Can. J. Math., Vol. XXVI, No. 3, 1974, pp. 734–745

A PURITY CRITERION FOR PAIRS OF LINEAR TRANSFORMATIONS

URI FIXMAN AND FRANK A. ZORZITTO

Introduction. In connection with the study of perturbation methods for differential eigenvalue problems, Aronszajn put forth a theory of *systems* $(X, Y; A, B)$ consisting of a pair of linear transformations $A, B: X \to Y$ (see [**1**]; cf. also [**2**]). Here X and Y are complex vector spaces, possibly of infinite dimension. The algebraic aspects of this theory, where no restrictions of topological nature are imposed, where developed in [**3**] and [**5**]. We hasten to point out that the category of $\mathbf{C}^2$-systems (definition in § 1) in which this algebraic investigation takes place is equivalent to the category of all right modules over the ring of matrices of the form

$$\begin{bmatrix} \beta & 0 & \alpha_1 \\ 0 & \beta & \alpha_2 \\ 0 & 0 & \gamma \end{bmatrix}, \quad \alpha_1, \alpha_2, \beta, \gamma \text{ complex numbers.}$$

Moreover, this category contains, not canonically, subcategories equivalent to the category of modules over the principal ideal domain of complex polynomials in one variable. We make no essential use of these equivalences. The language of systems (in the technical sense of § 1) is preferred because many concepts, not all dealt with here, are suggested by the context of pairs of linear transformations. However, the above equivalences explain why, as in the theory of abelian groups, a central role is played by the concept of a *pure* subsystem. (The terms "spectral", "quasi-spectral" and "quasi-spectrally irreducible" were used in the above references instead of "direct summand", "pure" and "purely simple" respectively.)

Relying on the reader's familiarity with purity in other contexts, we postpone its definition and just observe that this concept derives its importance from the following facts. Being a pure subsystem is a less demanding condition than being a direct summand. Yet for some kinds of subsystems purity implies being a direct summand; which facilitates the task of decomposing systems. The family of pure subsystems of a given system is inductive, and there are fewer isomorphism types of purely simple systems than there are of directly indecomposable ones. This enables one to regard the former as natural "building blocks" in terms of which isomorphism invariants for general systems can be defined.

Received January 8, 1973. This research was supported by the National Research Council of Canada under grant No. A3008 and by the Canada Council under grant No. W71 4966. The contribution of the second named author forms part of a doctoral thesis written at Queen's University in 1972.

For these reasons much of the algebraic study of systems is concerned with giving simple sufficient (and where possible necessary and sufficient) conditions for a given subsystem to be pure. In the case of abelian groups it is immediate that if H is a subgroup of G such that G/H is torsion-free, then H is pure in G. Our main result (Theorem 1 below) is the analog of this statement for systems, where however some restriction must be imposed on the subsystem corresponding to H and the proof is more involved. This result generalizes Theorem 5.6 of [**5**]. The need for it arose in an investigation of weighted shift operators in Hilbert space (see [**6**] and a paper under preparation by Zorzitto). It will doubtless have other applications.

In sections 2 and 3 we offer two distinct proofs of Theorem 1. The first generalizes the method of [**5**], while the second is based on a theorem of Zorzitto stating that purity is equivalent to the dual concept of co-purity [**6**; **7**]. This equivalence granted, the second proof is the simpler one. However, the first yields more information on the structure of torsion-free systems (Theorem 2), which is of interest mainly for purely simple torsion-free systems of rank higher than 1 (cf. [**5**, 5.7]). Finally, we show by means of examples in § 4 that no simple generalization of Theorem 1 to more than two transformations is to be expected.

1. Definitions and statement of main result. Although we are concerned here mainly with *pairs* of linear transformations, to be able to formulate Theorem A below in its full generality and to discuss counterexamples, we make the definitions for systems of N linear transformations, where N is any positive integer.

Denote by $\mathbf{C}^N$ the N-dimensional complex vector space of N-tuples of complex numbers. An (algebraic) $\mathbf{C}^N$-*system* (X, Y) is a pair of complex vector spaces X and Y together with a *system operation* which is a $\mathbf{C}$-bilinear map

$$(e, x) \mapsto ex$$

of $\mathbf{C}^N \times X$ into Y. If $(e_j)_{j=1}^N$ is the canonical basis of $\mathbf{C}^N$, the system operation determines and is determined by the N-tuple of linear transformations

$$A_j: x \mapsto e_j x, \quad j = 1, \ldots, N$$

of X into Y. As explained in detail in [**3**], it is technically more convenient to have the same space $\mathbf{C}^N$ acting in every $\mathbf{C}^N$-system, than to take as our objects directly the N-tuples of linear transformations $A_1, \ldots, A_N: X \to Y$. A *homomorphism* of a $\mathbf{C}^N$-system (S, T) into a $\mathbf{C}^N$-system (X, Y) is a pair (ϕ, ψ) of linear transformations $\phi: S \to X$ and $\psi: T \to Y$ such that

$$e\phi s = \psi e s$$

for all $e \in \mathbf{C}^N$, $s \in S$. Homomorphisms are composed componentwise giving rise to an abelian category (equivalent to a category of modules, in straight-

