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ASYMPTOTIC BEHAVIOUR OF THE LEAST ENERGY SOLUTIONS TO FRACTIONAL NEUMANN PROBLEMS

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Abstract

We study the asymptotic behaviour of the least energy solutions to the following class of nonlocal Neumann problems:

$$\begin{cases} d(-\Delta)^s u + u = |u|^{p-1}u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \mathcal{N}_s u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain of class $C^{1,1}$, $1 < p < (n+s)/(n-s)$, $n > \max\{1, 2s\}, 0 < s < 1$, $d > 0$ and $\mathcal{N}_s u$ is the nonlocal Neumann derivative. We show that for small d, the least energy solutions u_d of the above problem achieve an L^∞-bound independent of d. Using this together with suitable L^r-estimates on u_d, we show that the least energy solution u_d achieves a maximum on the boundary of Ω for d sufficiently small.

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1. Introduction

We discuss the asymptotic behaviour of nonconstant least energy solutions of the following problem:

$$\begin{cases} d(-\Delta)^s u + u = |u|^{p-1}u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \mathcal{N}_s u = 0 & \text{in } C\overline{\Omega}, \end{cases} \tag{1-1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain of class $C^{1,1}$, $1 < p < (n+s)/(n-s)$, $n > \max\{1, 2s\}, 0 < s < 1, d > 0$, $C\Omega := \mathbb{R}^n \setminus \Omega$ and $\mathcal{N}_s u$ is the nonlocal Neumann derivative, which is defined next. The nonlocal operator $(-\Delta)^s$ is called the fractional Laplacian, which is defined for smooth functions as follows:

$$(-\Delta)^s u(x) = c_{n,s}\, \mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy. \tag{1-2}$$

Here, by PV, we mean the Cauchy principal value and $c_{n,s}$ is a normalising constant, given by

$$c_{n,s} = \left(\int_{\mathbb{R}^n} \frac{1 - \cos x_1}{|x|^{n+2s}}\, dx \right)^{-1};$$

see for instance [12] for the details. Recently, Dipierro *et al.* [14] have introduced a new nonlocal Neumann condition $\mathcal{N}_s$, which is defined as follows:

$$\mathcal{N}_s u(x) := c_{n,s} \int_{\Omega} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy, \qquad x \in C\overline{\Omega}.$$

The advantage of this nonlocal Neumann condition is that it has simple probabilistic interpretation and (1-1) has a variational structure. Further, it naturally arises from the superposition of Brownian and Lévy processes; see [16] for the details. We recall that $\mathcal{N}_s u$ approaches the classical Neumann derivative $\partial_\nu u$ as s goes to 1.

In the last few decades, mathematical analysis of biological phenomena has gained much attention. For example, chemotaxis models, which are also known as Keller–Segel models [28], have been widely studied in different directions in many papers; see [3, 24, 25] for a survey on this subject. Chemotaxis refers to the movement of cells or organisms in response to chemical gradients in their environment. The analysis on the steady-state for a chemotactic aggregation model with linear or logarithmic sensitivity function was thoroughly done in many papers; see for instance [27, 31, 35].

Let us point out that the following semilinear Neumann problem is an example of the Keller–Segel model with a logarithmic chemotactic sensitivity:

$$\begin{cases} -d\Delta u + u = |u|^{p-1}u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-3}$$

where $d > 0$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and $1 < p \leq (n+2)/(n-2)$ if $n \geq 3$ and $1 < p < \infty$ if $p = 2$; see [31] for the details. Problem (1-3) admits a nonconstant solution for d sufficiently small; see [1, 30, 31]. Lin *et al.* [31] and Lin and Ni [30] established the existence of solutions to (1-3) in the subcritical case $1 < p < (n+2)/(n-2)$. In the critical case, when $p = (n+2)/(n-2)$,

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Adimurthi and Mancini [1] obtained a solution of (1-3). There have been developments on the asymptotic behaviour of solutions to such equations. In the subcritical case, $1 < p < (n+2)/(n-2)$, Ni and Takagi [34, 35] have studied the shape of the least energy solutions of (1-3). They have shown that the least energy solutions tend to zero as the diffusion constant d goes to zero except at a finite number of points. Moreover, the maximum of a solution u_d of (1-3) is attained at a unique point on the boundary of Ω. The critical case was examined by Adimurthi *et al.* [2] using blow-up analysis. We refer to [23] for the existence, nonexistence and the asymptotic behaviour of solutions to fractional Choquard equations with local perturbations.

