1/C_1S = \rho C_2/C_2S$.

Therefore S is the centre of similitude of the circles C_1 and C_2. In the same way it may be proved to be the centre of similitude of every other pair of circles whose centres lie on the same transversal.

(ii) Let SK, Π_1K_1 meet at Q, and let the circumcircles of $QS\Pi_1$, QKK_1 meet again in P. Then since P, K, K_1, Q are concyclic, PKS and $PK_1\Pi_1$ are equal angles and since P, S, Π_1, Q are concyclic, PSK and $P\Pi_1K_1$ are equal angles. Therefore PKS and $PK_1\Pi_1$ are similar triangles. - - - - (1).
Hence PKK_1 and $PS\Pi_1$ are similar triangles, and, therefore, since $KK_1/KK_2 = S\Pi_1/S\Pi_2$, PKK_2 and $PS\Pi_2$ are similar triangles.
Hence PKS and $PK_2\Pi_2$ are similar triangles. - - - (2).

Therefore, by (1) and (2) $KS/KP = K_1\Pi_1/K_1P = K_2\Pi_2/K_2P$.

But $\rho K/KS = \rho K_1/K_1\Pi_1 = \rho K_2/K_2\Pi_2 = r$.

Therefore $\rho K/KP = \rho K_1/K_1P = \rho K_2/K_2P$.

Therefore the circles (of the web) whose centres lie on the tranversal KK_1 are coaxaloid, with P as a Π-point.

(iii) $K\widehat{P}K_1 = K\widehat{Q}K_1 = C_1\widehat{K_1}\Pi_1$,

and $PK/PK_1 = KS/K_1\Pi_1 = K_1C_1/K_1\Pi_1$.

Therefore PKK_1 and $K_1C_1\Pi_1$ are similar triangles.

Therefore $$PK/K_1C_1 = KK_1/C_1\Pi_1,$$
i.e. $$PK/KS = SC_1/C_1\Pi_1.$$

But $$PK/PS = KK_1/S\Pi_1 = SC_1/S\Pi_1.$$

Therefore PKS and $SC_1\Pi_1$ are similar triangles.

Let r' denote the ratio of the system of circles C_1, $C_2 \ldots$, and r'' the ratio of the system $K, K_1, K_2 \ldots$.

Then $\rho C_1/C_1\Pi_1 = r$, and $\rho C_1/C_1S = r'$.

Therefore $r' = r \,.\, C_1\Pi_1/C_1S$.

And $\rho K/KS = r$, and $\rho K/KP = r''$.

Therefore $r'' = r \,.\, KS/KP$.

But $KS/KP = C_1\Pi_1/C_1S$.

Therefore $r'' = r'$.

Hence systems whose centres lie on parallel transversals have the same ratio.

Again, since PKS and $SC_1\Pi_1$ are similar triangles, the angle KSP is equal to the angle $C_1\Pi_1S$ which is constant so long as SC_1 retains its direction. Therefore the locus of P, as KK_1 moves parallel to SC_1 is a straight line through S. Hence it may be readily proved that the locus of the image of P with respect to KK_1, *i.e.*, the other Π-point, is also a straight line through S. Thus statement (iii) is completely proved.

(iv) Let K_1 be the centre of a variable circle of the web, having a constant radius ρ; and regard the circle as belonging in its different positions to the variable coaxaloid system whose line of centres is parallel to SK, and of which Π_1, O_1 (here regarded as variable) are respectively a Π-point, and the mid-point of the join of the Π-points. Let α denote the angle Π_1SO_1 and β the angle $S\Pi_1O_1$.

Then $$K_1\Pi_1 = \rho/r,$$
$$K_1\Pi_1^2 = K_1O_1^2 + \Pi_1O_1^2,$$
and $$\Pi_1O_1 = \frac{\sin\alpha}{\sin\beta} \,.\, SO_1$$

Therefore $$K_1O_1^2 + \frac{\sin^2\alpha}{\sin^2\beta} \,.\, SO_1^2 = \frac{\rho^2}{r^2}.$$

Therefore, taking SO_1 and SK as the axes of x and y respectively,

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we have as the equation of the locus of K_1,

$$y^2+\frac{\sin^2\alpha}{\sin^2\beta}\cdot x^2=\frac{\rho^2}{r^2},$$

which represents an ellipse having S as centre, and a pair of conjugate diameters coincident with SO_1 and SK. By making ρ vary, we obtain a series of concentric, similar and similarly placed ellipses. Thus statement (iv) is proved.