https://doi.org/10.4153/CJM-1974-068-8 Published online by Cambridge University Press

forward generalization of the case $N = 2$ noted in the introduction). Two systems in this category are isomorphic if and only if the corresponding N-tuples of linear transformations are equivalent in the classical sense. A $\mathbf{C}^N$-system (S, T) is a *subsystem* of the $\mathbf{C}^N$-system (X, Y) in case S and T are subspaces of X and Y respectively, and the pair of inclusions is a homomorphism of (S, T) into (X, Y), or, equivalently $es \in T$ for all $s \in S$, $e \in \mathbf{C}^N$. The *quotient* $(X, Y)/(S, T)$ is the $\mathbf{C}^N$-system given by the pair of spaces $(X/S, Y/T)$ and the system operation $(e, x + S) \mapsto ex + T$ for all $x \in X$, $e \in \mathbf{C}^N$. A subsystem (S, T) of (X, Y) is a *direct summand* of (X, Y) in case there exists a *subsystem* (U, Z) of (X, Y) such that $X = S + U$, $S \cap U = 0$, $Y = T + Z$ and $T \cap Z = 0$. We then call (U, Z) a *supplement* of (S, T) in (X, Y) and write $(X, Y) = (S, T) \dotplus (U, Z)$. A non-zero $\mathbf{C}^N$-system with no non-trivial direct summands is said to be *indecomposable*. A $\mathbf{C}^N$-system (X, Y) is said to be *finite-dimensional* in case X and Y are finite-dimensional.

A subsystem (S, T) of (X, Y) is said to be *pure in* (X, Y) provided for every intermediate subsystem (U, Z), $(S, T) \subset (U, Z) \subset (X, Y)$, such that $(U, Z)/(S, T)$ is finite-dimensional, (S, T) is a direct summand of (U, Z). Dually, (S, T) is said to be *copure in* (X, Y) provided, for every subsystem (U, Z) of (S, T) such that $(S, T)/(U, Z)$ is finite-dimensional, we have that $(S, T)/(U, Z)$ is a direct summand of $(X, Y)/(U, Z)$.

These two concepts are equivalent. That *a pure subsystem is copure* is an easy consequence of the fact that *a finite-dimensional pure subsystem is a direct summand.* (See [**3**] for the case $N = 2$. The same proof works for general N.) Using topological considerations, the second named author has proved the following theorem [**6**; **7**].

THEOREM A. *If (S, T) is a copure subsystem of a $\mathbf{C}^N$-system (X, Y), then (S, T) is pure in (X, Y).*

A $\mathbf{C}^N$-system (X, Y) is said to be *torsion-free*† in case all the linear transformations $x \mapsto ex$, $0 \neq e \in \mathbf{C}^N$ are injective. In the case $N = 2$ the finite-dimensional torsion-free systems can be described as follows. Let (X, Y) be a $\mathbf{C}^2$-system, (a, b) a basis of $\mathbf{C}^2$ and p, q rational integers with $p - 1 \leqq q$. We denote by $C^{p,q}(a, b; X, Y)$ the set of all pairs of sequences $((x_k)_{k=p}^{q}, (y_k)_{k=p}^{q+1})$, with $x_k \in X$, $y_k \in Y$, which satisfy the conditions

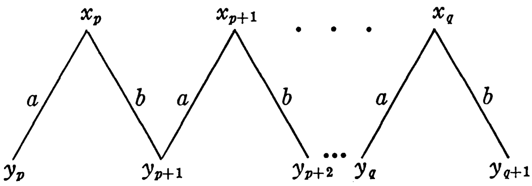
$$ax_k = y_k,\ bx_k = y_{k+1} \quad \text{for } p \leqq k \leqq q.$$

(If $p - 1 = q$, the first sequence is empty.) Such a pair of sequences is called

†In spite of the connection to eigenvalues we prefer this term to "eigenvalue-free" used in [**3**] mainly because it is linguistically more convenient in combination with other terms. Its use is justified by the facts that a torsion-free module over $\mathbf{C}[\lambda]$ can be construed as a torsion-free $\mathbf{C}^2$-system, and that for general N the torsion-free systems actually form a torsion-free class (see e.g. [**4**]). We modify similarly the derived terms.

https://doi.org/10.4153/CJM-1974-068-8 Published online by Cambridge University Press

a *chain* with *domain sequence* $(x_k)_p^q$ and *range sequence* $(y_k)_p^{q+1}$. A chain can be represented diagramatically thus:



$C^{p,q}(a, b; X, Y)$ is endowed with the structure of a complex vector space by defining the operations componentwise for the domain and range sequences. A $\mathbf{C}^2$-system (X, Y) is said to be *of type* III^m, $m = 1, 2, \ldots$, in case for some basis (a, b) of $\mathbf{C}^2$ there exists a chain in $C^{1,m-1}(a, b; X, Y)$ whose first and second component sequences are bases of X and Y respectively. (We follow here the notations of [**3**]. The notations are somewhat different in our other references.) It is shown in [**3**, Lemma 2.5] that this definition is independent of choice of basis (a, b), so that III^m denotes a definite isomorphism type. Moreover, it follows from the results 2.6, 2.2, 4.3 and 9.3 (a) of [**3**] that:

THEOREM B. *$\mathbf{C}^2$-systems of the types III^m are indecomposable. Every torsion-free finite-dimensional $\mathbf{C}^2$-system is a direct sum of a finite number of subsystems of the types III^m.*

We can now formulate our main result.