We mention that Problem (1-1), which we explore in this paper is a nonlocal analogue of the classical problem (1-3).

The substitution of standard diffusion with fractional diffusion is a perceived approach in modelling feeding procedures across a wide range of organisms. In many situations observed in nature, Lévy flights are often used as an accomplished search strategy by living organisms [5, 29]. Since the fractional Laplacian $(-\Delta)^s$ is an infinitesimal generator of a Lévy process, dispersal is better modelled by the nonlocal operator $(-\Delta)^s$. The generalised Keller–Segel model with nonlocal diffusion term $d(-\Delta)^s$, where d is a positive constant is used to investigate chemotaxis with anomalous diffusion. For the fractional Keller–Segel model, we refer to [18, 26]. In [26], Huang and Liu studied the existence, stability, uniqueness and regularity of solutions for the following model in dimension $n \geq 2$:

$$\begin{cases} u_t = d(-\Delta)^s u - \nabla \cdot (u \nabla \phi), & x \in \mathbb{R}^n, \quad t \geq 0, \\ -\Delta \phi = u, \\ u(x,0) = u_0(x), \end{cases}$$

where d is a positive constant, $u(t,x)$ is the density of some biological cells and $\phi(t,x)$ is the chemical substance concentration. We mention the work [9], where the authors have investigated the asymptotic behaviour of solutions for nonlinear elliptic problems for fractional Laplacians with Dirichlet boundary conditions. We refer to [15] for the regularity, monotonicity and other results on fractional equations in Lipschitz sets, [22] for the existence of solutions to critical Neumann problems and [32] for an in-depth treatment of variational methods to nonlocal fractional problems.

Motivated by the above literature, the works on the fractional Laplacian [33, 36, 38, 39] and the very recent works on the nonlocal Neumann problem for fractional Laplacians and its connections with fractional Keller–Segel models, we have the following natural question to ask.

QUESTION. Can we establish the asymptotic behaviour of the least energy solutions of (1-1)?

The aim of this paper is to answer the above question. More precisely, we discuss the asymptotic behaviour of the least energy solutions of (1-1).

A weak solution of (1-1) can be obtained as a critical point of the following energy functional J_d:

$$J_d(u) := \frac{1}{2}\Big[\frac{dc_{n,s}}{2}\int_{T(\Omega)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega u^2\,dx\Big] - \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx, \quad u \in H^s_\Omega.$$

In the above equation, $T(\Omega) = \mathbb{R}^{2n} \setminus (C\Omega)^2$ and the space H^s_Ω is defined in (2-1). The functional J_d is well defined and of class C^2 by Theorem 2.1, stated next. An application of the *Mountain-Pass lemma* applied to the functional J_d yields that

$$c_d := \inf_{\gamma\in\Gamma}\max_{[0,1]} J_d(\gamma(t)) \tag{1-4}$$

is a critical value of J_d. In the above equation, by Γ, we mean the following set:

$$\Gamma = \{\gamma \in C([0,1]; H^s_\Omega) \mid \gamma(0) = 1,\ \gamma(1) = u\},$$

where $u \in H^s_\Omega$, and $u > 0$ satisfies $J_d(u) = 0$. It turns out that c_d is the least positive critical value; see Lemma 3.3. For the details, one may refer to [4, Theorem 6.1] and [7, Theorem 1.1], where the authors have obtained a nonnegative weak solution u_d of (1-1) with critical value c_d, provided d is sufficiently small. Moreover, u_d satisfies

$$0 < J_d(u_d) \le Cd^{n/2s},$$

where the constant C is independent of d. Consequently, u_d is nonconstant. From the proof of [7, Theorem 1.1], it is immediate to see that the critical points of J_d are not sign-changing in Ω. In fact, when $u_d \le 0$, we can choose $-u_d$ to have a nonnegative solution of (1-1). By the strong maximum principle (see [10, Theorem 2.6]), one can see that $u_d > 0$ almost everywhere (a.e.) in Ω. Further, since u_d satisfies the Neumann condition, $\mathcal{N}_s u_d(x) = 0$ in $C\Omega$, which implies that $u_d > 0$ a.e. in $\mathbb{R}^n$.

DEFINITION 1.1. We call a critical point u_d of J_d with $J_d(u_d) = c_d$ the *least energy* solution or *Mountain-Pass* solution of (1-1).