It may be noted that in the case when $S\Pi=S\Pi'$, the lines of centres (SK, O_1K_1, O_2K_2, etc.) of the generating systems coincide, so that the web obtained is such that any point on SO_1 is the centre of a concentric system.

A coaxaloid web of the second kind may be generated in a great many different ways, of which the following one may be mentioned. In connection with Fig. 6 it was shown that if ABC remain fixed, while PQR varies so as to remain homothetic with itself with respect to S as homothetic centre, we obtain a coaxaloid system $LlMmNn$ connected with ABC. If now ABC also be made to vary so as to remain homothetic with itself with respect to S as homothetic centre, we obtain a series of coaxaloid systems having the same S-point, and forming a web of the second kind; for as may easily be proved by similar triangles, their ratios are the same, their Π-points lie on two straight lines intersecting in S, and their line of centres are either parallel or, as is the case when the Π-points are the Brocard points of ABC, coincident.

Note referring to Art. 17.

Let K be the centre (Fig. 12) and TT′ the diameter through Π and Π′ of the minimum circle of a coaxaloid system whose Π-points (real) are Π and Π′. On the line of centres of the system, on opposite sides of K, take two points P and P′ such that $KP=KP'=K\Pi$. Suppose that a system of coaxal circles is constructed with TT′ as radical axis, and P, P′ as limiting points; and suppose that the radii of these are increased or diminished, as the case may be, in the ratio $KT/K\Pi$, denoted by r, so as to give rise to a coaxaloid system with imaginary Π-points. Let K′ be the centre of any circle of this system, and K′W a radius of it parallel to any fixed direction KV. Draw K′R a radius of the circle of the

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original coaxal system, whose centre is K′, parallel to TT′; and draw RM perpendicular to TT′, and join MP, MK′. Then if x and y be the coordinates of W, with respect to KV, KP as axes, we have

$$x^2 = \mathrm{K'W}^2 = r^2\mathrm{K'R}^2 = r^2(\mathrm{K'M}^2 - \mathrm{RM}^2).$$

But RM = PM, both being tangents from a point on the radical axis to circles of the coaxal system. Therefore

$$x^2 = r^2(\mathrm{K'M}^2 - \mathrm{PM}^2) = r^2(\mathrm{K'K}^2 - \mathrm{PK}^2) = (y^2 - \mathrm{K\Pi}^2)\mathrm{KT}^2/\mathrm{K\Pi}^2.$$

Hence the equation of the locus of W is

$$y^2/\mathrm{K\Pi}^2 - x^2/\mathrm{KT}^2 = 1.$$

It will be noticed that the hyperbolas in this case are the conjugates of those associated with the coaxaloid system whose Π-points (real) are Π and Π′, and whose ratio is also KT/KΠ.

It may be remarked that the foregoing proof and that of Art. 17 implicitly contain the following theorem which I have not seen stated elsewhere: If K′W (Fig. 5) be a variable line of fixed direction, such that its extremity K′ moves on a fixed line (KK″) and such that K′W/K′Π, where Π is a fixed point, is constant, the locus of W is a hyperbola.

Systems of Conics connected with the Triangle.

By J. A. Third, M.A.

[The references in square brackets are to the articles of my paper on Systems of Circles analogous to Tucker Circles.]

I.

Systems of Six-point Conics.

1. It has been shown [4] that if two triangles ABC and PQR (Fig. 6) be in perspective with respect to any point S as centre of perspective, the sides of PQR cut the non-corresponding sides of ABC in three pairs of points which lie on a circle, provided that the angles made by the sides of PQR with those of ABC possess certain

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