THEOREM 1. *Let (S, T) be a subsystem of a $\mathbf{C}^2$-system (X, Y). Suppose that (S, T) has no direct summand of any of the types III^m, $m = 1, 2, \ldots$, and that $(X, Y)/(S, T)$ is torsion-free. Then (S, T) is pure in (X, Y).*

The isomorphism types of finite-dimensional indecomposable $\mathbf{C}^2$-systems which are not torsion-free were denoted in [**3**] by the symbols I^m and $\mathrm{II}_{\dot{e}}^m$ ($\dot{e}$ is the point of the projective complex line $P^1(\mathbf{C})$ generated by $0 \neq e \in \mathbf{C}^2$). We refer to [**3**] for descriptions of these types by means of chains (not needed here) and for a simple proof of the theorem, essentially due to Kronecker, that *every finite-dimensional $\mathbf{C}^2$-system is a direct sum of subsystems of the above types* [**3**, Theorem 4.3]. We shall use however the following part of [**3**, Proposition 6.9].††

THEOREM C. *Let (S, T) be a subsystem of the $\mathbf{C}^2$-system (X, Y). If $(X, Y)/(S, T)$ is of type III^n and (S, T) is of one of the types I^m, $II_{\dot{e}}^m$ (m any positive integer) or of type III^m with $n \leqq m + 1$, then (S, T) is a direct summand in (X, Y).*

††This proposition and the related result [**3**, Lemma 3.6] may be improved by making the following changes in [**3**]: p. 295, l. 4: replace "$m \leqq n$" by "$m \leqq n + 1$"; l. 23: replace "the first, that $m \leqq n$. If $m \leqq n$" by "that $m \leqq n$. If $m \leqq n + 1$"; p. 310, l. 5 from bottom: replace "$n \geqq m$" by "$n + 1 \geqq m$"; p. 310, last line: replace "$n \leqq m$" by "$n \leqq m + 1$".

2. First proof of Theorem 1. The proof will follow from the following two results.

THEOREM 2. *Let* (X, Y) *be a non-zero torsion-free* $\mathbf{C}^2$*-system which has no direct summand of any of the types* III^m, $m = 1, 2, \ldots$. *Then for every finite-dimensional subsystem* (S, T) *of* (X, Y) *and positive integer* n, *there exists a subsystem* (U, Z) *of* (X, Y) *containing* (S, T) *such that* (U, Z) *is a finite direct sum of subsystems of types* III^{m_j} *with* $m_j \geqq n$ *for all* j.

Proof. Since (X, Y) is non-zero and torsion-free, $Y \neq 0$. Thus if $(S, T) = (0, 0)$, we can embed it first in a subsystem $(0, \mathbf{C}y)$ with $0 \neq y \in Y$. Hence we may assume that $(S, T) \neq (0, 0)$. Then by Theorem B, (S, T) is a non-empty finite direct sum

$$(S, T) = \sum_{j=1}^{g} \cdot (S^j, T^j),$$

where (S^j, T^j) is of type III^{l_j}, $j = 1, \ldots, g$. Put

$$l = \min \{l_j : 1 \leqq j \leqq g\}.$$

Using induction, it suffices to embed (S, T) in a subsystem (U, Z) of (X, Y), which has a finite decomposition

$$(*) \quad (U, Z) = \sum_{k=1}^{h} \cdot (U^k, Z^k),$$

where (U^k, Z^k) is of type III^{m_k} and

$$(**) \quad m_k > l \quad \text{for all } k = 1, \ldots, h.$$

Actually, the m_k's are uniquely determined by (U, Z) up to order (see [**3**, p. 309]); but this is immaterial for the present proof.

Fix a basis (a, b) of $\mathbf{C}^2$ and let Γ^j be a chain in $C^{1, l_j - 1}(a, b; S^j, T^j)$ which spans (S^j, T^j). Since by assumption (S^j, T^j) is not a direct summand of (X, Y), it follows from Theorems 5.5 and 6.6 of [**3**] that $\Gamma^j \in \hat{C}\mathrm{III}^{l_j}(a, b; X, Y)$; namely, that Γ^j is the sum of restrictions of two longer chains $\Gamma^j{}_1 \in C^{0, l_j - 1}(a, b; X, Y)$ and $\Gamma^j{}_2 \in C^{1, l_j}(a, b; X, Y)$. Here the restrictions of the domain sequences are to the interval $[1, l_j - 1]$ which is empty if $l_j = 1$) and those of the range sequences are to the interval $[1, l_j]$. Let $(U^j{}_1, Z^j{}_1)$ and $(U^j{}_2, Z^j{}_2)$ be the subsystems spanned by $\Gamma^j{}_1$, and $\Gamma^j{}_2$ respectively (i.e., $U^j{}_1$ is the subspace spanned by the domain sequence of $\Gamma^j{}_1$, etc.). Define (U, Z) as the (not necessarily direct) sum

$$(U, Z) = \sum_{j=1}^{g} ((U^j{}_1, Z^j{}_1) + (U^j{}_2, Z^j{}_2)).$$

This subsystem clearly contains (S, T). By Theorem B, (U, Z) has a decomposition of the form $(*)$. We claim that any such decomposition satisfies the condition $(**)$.