We show the asymptotic behaviour of the least energy solutions of (1-1) following a similar approach to that of Ni and Takagi [35] for (1-3). They used a positive solution w of the nonlinear Schrödinger equation

$$-\Delta u + u = |u|^{p-1}u \text{ in } \mathbb{R}^n, \quad 1 < p < \frac{n+2}{n-2}$$

to study the asymptotic behaviour of the least energy solutions of (1-3). The fractional nonlinear Schrödinger equation

$$(-\Delta)^s u + u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n, \tag{1-5}$$

where $1 < p < (n+2s)/(n-2s), n > \max\{1, 2s\}$, $0 < s < 1$, is thoroughly studied; see for instance [8, 13, 20, 21] and the references therein.

Let us discuss the main idea of this work, which goes as follows.

Let c_d be the critical value of J_d, which is defined in (1-4). Following the arguments of [35], we use a positive solution w of (1-5) to observe the asymptotic behaviour of

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c_d as $d \downarrow 0$. More specifically, w is used to build a suitable function ϕ_d to compare c_d with $\max_{t\geq 0} J_d(t\phi_d)$. In particular, we obtain an inequality

$$c_d < \frac{d^{n/2s}}{2} F(w)$$

for d sufficiently small, where F is the functional associated with (1-5), defined in (2-2). This is closely related to the location of the maximum point of a solution u_d of (1-1) on the boundary of Ω.

Now, we summarise the above discussion in terms of the following three main theorems. *A priori*, it is known that for $1 \leq p < (n+s)/(n-s)$, any weak solution u of (1-1) satisfies

$$\|u\|_{L^\infty(\Omega)} \leq K,$$

where $K > 0$ is some constant depending on Ω, p and d; see [33, Theorem 3.1]. In the next result, we obtain a bound for the least energy solution u_d of (1-1), which is independent of d.

THEOREM 1.2. *Let u_d be the least energy solution of (1-1). Then*

$$d\frac{c_{n,s}}{2}\int_{T(\Omega)} \frac{|u_d(x) - u_d(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega u_d^2\,dx = \int_\Omega u_d^{p+1}\,dx \leq C_0 d^{n/2s}, \tag{1-6}$$

where $C_0 > 0$ is some constant depending on p. Moreover, there is a constant $C_1 > 0$ depending only on p and Ω such that

$$\sup_\Omega u_d(x) \leq C_1.$$

In the next theorem, we show that the L^r-norm of the least energy solution u_d is bounded by $d^{n/2s}$ times some constant independent of d.

THEOREM 1.3. *Let u_d be the least energy solution of (1-1). Then*

$$b(r)d^{n/2s} \leq \int_\Omega u_d^r\,dx \leq B(r)d^{n/2s} \quad \text{if } 1 \leq r \leq \infty, \tag{1-7}$$

$$b(r)d^{n/2s} \leq \int_\Omega u_d^r\,dx \leq B(r)d^{nr/2s} \quad \text{if } 0 < r < 1, \tag{1-8}$$

where $b(r)$ and $B(r)$ are positive constants such that $b(r) < B(r)$ and are independent of d.

We show the asymptotic behaviour in the next theorem.

THEOREM 1.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^{1,1}$. Let u_d be the least energy solution of (1-1). If u_d achieves a maximum at a point $z_d \in \overline{\Omega}$, then for all d sufficiently small, we have $z_d \in \partial\Omega$.*

The plan of the paper is as follows. In Section 2, we recollect known results that are useful for our analysis. In Section 3, we study the regularity of the least energy solution

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of (1-1) and complete the proof of Theorem 1.2. In Section 4, we derive L^r-estimates for the least energy solutions of (1-1). Section 5 contains the proof of Theorem 1.4. The proof of inequality (3-8) is a part of Appendix A.

2. Auxiliary results

Let us recall some important results that are used in this paper.

THEOREM 2.1 (Fractional Sobolev embedding [12]). *Let $n > 2s$ and $2_s^* = 2n/(n-2s)$ be the fractional critical exponent. Then, we have the following inclusions.*

(1) *For any function $u \in C_0(\mathbb{R}^n)$ and for $q \in [0, 2_s^* - 1]$,*

$$\|u\|^2_{L^{q+1}(\mathbb{R}^n)} \leq B(n,s) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy$$

for some positive constant B. That means $H^s(\mathbb{R}^n)$ is continuously embedded in $L^{q+1}(\mathbb{R}^n)$.

