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Bifilar Suspension treated by the method of Contour Lines.

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The object of the following paper is to illustrate how readily some problems, which frequently occur in physical work, may be solved by the application of certain methods which are not very generally employed in mathematical investigations. The method on which the following proof is based is that of contour lines. These are a device enabling us to represent the third dimension on a plane, and are such that all the points on one contour line are at one and the same height above the level of the base plane. Probably the best known application of the principle of contours is to be found in the Survey Maps, where successive (closed) curves pass through all points at heights of 100, 200, 300, &c., feet above the sea-level; but they are of equal value in many physical diagrams, where they appear as equipotential lines, isothermals, lines of equal illumination, lines of equal force in a magnetic field, as well as in many other forms. In this paper they are employed as representing the third dimension merely. When contour lines are placed so as to indicate an equal rise in the intervals between each successive pair [or, in fact, according to any definite system], we can, with their aid, tell two things about a surface; the one—what is the steepness, or gradient, in any direction from a given point; the other—which is the direction of steepest slope at any point. The absolute steepness is measured by the amount of vertical motion per unit of horizontal motion of a point; *i.e.*, by the tangent of the angle between the horizontal plane and the line in which the point moves. This is a fact with which everyone is acquainted; every person knows what is meant by saying that the gradient along a line of railway is 1 in 50 or 1 in 500, as the case may be. To find the steepness at any point we merely need to know, in terms of the length of the line taken to represent unit steepness, the length of the line passing through the point in the given direction, and terminated by the two contours on either side of that point. The length of the line will then be *inversely* as the steepness.

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The direction of steepest slope—the "Stream-line"—is that which cuts the contours at right angles. This is obvious when we consider that such a line is the shortest which can be drawn between two contours, one very close on each side of the given point and passing through the given point.

We may now proceed to apply the principles indicated above. and first we shall consider the case of the intersection of a vertical cylinder with an inclined plane (fig. 37). To save employing a rather lengthy phrase, I shall call the actual line of intersection of the cylinder with the plane, or other figure under discussion, the "path," and its representation on the contour diagram the "projected path." A point on the path and its projection are said to correspond.

In a plane the contours marking equal increments of height are equidistant, parallel, straight lines, and therefore the *direction* of maximum steepness is the same at any point in the plane, and the actual *maximum* steepness is the same for every point.

Hence, in the intersection of a plane and cylinder, *wherever* the projected path, which is obviously circular, is situated, the steepest part will always be in the same direction, and will be that which cuts the contours at right angles. It will, therefore, be situated at a point distant from that of no steepness by one quarter of the circumference of the projected path. The direction of motion at that point of the path is that of the diameter joining the highest and lowest points of the path, *i.e.*, of the shortest, and therefore the steepest line joining them.

Further, the measure of the steepness of the path at any intermediate point, in terms of the steepness at the point of maximum gradient, will be inversely as the length of the tangent to the corresponding point intercepted between two contours is to the length of the tangent at the steepest point similarly intercepted; or, if these distances be represented by l and l', inversely as $\frac{l}{l'}$; *i.e.*, as $\frac{l'}{l}$. But $\frac{l'}{l}$ is the sine of the angle subtended at the centre by that part which lies between the point of no steepness and the point under consideration Hence, if we call the maximum steepness unity, the steepness of the path varies from point to point, starting from that of no steepness, as the sine of the angle through which the corresponding point has turned.

The next case which may be taken as leading up to the final one is that of the intersection of a cylinder with a cone (fig. 38). In a cone the shape of the contours will, of course, be changed. They now are a series of concentric circles whose radii increase in A.P.—the common centre being the point of projection of the axis of the cone. Under these circumstances the perpendicular distance between any one pair of contours is the same as that between any other, and therefore every point has the same value for maximum gradient. Hence, when we have a cylinder intersecting a cone (both axes being supposed vertical), the projected path is steepest at its point of orthogonal intersection with the contours. The position of this point depends on the distance between the two axes, in terms of the cylinder radius. When the distance is great, the point of maximum steepness approaches that of the path traced by the intersecting plane and cylinder; when the distance is diminished, the point approaches that position of minimum (zero) steepness which is nearest the axis of the cone, and eventually, when the circumference of the cylinder cuts the axis of the cone, this point lies on the axis itself.

Lastly, if we take the intersection of a cylinder and a sphere (fig. 39), and consider it by the same method, we note that the contours of a sphere are concentric circles, having the projection of the axis (vertical) of the sphere for their common centre; but they are no longer spaced equally—they become more and more close to each other as they recede. In fact, since the steepness from point to point along the sphere varies as the tangent of the angle at the centre of the sphere between the axis and the line joining that point with the centre, the spacing of the contours varies from the centre as the cotangent of the same angle [θ].