Let $(\pi_k)_{k=1}^h$ and $(\rho_k)_{h=1}^k$ be the sequences of projections associated to the decompositions

$$U = \sum_{k=1}^{h} \cdot U^k \quad \text{and} \quad Z = \sum_{k=1}^{h} \cdot Z^k$$

respectively. Then it is easy to verify that the pairs (π_k, ρ_k) are endomorphisms of the $\mathbf{C}^2$-system (U, Z). It follows that by applying π_k to the elements of the domain sequence of $\Gamma^j{}_1$ and ρ_k to the elements of its range sequence one obtains a chain $(\pi_k, \rho_k)\Gamma^j{}_1 \in C^{0,l_j-1}(a, b; U^k, Z^k)$. Similarly, $(\pi_k, \rho_k)\Gamma^j{}_2 \in C^{1,l_j}(a, b; U^k, Z^k)$.

We now observe that, in general, if (V, W) is a $\mathbf{C}^2$-system of type III^p and Γ is a chain in $C^{1,q}(a, b; V, W)$ for some integer $q \geqq p$, then Γ is a zero chain. Indeed, by passing to an isomorphic system, we may assume that V is the space of all complex polynomials in the indeterminate z of degree at most $p - 2$ ($V = 0$ if $p = 1$), W is the space of polynomials of degree at most $p - 1$ and the system operation is defined as follows. For $v(z) \in V, \alpha a + \beta b \in \mathbf{C}^2$ let

$$(\alpha a + \beta b)v(z) = (\alpha + \beta z)v(z).$$

The chain

$$((z^{k-1})_{k=1}^{p-1}, (z^{k-1})_{k=1}^{p}),$$

which belongs to $C^{1,p-1}(a, b; V, W)$, shows that this (V, W) is actually of type III^p (if $p = 1$, the domain sequence is empty). Now if $\Gamma = ((v_k(z))_1^q, (w_k(z))_1^{q+1})$, we have the relations

$$w_k(z) = av_k(z) = v_k(z), \quad w_{k+1}(z) = bv_k(z) = zv_k(z), \quad k = 1, \ldots, q.$$

Thus $w_{q+1}(z) = z^q w_1(z)$. Since W contains no non-zero polynomial of degree exceeding $p - 1$, it follows that $w_1(z) = 0$. The above relations then imply that all the chain elements vanish.

In our case we thus get that if $m_k \leqq l$, and hence $m_k \leqq l_j$, then $(\pi_k, \rho_k)\Gamma^j{}_1$ and $(\pi_k, \rho_k)\Gamma^j{}_2$ vanish. Hence for every $j = 1, \ldots, g$ the elements of the domain and range sequences of $\Gamma^j{}_1$ and $\Gamma^j{}_2$ belong to $\sum\{U^k : m_k > l\}$ and $\sum\{Z_k : m_k > l\}$ respectively. It follows that

$$(U, Z) \subset \sum\{(U_k, Z_k) : m_k > l\}.$$

As the reverse inclusion is obvious, we actually have here equality. Since none of the subspaces Z_k, $1 \leqq k \leqq h$, is a zero space, this and (*) imply (**).

We observe that if (S, T) is a pure subsystem of a $\mathbf{C}^N$-system (X, Y), then every equation $ex = t$, $t \in T$, $e \in \mathbf{C}^N$, which is solvable by some $x \in X$ is also solvable by some $s \in S$. This is because (S, T) is a direct summand in $(S + \mathbf{C}x, T + \mathbf{C}^N x)$. It follows that whenever (S, T) is pure in a torsion-free $\mathbf{C}^N$-system

(X, Y), the quotient $(X, Y)/(S, T)$ is also torsion-free. The following partial converse is already a generalization of [**5**, Theorem 5.6].

LEMMA 1. *Let (S, T) be a torsion-free subsystem of a $\mathbf{C}^2$-system (X, Y). Suppose that (S, T) has no direct summand of any of the types III^m, $m = 1, 2, \ldots$, and that $(X, Y)/(S, T)$ is torsion-free. Then (S, T) is pure in (X, Y).*

Proof. A sufficient condition for purity is that (S, T) be a direct summand in every intermediate extension with a finite-dimensional indecomposable quotient (see [**3**, Proposition 5.2]). Since $(X, Y)/(S, T)$ is torsion-free we see from Theorem B that only quotients of the types III^n come into account here. So it suffices to consider the case that $(X, Y)/(S, T)$ is itself of type III^n and prove that (S, T) is a direct summand in (X, Y). Let (P, Q) be a finite-dimensional subsystem of (X, Y) such that