(2) *Let $\Omega \subset \mathbb{R}^n$ be a bounded extension domain for $H^s(\Omega)$. Then, the space $H^s(\Omega)$ is continuously embedded in $L^{q+1}(\Omega)$ for any $q \in [0, 2_s^* - 1]$, that is,*

$$\|u\|^2_{L^{q+1}(\Omega)} \leq B(n,s,\Omega)\|u\|^2_{H^s(\Omega)}$$

for some positive constant B. Further, the above embedding is compact for any $q \in [0, 2_s^* - 1)$.

Let $T(\Omega) := \mathbb{R}^{2n} \setminus (\mathbb{R}^n \setminus \Omega)^2$ be a cross-shaped set on a bounded domain $\Omega \subset \mathbb{R}^n$. Define

$$H^s_\Omega := \{u : \mathbb{R}^n \longrightarrow \mathbb{R} \text{ measurable} : \|u\|_{H^s_\Omega} < \infty\}, \tag{2-1}$$

which is equipped with the norm

$$\|u\|_{H^s_\Omega} := \left(\|u\|^2_{L^2(\Omega)} + \int_{T(\Omega)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2}.$$

REMARK 2.2. Here, H^s_Ω is a Hilbert space (see [14, Proposition 3.1]).

Let us define the following set:

$$\mathcal{L}_s := \left\{u : \mathbb{R}^n \longrightarrow \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}}\,dx < \infty\right\}.$$

The condition $u \in \mathcal{L}_s$ is useful to give a sense to the pointwise definition of fractional Laplacians (1-2).

LEMMA 2.3 [10, Lemma 2.3]. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set. Then, $H^s_\Omega \subset \mathcal{L}_s$.*

Next, we recall a few known results about the fractional Schrödinger equation (1-5).

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DEFINITION 2.4. A measurable function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called a weak solution of (1-5) if it satisfies the following equation:

$$\frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_{\mathbb{R}^n} u(x)\psi(x)\,dx = \int_{\mathbb{R}^n}|u(x)|^{p-1}u(x)\psi(x)\,dx$$

for all $\psi \in C_0^1(\mathbb{R}^n)$.

We define the corresponding energy functional $F : H^s(\mathbb{R}^n) \longrightarrow \mathbb{R}$ as follows:

$$F(u) := \frac{1}{2}\Big[\frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_{\mathbb{R}^n} u^2\,dx\Big] - \frac{1}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}\,dx. \tag{2-2}$$

The weak solutions of (1-5) correspond to the critical points of F.

DEFINITION 2.5. A function $u \in \mathcal{L}_s(\mathbb{R}^n) \cap C^{2s+\epsilon}(\mathbb{R}^n)$, when $0 < s < \frac{1}{2}$, $2s + \epsilon < 1$, or $u \in C^{1,2s+\epsilon-1}(\mathbb{R}^n) \cap \mathcal{L}_s(\mathbb{R}^n)$, when $\frac{1}{2} \le s < 1$, $2s + \epsilon - 1 < 1$, is said to be a classical solution of (1-5) if it satisfies (1-5) pointwise in $\mathbb{R}^n$.

The next result gives us a positive, radially symmetric solution of (1-5), which decays at infinity.

THEOREM 2.6 [20, Theorem 3.4]. *Let u be the weak solution of (1-5). Then, $u \in L^q(\mathbb{R}^n) \cap C^\alpha(\mathbb{R}^n)$ for some $q \in [2,\infty)$ and $\alpha \in (0,1)$. Moreover,*

$$\lim_{|x|\to\infty} u(x) = 0.$$

THEOREM 2.7 [20, Theorem 1.3]. *Equation (1-5) has a weak solution in $H^s(\mathbb{R}^n)$, which satisfies $u \ge 0$ a.e. in $\mathbb{R}^n$. Moreover, u is a classical solution, which satisfies $u > 0$ in $\mathbb{R}^n$.*

The following theorem shows that the solutions of (1-5) have a power type of decay at infinity.

THEOREM 2.8 [20, Theorem 1.5]. *Let u be a positive classical solution of (1-5) such that*

$$\lim_{|x|\to\infty} u(x) = 0.$$

Then, there exist constants $0 < C_1 \le C_2$ such that

$$\frac{C_1}{|x|^{n+2s}} \le u(x) \le \frac{C_2}{|x|^{n+2s}} \quad \text{for all } |x| \ge 1.$$

One can see that there exist some $m > 0$ and $s_0 > 0$ such that for $f(u) = u^p - u$,

$$\frac{f(v)-f(u)}{v-u} \le \frac{v^p-u^p}{v-u} \le C(v+u)^m \quad \text{for all } 0 < u < v < s_0,$$

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where $C > 0$ is some constant. Also, it is simple to see that $f : [0, \infty) \to \mathbb{R}$ is locally Lipschitz. Consequently, we have the following result on the radial symmetry and monotonicity property of positive solutions of (1-5).