Now, it is obvious that, until the point of orthogonal intersection is reached, both the slope of the sphere and the curvature of the cylinder conspire to increase the steepness of the path, and, thereafter, the increasing slope of the sphere tends still to increase the steepness, while the motion of the point moving round its path draws it away from the line of maximum steepness,—that perpendicular to the contours,—which it had at the instant of orthogonal intersection.

But we have already seen that the increase of steepness, due to the changing slope of the sphere, varies as tan θ, and also it is easy to see that the steepness of the path depends on a function of the angle through which the point turns round the cylinder. This function has a value equal to unity at the point of orthogonal intersection, and to zero at the lowest point of the path. If, then, we call this function $f(\phi)$, the steepest point on the path is that at which

$\theta \times f(\phi)$ is a maximum. This can be found when we know the relations between θ and ϕ, which is the case when we know the radius of the sphere, the radius of the cylinder, and the distance between their axes.

If the radius of the sphere be very large compared with the radius of the cylinder, we may consider—for all practical cases—that the path is formed by the intersection of the cylinder with a plane which is the tangent plane to the sphere at the point where the axis of the cylinder cuts it.

Now, to apply these facts to bifilar suspension, we need only add to what we have already observed, the fact that the steeper a track is, the greater is the force required to drive a body of given mass up it against gravity. If we suppose a force tends to drive a body along the track of intersection of a sphere and a cylinder, or, since in all practical instruments the radius of the sphere is very large compared with that of the cylinder, the intersection of a plane and a cylinder, then by the length of the track over which the force was able to drive the body, we could measure the force. Forces of every amount from zero up to that which moves the body one quarter of the way round the track, can thus be measured by such an instrument; [all the forces less than that being, at some definite point or other, balanced by the component of gravity acting in the opposite direction *down* the path.] Such an instrument might be constructed by making a bar capable of rotating round, and sliding up and down a vertical rod; while to the further end of the bar is attached a cord whose other end is fixed to some point above the bar, and not in line of the vertical rod. In this case, evidently the lower end of the cord can move only on the surface of a sphere, having for centre the point of attachment: while, at the same time, because it is there fastened to the bar, it must move along the surface of a cylinder, and, therefore, it traces out the line common to both.

The inevitable friction involved in such a form of apparatus would, however, render it practically useless, and, therefore, the following substitute is employed. A bar has a cord attached to each end, these cords are of the same length and are fixed at equal heights above the bar, *i.e.*, in the same horizontal line. If, then, a vertical line be drawn through the centre of the bar, it will be the axis of symmetry of the system, and the bar will rotate round it and move up and down it as if it were a vertical rod passing through the bar. Hence the two ends of the bar move respectively along the lines of intersection of a cylinder with two equal and similarly situated spheres, the diameter of the cylinder being the length of the bar, and the radius of each sphere the length of either cord.

The proof, then, may be given concisely as follows:—The point of attachment of the moving bar traces out the intersection of a cylinder with a sphere, this—in practical cases—is very approximately that of a cylinder with an inclined plane; the steepness in this case is as the sine of the angle through which the bar is turned. This is directly as the magnitude of the force tending to turn it.

The ordinary trigonometrical treatment may be found in Wiedemann's Galvanismus, B. II., Th. I., s. 289.

Abstract of one of Euler's papers.

By J. S. Mackay, M.A.

The paper is entitled *Solutio facilis problematum quorumdam geometricorum difficillimorum*, and is printed in *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae*, Tom. xi., pp. 103-123. The volume is for the year 1765; the title-page is dated 1767.

The investigation is concerned with a plane triangle and its four points the orthocentre, the centroid, the inscribed centre, and the circumscribed centre.

Not to complicate a single figure with too many lines, four figures are exhibited. In fig. 40, AM, BN, CP are the perpendiculars from A, B, C on the opposite sides, and E is the orthocentre. In fig. 41, Aa, Bb, Cc are the medians from A, B, C, and F is the centroid; FQ and CP are perpendicular to AB. In fig. 42, Aα, Bβ, Cγ are the bisectors of angles A, B, C, and G is the inscribed centre; GR is perpendicular to AB. In fig. 43, S, T, V are the middle points of AB, BC, CA, and SH, TH, VH respectively perpendicular to those sides meet in H the circumscribed centre; AM, CP are perpendicular to BC, AB, and AH is joined.

The sides of the triangle BC, CA, AB are denoted respectively by a, b, c, [see the notation for triangle FGH on p. 54] and the area by A.

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