(i) $\quad (X, Y) = (S, T) + (P, Q).$

We neglect the trivial case $(S, T) = (\mathbf{0}, \mathbf{0})$. Then by Theorem 2 there exists a subsystem (U, Z) of (S, T) containing $(S, T) \cap (P, Q)$ such that

(ii) $$(U, Z) = \sum_{j=1}^{r} \cdot (U^j, Z^j)$$

where (U^j, Z^j) is of type III^{m_j} and $m_j + 1 \geqq n$ for all $j = 1, \ldots, r$. By the choice of (U, Z) we have $(S, T) \cap (P, Q) \subset (U, Z) \cap (P, Q)$. Since $(S, T) \supset (U, Z)$, we have in fact $(S, T) \cap (P, Q) = (U, Z) \cap (P, Q)$. This implies by the isomorphism theorem that

$$((U,Z) + (P,Q))/(U,Z) \cong ((S,T) + (P,Q))/(S,T) = (X, Y)/(S,T),$$

which is of type III^n. Using (ii), Theorem C, and the fact that Ext commutes with finite direct sums in its second variable, we conclude that we have a decomposition

(iii) $\quad (U, Z) + (P, Q) = (U, Z) \dot{+} (K, L).$

The last decomposition can also be derived directly, by showing by induction on $k = 1, 2, \ldots, r$ that $(U, Z)/\sum_{j=1}^{r-k}(U^j, Z^j)$ is a direct summand in $((U, Z) + (P, Q))/\sum_{j=1}^{r-k}(U^j, Z^j)$.

From (i), (iii) and $(P, Q) \cap (S, T) \subset (U, Z) \subset (S, T)$ it easily follows that $(X, Y) = (S, T) \dot{+} (K, L)$.

Conclusion of proof of Theorem 1. We show that the assumption in Lemma 1, that (S, T) be torsion-free, can be deleted, whence we are done.

Indeed, let (X, Y) and (S, T) be as in the statement of Theorem 1. We denote by $t(X, Y)$ the *torsion part* (see footnote) of (X, Y), which is the smallest subsystem (P, Q) of (X, Y) such that $(X, Y)/(P, Q)$ is torsion-free (see [**3**, p. 324]). The subsystem $t(X, Y)$ is pure in (X, Y) by [**3**, Proposition 9.12], and since it is contained in (S, T) it is pure also in (S, T). The subsystem

$(S, T)/t(X, Y)$ inside $(X, Y)/t(X, Y)$ satisfies the hypotheses of Lemma 1. Clearly $((X, Y)/t(X, Y))/((S, T)/t(X, Y)) \cong (X, Y)/(S, T)$ is torsion-free. In addition, if $(K, L)/t(X, Y)$ is a direct summand in $(S, T)/t(X, Y)$ of type III^m for some m, then $(K, L) = t(X, Y) \dotplus (G, H)$, where (G, H) is of the same type. By [**3**, Proposition 5.3], which gives the usual properties of pure subsystems, (K, L) is pure in (S, T) and hence (G, H) is pure in (S, T). From [**3**, Theorem 5.5] it follows that, being finite-dimensional, (G, H) is a direct summand in (S, T). This is against the hypothesis on (S, T). Thus Lemma 1 implies that $(S, T)/t(X, Y)$ is pure in $(X, Y)/t(X, Y)$. Using [**3**, Proposition 5.3] again, it follows that (S, T) is pure in (X, Y).

3. Second proof of Theorem 1. We break this proof into three parts.

LEMMA 2. *A subsystem (S, T) of a $\mathbf{C}^2$-system (X, Y) is copure in (X, Y) if and only if for every subsystem (U, Z) of (S, T) such that $(S, T)/(U, Z)$ is finite-dimensional and indecomposable we have that $(S, T)/(U, Z)$ is a direct summand of $(X, Y)/(U, Z)$.*

Proof. This lemma is clearly the dual statement of [**3**, Proposition 5.2], and we shall use that result to prove it.

The condition is evidently necessary. We show that it is sufficient. Consider the dual $\mathbf{C}^2$-system (Y^*, X^*) of (X, Y), where Y^* is the space of linear functionals on Y, and X^* the functionals on X. (In [**3**] the dual is defined by antilinear functionals instead of linear ones, but for the purpose at hand it does not matter which are taken.) The system operation for (Y^*, X^*) is given by