THEOREM 2.9 [21, Theorem 1.2]. *Let u be a positive classical solution of (1-5) such that*

$$\lim_{|x|\to\infty} u(x) = 0.$$

Further, assume that there exists

$$t > \max\left\{\frac{2s}{m}, \frac{n}{m+2}\right\}$$

such that u satisfies $u(x) = O(1/|x|^t)$ as $|x| \to \infty$. Then, u is radially symmetric and strictly decreasing about some point in $\mathbb{R}^n$.

REMARK 2.10. Since

$$\frac{C_1}{|x|^{n+2s}} \le u(x) \le \frac{C_2}{|x|^{n+2s}} \quad \text{for all } |x| \ge 1,$$

we can take $t = n + 2s$ in the above theorem.

Now, [37, Proposition 4.1] ascertains that if $u \in \mathbb{R}^n$ is a weak solution of (1-5), then u satisfies the following Pohozaev identity:

$$\mathcal{P}(u) := \frac{(n-2s)c_{n,s}}{4} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \frac{n}{2}\int_{\mathbb{R}^n} u^2\,dx - \frac{n}{p+1}\int_{\mathbb{R}^n} u^{p+1} = 0.$$

Let us define

$$\mathcal{G} := \{u \in H^s(\mathbb{R}^n) \setminus \{0\} \mid \mathcal{P}(u) = 0\}.$$

In [8], the authors have obtained a weak solution $w \in H^s(\mathbb{R}^n)$ of (1-5) with the least energy among all other solutions. In particular, they have proved the following result.

THEOREM 2.11 [8, Theorem 1.2]. *Equation (1-5) has a weak solution $w \in H^s(\mathbb{R}^n)$ such that*

$$0 < F(w) = \inf_{u\in\mathcal{G}} F(u).$$

Combining Theorems 2.7, 2.8, 2.9 and 2.11, we have the following result.

THEOREM 2.12. *Equation (1-5) has a positive classical solution $w \in H^s(\mathbb{R}^n)$ satisfying:*

(a) *w has a power type of decay at infinity, that is, there exist constants $0 < C_1 \le C_2$ such that*

$$\frac{C_1}{|x|^{n+2s}} \le w(x) \le \frac{C_2}{|x|^{n+2s}} \quad \textit{for all } |x| \ge 1;$$

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(b) *w is radially symmetric, that is,* $w(x) = w(r)$ *with* $r = |x|$;
(c) *for any nonnegative classical solution* $u \in H^s(\mathbb{R}^n)$ *of (1-5),* $0 < F(w) \leq F(u)$ *holds unless* $u = 0$.

DEFINITION 2.13. We call w, given by Theorem 2.12, a ground state solution of (1-5).

3. Regularity and bounds for the least energy solution u_d

Let $s \in (0, 1)$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^{1,1}$.

DEFINITION 3.1. A measurable function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be a weak solution of (1-1) if it satisfies the following equation:

$$\frac{dc_{n,s}}{2}\int_{T(\Omega)}\frac{(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega u(x)\psi(x)\,dx = \int_\Omega |u(x)|^{p-1}u(x)\psi(x)\,dx \tag{3-1}$$

for all $\psi \in H^s_\Omega$.

We have the following result on the existence of a weak solution of (1-1).

THEOREM 3.2 ([4, Theorem 6.1], [7, Theorem 1.1]). *There exists a nonnegative weak solution* u_d *of (1-1) with critical value* c_d*, provided* d *is sufficiently small. Moreover,* u_d *satisfies*

$$0 < J_d(u_d) \leq Cd^{n/2s},$$

where the constant C *is independent of* d*. Consequently,* u_d *is nonconstant.*

Define

$$M[v] := \sup_{t\geq 0} J_d(tv), \qquad v \in H^s_\Omega.$$

In the next lemma, we indicate a useful characterisation of the critical value c_d. We follow similar lines of proof to [35, Lemma 3.1].

LEMMA 3.3. *The critical value* c_d *is independent of the choice of* $u \in H^s_\Omega$ *such that* $u \geq 0$, $u \not\equiv 0$ *and* $J_d(u) = 0$*. In fact,* c_d *is the least positive critical value of* J_d*, which is given by*

$$c_d = \inf\{M[v] \mid v \in H^s_\Omega,\ v \not\equiv 0, v \geq 0 \text{ in } \Omega\}. \tag{3-2}$$

PROOF. For $v \in H^s_\Omega$, let

$$\Omega^+ = \{x \in \Omega \mid v(x) > 0\}.$$

Now, for all those v satisfying $|\Omega^+| > 0$, define

$$g_d(t) := J_d(tv) \quad \text{for } t \geq 0.$$

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First, we show that $g_d(t)$ has a unique maximum. For this,

$$g_d'(t) = t\Big[\frac{dc_{n,s}}{2}\int_{T(\Omega)}\frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega v^2\,dx\Big] - t^p\int_\Omega v^{p+1}\,dx.$$

Therefore, $g_d'(t_0) = 0$ for some $t_0 > 0$ if and only if

$$\frac{dc_{n,s}}{2}\int_{T(\Omega)}\frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega v^2\,dx = t_0^{p-1}\int_\Omega v^{p+1}\,dx.$$

Note that the right-hand side is strictly increasing in t_0. And hence there exists a unique $t_0 > 0$ such that $g_d'(t_0) = 0$. Since $g_d(t) > 0$ for $t > 0$ small and $g_d(t) \to -\infty$ as $t \to +\infty$, one easily find that $g_d(t)$ has a unique maximum.

Let us fix a function $u \not\equiv 0, u \geq 0$ in H^s_Ω with $J_d(u) = 0$. Let u_d be a positive solution of (1-1) obtained by applying the *Mountain-Pass lemma* and c_d the corresponding critical value. We have $J_d(u_d) = c_d$ and $J_d^{'}(u_d) = 0$. Since $u_d > 0$ and $J_d^{'}(u_d) = 0$,

$$M[u_d] = c_d, \tag{3-3}$$

and hence

$$c_d \geq \inf\{M[v] \mid v \in H^s_\Omega,\ v \not\equiv 0, v \geq 0 \text{ in } \Omega\}. \tag{3-4}$$

In contrast, assume that strict inequality occurs in (3-4). Then,

$$M[v_0] < c_d,$$

for some $v_0 \geq 0$, $v_0 \not\equiv 0$ in H^s_Ω. Therefore, there exists some $t_1 > 0$ such that $t_1 v_0 = u_0$ satisfies $J_d(u_0) = 0$. Denote by U the subspace of H^s_Ω spanned by u and u_0. Consider the subset of U defined as follows:

$$U^+ := \{\alpha u + \beta u_0 \mid \alpha,\ \beta \geq 0\}.$$

Suppose S is a circle on U of radius R so large that $R > \max\{\|u\|, \|u_0\|\}$ and $J_d \leq 0$ on $S \cap U^+$. Assume that γ is the path made up of the line segment with endpoints 0 and $Ru_0/\|u_0\|$, the circular arc $S \cap U^+$ and the line segment with endpoints $Ru/\|u\|$ and u. One can easily see that, along γ, J_d is positive only on the line segment joining 0 and u_0. Hence,

$$\max_{v\in\gamma} J_d(v) = M[v_0] < c_d,$$

which is a contradiction to (1-4). Thus, we have equality in (3-4), that is,

$$c_d = \inf\{M[v] \mid v \in H^s_\Omega,\ v \not\equiv 0, v \geq 0 \text{ in } \Omega\}.$$

Note that $J_d(v) = J_d(-v)$ for any $v \in H^s_\Omega$. Since any nontrivial critical point of J_d is either positive or negative a.e. in Ω, from the above discussion, one can see that c_d is the least positive critical value of J_d. This completes the proof. □

The following lemma gives us the regularity estimate. A similar result is already proved in [10, Lemma 3.6] and [11, Remark 4.9].

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LEMMA 3.4. *Let $u \in H^s_\Omega$ be a weak solution of (1-1). Let $u \in L^\infty(\Omega)$, then $u \in L^\infty(\mathbb{R}^n)$. Moreover:*

(1) *for $0 < s < \frac{1}{2}$, $u \in C^2(\Omega)$ if $p > 3 - 2s$ and $u \in C^{1,p-2+2s}(\Omega)$ if $2 < p \leq 3 - 2s$;*
(2) *for $\frac{1}{2} \leq s < 1$, $u \in C^2(\Omega)$.*

Now, we prove that the least energy solution u_d is bounded by some constant independent of d.