$$(ey^*)(x) = y^*(ex), \quad e \in \mathbf{C}^2, y^* \in Y^*, x \in X.$$

The polars $T^\perp = \{y^* \in Y^* : y^*(T) = 0\}$ and $S^\perp = \{x^* \in X : x^*(S) = 0\}$ inside Y^* and X^* respectively determine a subsystem $(T^\perp, S^\perp)$ of (Y^*, X^*). One may check in a routine way or from [**7**] that (S, T) is copure in (X, Y) if (and only if) $(T^\perp, S^\perp)$ is pure in (Y^*, X^*). Let (Q, P) be a subsystem of (Y^*, X^*) containing the polar $(T^\perp, S^\perp)$ inside (Y^*, X^*) such that $(Q, P)/(T^\perp, S^\perp)$ is finite-dimensional and *indecomposable*. By [**3**, Proposition 5.2] to show that $(T^\perp, S^\perp)$ is pure in (Y^*, X^*) it suffices to show that $(T^\perp, S^\perp)$ is a direct summand in every such (Q, P). The polar $(P^\perp, Q^\perp)$ is inside (S, T) and is such that $(S, T)/(P^\perp, Q^\perp)$ is finite-dimensional and indecomposable. Here $P^\perp$ is defined as $\{x \in X : x^*(x) = 0 \text{ for all } x^* \in P\}$; similarly for $Q^\perp$. By hypothesis, $(S, T)/(P^\perp, Q^\perp)$ is a direct summand in $(X, Y)/(P^\perp, Q^\perp)$ with a supplement $(K, L)/(P^\perp, Q^\perp)$ for some (K, L) in (X, Y). Then $(L^\perp, K^\perp)$ in (Y^*, X^*) serves as a supplement to $(T^\perp, S^\perp)$ inside (Q, P).

LEMMA 3. *Let (S, T) be a subsystem of a $\mathbf{C}^2$-system (X, Y). Suppose (S, T) has no homomorphic images of any of the types III^m, $m = 1, 2, \ldots,$ and that $(X, Y)/(S, T)$ is torsion-free. Then (S, T) is pure in (X, Y).*

Proof. Let (P, Q) be an intermediate subsystem $(S, T) \subset (P, Q) \subset (X, Y)$ such that $(P, Q)/(S, T)$ is finite-dimensional and indecomposable. Since

$(X, Y)/(S, T)$ is torsion-free it follows from Theorem B that $(P, Q)/(S, T)$ must be of type III^m for some m. To show that (S, T) is a direct summand of (P, Q) it clearly suffices to show it is pure in (P, Q). By Theorem A it will do to show (S, T) is copure in (P, Q). Testing for this according to Lemma 2 we let (U, Z) be a subsystem of (S, T) such that $(S, T)/(U, Z)$ is finite-dimensional and indecomposable. By hypothesis $(S, T)/(U, Z)$ is never of type III^n for any n, and thus must be of one of the types I^n or II_e^n. From Theorem C we deduce that $(S, T)/(U, Z)$ is a direct summand in $(P, Q)/(U, Z)$ and hence that (S, T) is copure in (P, Q), yielding our result.

THEOREM 3. *If (S, T) is a $\mathbf{C}^2$-system with a homomorphic image of type III^n for some n, then it has a direct summand of type III^m for some $m \leqq n$.*

Proof. The proof could be derived from the dual statement which is implicit in [**3**]. However, a direct proof is almost as simple. Let (U, Z) be a subsystem of (S, T) such that $(S, T)/(U, Z)$ is of type III^m with m minimal. We shall prove that (U, Z) is a direct summand of (S, T), in which case the other summand will be of type III^m. By Theorem A one needs only to show (U, Z) is copure in (S, T). In view of Lemma 2 consider (P, Q) inside (U, Z) such that $(U, Z)/(P, Q)$ is finite-dimensional and indecomposable, and hence of type I^k, II_e^k or III^k for some $k = 1, 2, \ldots$. In the cases I^k, II_e^k and III^j where $j \geqq m$, Theorem C guarantees that $(U, Z)/(P, Q)$ is a direct summand in $(S, T)/(P, Q)$. The remaining case where $j < m$ cannot occur. To see this suppose to the contrary that $(U, Z)/(P, Q)$ is a subsystem of $(S, T)/(P, Q)$, which is of type III^j with $j < m$. The system $(S, T)/(P, Q)$ is finite-dimensional and torsion-free and thus by Theorem B

$$(S, T)/(P, Q) = \sum_{i=1}^{r} \cdot (V_i, W_i),$$

where for each $i = 1, \ldots, r$ (V_i, W_i) is of type III^{n_i} for some n_i. If we had $n_i < m$ for some i, then (V_i, W_i) would be a homomorphic image of (S, T), contrary to the minimality of m. So $n_i \geqq m$ for all i, but since $j < m$ and

$$m + j = \dim T/Q = \sum_{i=1}^{r} n_i$$

it follows that $r = 1$ and $n_1 = m + j$. Thus $(S, T)/(P, Q)$ is of type III^{m+j} and has $(S, T)/(U, Z)$ of type III^m as a homomorphic image. This contradicts the easily verifiable fact that systems of type III^n do not have homomorphic images of type III^k when $k < n$. Hence $j < m$ cannot occur. So (U, Z) is a direct summand of (X, Y).

The statement of Theorem 1 now follows immediately by applying Theorem 3 to Lemma 3.

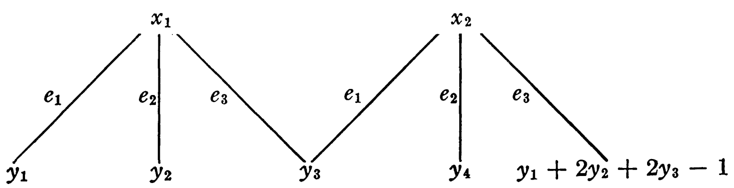
4. Counterexamples. The statement in Theorem 1 or in Lemma 1 that (S, T) has no direct summands of any of the types III^m is tantamount to saying that (S, T) has no finite-dimensional, torsion-free, indecomposable (non-zero) direct sumands. With this rewording, the statements of Theorem1 and Lemma 1 make sense with any N replacing $N = 2$. However, the examples to follow show these generalizations to be false when $N > 2$.