PROOF OF THEOREM 1.2. The proof of the first inequality of Theorem 1.2 is fairly standard and simple, and can be seen in the literature; for instance, see [8, Theorem 1.1]. Since it is short, for the sake of completeness, we include it here. For this,

$$J_d(u_d) := \frac{1}{2}\Big[\frac{c_{n,s}d}{2}\int_{T(\Omega)} \frac{|u_d(x) - u_d(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega u^2\,dx\Big] - \frac{1}{p+1}\int_\Omega u_d^{p+1}\,dx.$$

Since u_d is a critical point of J_d,

$$J'_d(u_d) = 0 \quad \text{on } H^s_\Omega.$$

This implies that

$$d\frac{c_{n,s}}{2}\int_{T(\Omega)} \frac{|u_d(x) - u_d(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega u_d^2\,dx = \int_\Omega u_d^{p+1}\,dx. \tag{3-5}$$

Hence, from the above equations,

$$\begin{aligned} J_d(u_d) &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_\Omega u_d^{p+1}\,dx \\ &= \frac{(p-1)}{2(p+1)}\int_\Omega u_d^{p+1}\,dx. \end{aligned} \tag{3-6}$$

Now, by Theorem 3.2, we have $J_d(u_d) \leq Cd^{n/2s}$, where the constant C depends only on p. Using this inequality in the above equation,

$$\int_\Omega u_d^{p+1}\,dx \leq \frac{2(p+1)}{p-1}Cd^{n/2s}.$$

Taking $C_0 = 2(p+1)/(p-1)C$ proves the first inequality of Theorem 1.2. The proof of the second inequality of Theorem 1.2 is a little constructive. We claim that

$$\sup_\Omega u_d(x) \leq C_1$$

for some constant $C_1 > 0$ depending on p and Ω only. Multiplying (1-1) by u_d^{2t-1} and integrating over Ω,

$$\frac{c_{n,s}d}{2}\int_{T(\Omega)} \frac{(u_d(x) - u_d(y))(u_d^{2t-1}(x) - u_d^{2t-1}(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega u_d^{2t}\,dx = \int_\Omega u_d^{p+2t-1}\,dx. \tag{3-7}$$

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Now, we use the following inequality. We give the proof of this inequality in the Appendix. Let $x, y \geq 0$ be real numbers and $k \geq 1$, then

$$\frac{1}{k}(x^k - y^k)^2 \leq (x-y)(x^{2k-1} - y^{2k-1}). \tag{3-8}$$

Consequently,

$$\frac{1}{t}\int_{T(\Omega)} \frac{(u_d^t(x) - u_d^t(y))^2}{|x-y|^{n+2s}}\,dx\,dy \leq \int_{T(\Omega)} \frac{(u_d(x) - u_d(y))(u_d^{2t-1}(x) - u_d^{2t-1}(y))}{|x-y|^{n+2s}}\,dx\,dy. \tag{3-9}$$

From (3-7) and (3-9),

$$\frac{dc_{n,s}}{2t}\int_{T(\Omega)} \frac{(u_d^t(x) - u_d^t(y))^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega u_d^{2t}\,dx \leq \int_\Omega u_d^{p+2t-1}\,dx. \tag{3-10}$$

Further, by the fractional Sobolev embedding (Theorem 2.1),

$$\left(\int_\Omega |v|^{2_s^*}\right)^{2/2_s^*} \leq \frac{A}{d}\left(d\frac{c_{n,s}}{2}\int_\Omega\int_\Omega \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega |v|^2\,dx\right), \tag{3-11}$$

where $d \in (0, d_0)$ for some $d_0 > 0$, $A > 0$ some constant, $v \in H^s(\Omega)$ and $2_s^* = 2n/(n-2s)$. The embedding constant A depends only on n, s, d_0 and Ω. To see this, let us define

$$\Omega_d := \left\{y : \frac{y}{d^{1/2s}} \in \Omega\right\} \quad \text{and} \quad w(y) := v\left(\frac{y}{d^{1/2s}}\right), \quad \text{where } y \in \Omega_d.$$