Example 1. We present an infinite-dimensional indecomposable torsion-free $\mathbf{C}^3$-system (S, T) embedded inside a $\mathbf{C}^3$-system (X, Y) such that $(X, Y)/(S, T)$ is torsion-free and finite-dimensional. Yet (S, T) will fail to be a direct summand, and hence fail to be pure, in (X, Y). With this we negate the proposed generalization of Lemma 1 and of the stronger Theorem 1 to $N = 3$.

Let X be complex vector space containing $\mathbf{C}[z]$, the space of complex polynomials, as a subspace such that $X/\mathbf{C}[z]$ is 2-dimensional. Let Y be a space containing $\mathbf{C}[z]$ as a subspace such that $Y/\mathbf{C}[z]$ is 4-dimensional. We select bases $\{1, z, z^2, \ldots ; x_1, x_2\}$, $\{1, z, z^2, \ldots ; y_1, y_2, y_3, y_4\}$ and $\{e_1, e_2, e_3\}$ of X, Y and $\mathbf{C}^3$ respectively. We define the system operation of $\mathbf{C}^3$ from X to Y as follows. For $f(z) \in \mathbf{C}[z]$, $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \in \mathbf{C}^3$, let

$$(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3) f(z) = (\alpha_1 + \alpha_2 z + \alpha_3 z^2) f(z).$$

For $x_1, x_2 \in X$ let e_1, e_2, e_3 act according to the diagram



That is, $e_1 x_1 = y_1, \ldots, e_3 x_2 = y_1 + 2y_2 + 2y_3 - 1$. Then we extend the operation of $\mathbf{C}^3$ to all of X linearly to define the $\mathbf{C}^3$-system (X, Y).

Evidently $(\mathbf{C}[z], \mathbf{C}[z])$ is a torsion-free subsystem of (X, Y), and $(X, Y)/(\mathbf{C}[z], \mathbf{C}[z])$ is finite-dimensional. This quotient system is also torsion-free; for if

$$(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3)(\beta_1 x_1 + \beta_2 x_2 + \mathbf{C}[z]) = 0$$

in $Y/\mathbf{C}[z]$ with a non-zero $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \in \mathbf{C}^3$ and $\beta_1, \beta_2 \in \mathbf{C}$, then

$$(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3)(\beta_1 x_1 + \beta_2 x_2) = \gamma \cdot 1$$

for some $\gamma \in \mathbf{C}$. By expanding the left-hand side and comparing coefficients of the five linearly independent vectors $y_1, y_2, y_3, y_4, 1$, one sees that this implies $\beta_1 = \beta_2 = 0$.

In addition $(\mathbf{C}[z], \mathbf{C}[z])$ has no finite-dimensional, non-zero direct summands. (In fact, it is indecomposable.) Indeed, if (P, Q) is a direct summand of $(\mathbf{C}[z], \mathbf{C}[z])$ the fact that $e_1 \in \mathbf{C}^3$ acts as the identity on $\mathbf{C}[z]$ implies that $P = Q$. If $0 \neq f(z) \in P$, then, by the action of e_2 on $\mathbf{C}[z]$, we see that $z^n f(z) \in P$

https://doi.org/10.4153/CJM-1974-068-8 Published online by Cambridge University Press

for all n. Thus P is not finite-dimensional. (One could show that in fact $P = \mathbf{C}[z]$.)

Finally, we check that $(\mathbf{C}[z], \mathbf{C}[z])$ is not a direct summand of (X, Y). The equation

$$(e_1 + 2e_2 + e_3)\xi_1 + (e_1 - e_3)\xi_2 = 1$$

with $1 \in \mathbf{C}[z]$ is solved by $\xi_1 = x_1$, $\xi_2 = x_2$ in X. If $(\mathbf{C}[z], \mathbf{C}[z])$ were a direct summand in (X, Y) one could also solve this equation by some $f_1(z), f_2(z) \in \mathbf{C}[z]$ (see [**3**, Proposition 3.4]). However $(e_1 + 2e_2 + e_3)f_1(z) + (e_1 - e_3)f(z) = (1 + 2z + z^2)f_1(z) + (1 - z^2)f_2(z)$, which is never 1 due to the common divisor $1 + z$ of $1 + 2z + z^2$ and $1 - z^2$.

Suppose now that $N > 3$. Let $\{e_j\}_{j=1}^N$ be a basis of $\mathbf{C}^N$. We replace the former $\mathbf{C}^3$-system (X, Y) by the $\mathbf{C}^N$-system (X, W), where W is the external direct sum of Y with $N - 3$ copies of X. We define the system operation in (X, W) by

$$\left(\sum_{j=1}^{N} \alpha_j e_j\right)x = ((\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3)x, \alpha_4 x, \ldots, \alpha_N x),$$

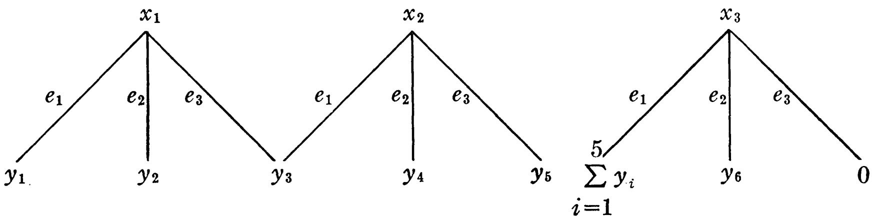
where the first component on the right hand side is computed as in (X, Y). If

$$Z = \{(t, s_4, \ldots, s_N) : t \in T;\ s_4, \ldots, s_N \in S\},$$

then (S, Z) is a subsystem of (X, W). It is easy to verify that (S, Z) and (X, W) provide the required counterexample for the case of $\mathbf{C}^N$-systems.

Example 2. Now we negate again the tentative generalization of Theorem 1, this time by use of a finite-dimensional $\mathbf{C}^3$-system.

Let X be a 3-dimensional complex space with a base $\{x_1, x_2, x_3\}$ and Y a 6-dimensional space with a base $\{y_1, \ldots, y_6\}$. Let $\{e_1, e_2, e_3\}$ be a base of $\mathbf{C}^3$. We determine a $\mathbf{C}^3$-system (X, Y) by the following diagram



and extending these assignments linearly to an action of $\mathbf{C}^3$ from X to Y.

Clearly, the subsystem $(\mathbf{C}x_3, \mathbf{C}\sum_{i=1}^5 y_i + \mathbf{C}y_6)$ of (X, Y) is not torsion-free, and, being indecomposable, does not have any finite-dimensional, torsion-free, indecomposable, non-zero direct summands. The quotient $(X, Y)/(\mathbf{C}x_3, \mathbf{C}\sum_{i=1}^5 y_i + \mathbf{C}y_6)$ is torsion-free. For if $0 \neq \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \in \mathbf{C}^3$ and $\beta_1(x_1 + \mathbf{C}x_3) + \beta_2(x_2 + \mathbf{C}x_3) \in X/\mathbf{C}x_3$ are such that

$$(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3)(\beta_1(x_1 + \mathbf{C}x_3) + \beta_2(x_2 + \mathbf{C}x_3)) = 0$$

in $Y/(\mathbf{C}\sum_{i=1}^{5}y_i + \mathbf{C}y_6)$, then

$$(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3)(\beta_1 x_1 + \beta_2 x_2) = \gamma \sum_{i=1}^{5} y_i$$

for some $\gamma \in \mathbf{C}$. Expansion of the left-hand side and comparison of coefficients of the 6 independent $y_1, \ldots, y_6$ reveals that $\beta_1 = \beta_2 = \gamma = 0$ and proves our claim. Thus $(\mathbf{C}x_3, \mathbf{C}\sum_{i=1}^{5}y_i + \mathbf{C}y_6)$ in (X, Y) satisfies all the hypotheses of the proposed generalization of Theorem 1.

However, $(\mathbf{C}x_3, \mathbf{C}\sum_{i=1}^{5}y_i + \mathbf{C}y_6)$ is not a direct summand (equivalent to pure in this case) in (X, Y). This follows from the fact that the equation

$$(e_1 + e_2)\xi_1 + (e_1 + e_2 + e_3)\xi_2 = \sum_{i=1}^{5} y_i$$

can be solved by x_1, x_2 in X but not by a pair $\lambda_1 x_3$, $\lambda_2 x_3$ in $\mathbf{C}x_3$. Indeed, the equation $(e_1 + e_2)\lambda_1 x_3 + (e_1 + e_2 + e_3)\lambda_2 x_3 = \sum_{i=1}^{5} y_i$ would imply the contradictory equations $0 = \lambda_1 + \lambda_2 = 1$.

The procedure outlined at the end of Example 1 can be used also here to provide a counterexample with a finite-dimensional $\mathbf{C}^N$-system where $N > 3$.

References

1. N. Aronszajn, *Quadratic forms on vector spaces*, Proc. Intern. Symposium on Linear Spaces 1960, Jerusalem, 1961.
2. N. Aronszajn and R. D. Brown, *Finite-dimensional perturbaions of spectral problems and variational approximation methods for eigenvalue problems. I: Finite-dimensional perturbations*, Studia Math. *36* (1970), 1–76.
3. N. Aronszajn and U. Fixman, *Algebraic spectral problems*, Studia Math. *30* (1968), 273–338.
4. S. E. Dickson, *A torsion theory for abelian categories*, Trans. Amer. Math. Soc. *121* (1966), 223–235.
5. U. Fixman, *On algebraic equivalence between pairs of linear transformations*, Trans. Amer. Math. Soc. *113* (1964), 424–453.
6. F. A. Zorzitto, *Topological decompositions in systems of linear transformations*, Ph.D. Thesis, Queen's University, 1972.
7. ——— *Purity and copurity in systems of linear transformations* (to appear).

Queen's University,
Kingston, Ontario