Now,

$$\begin{aligned}
d&\int_\Omega\int_\Omega \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega v^2\,dx \\
&= \frac{1}{d^{n/2s}}\left[\int_{\Omega_d}\int_{\Omega_d} \frac{|v(\frac{x'}{d^{1/2s}}) - v(\frac{y'}{d^{1/2s}})|^2}{|x'-y'|^{n+2s}}\,dx'\,dy' + \int_{\Omega_d} v\left(\frac{x'}{d^{1/2s}}\right)^2 dx'\right] \\
&= \frac{1}{d^{n/2s}}\left[\int_{\Omega_d}\int_{\Omega_d} \frac{|w(x')-w(y')|^2}{|x'-y'|^{n+2s}}\,dx'\,dy' + \int_{\Omega_d} w(x')^2\,dx'\right] \\
&\geq \frac{A}{d^{n/2s}}\left(\int_{\Omega_d} |w|^{2_s^*}\,dx'\right)^{2/2_s^*} \\
&= Ad^{(2/2_s^*-1)n/2s}\left(\int_\Omega |v|^{2_s^*}\,dx\right)^{2/2_s^*}.
\end{aligned}$$

Therefore, we observe that A is uniform for $d \in (0, d_0)$.

It is easy to see that $\Omega \times \Omega \subset T(\Omega)$. Then, by virtue of (3-10) and (3-11),

$$\left(\int_\Omega |u_d|^{t2_s^*}\right)^{2/2_s^*} \leq \frac{tA}{d}\int_\Omega u_d^{p+2t-1}\,dx. \tag{3-12}$$

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Now, we define two sequences $\{L_j\}$ and $\{M_j\}$ by the following recurrence relations:

$$\begin{aligned} p-1+2L_0 &= 2_s^*, \\ p-1+2L_{j+1} &= 2_s^* L_j, \quad j=0,1,2,\ldots \end{aligned} \tag{3-13}$$

$$\begin{aligned} M_0 &= (AC_0)^{2_s^*/2}, \\ M_{j+1} &= (AL_jM_j)^{2_s^*/2}, \quad j=0,1,2,\ldots \end{aligned} \tag{3-14}$$

We note that L_j is explicitly given by

$$L_j = \frac{1}{(2_s^*-2)}\bigg(\Big(\frac{2_s^*}{2}\Big)^{j+1}(2_s^*-p-1)+p-1\bigg). \tag{3-15}$$

Since $1 < p < 2_s^*-1$, it follows that $L_j \geq 1$ for all $j \geq 0$ and $L_j \to \infty$ as $j \to \infty$. We show that

$$\int_\Omega u_d^{p-1+2L_j}\,dx \leq M_j d^{n/2s} \quad \text{for all } j \geq 0, \tag{3-16}$$

and

$$M_j \leq e^{mL_{j-1}} \tag{3-17}$$

for some constant $m > 0$. Then,

$$\sup_\Omega u_d(x) \leq C_1,$$

where $C_1 > 0$ depends only on C_0 and Ω. In fact, (3-15) and (3-16) give

$$\begin{aligned} \|u\|_{L^{2_s^*L_{j-1}}(\Omega)} &\leq (e^{mL_{j-1}}d^{n/2s})^{1/(2_s^*L_{j-1})} \\ &= e^{m/2_s^*}d^{(n-2s)/4L_{j-1}} \end{aligned}$$

and hence letting $j \to \infty$,

$$\|u\|_{L^\infty(\Omega)} \leq e^{m/2_s^*}.$$

First, we verify (3-16). By virtue of (1-6) and (3-11),

$$\begin{aligned} \Big(\int_\Omega |u_d|^{2_s^*}\Big)^{2/2_s^*} &\leq \frac{A}{d}\Big(\frac{c_{n,s}d}{2}\int_{T(\Omega)}\frac{|u_d(x)-u_d(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega |u_d|^2\,dx\Big) \\ &\leq \frac{A}{d}C_0 d^{n/2s} \\ &= AC_0 d^{n/s2_s^*}. \end{aligned}$$

Hence, (3-16) holds for $j = 0$. Suppose that we have proved (3-16) for $j \geq 0$. Then, by (3-12),

$$\begin{aligned}\int_\Omega |u_d|^{p-1+2L_{j+1}}\,dx &\le \Big(\frac{L_jA}{d}\int_\Omega u_d^{p+2L_j-1}\,dx\Big)^{2_s^*/2}\\ &\le (AL_jd^{-1}M_jd^{n/2s})^{2_s^*/2}\\ &= (AL_jM_j)^{2_s^*/2}d^{n/2s}.\end{aligned}$$

This implies that (3-16) is also true for $j+1$. Therefore, it remains to show (3-17). Put

$$\lambda_j = \frac{2_s^*}{2}\cdot\log(AL_j) \quad\text{and}\quad \eta_j = \log(M_j). \tag{3-18}$$

Hence,

$$\eta_{j+1} = \frac{2_s^*}{2}\cdot\eta_j + \lambda_j.$$

The explicit value of L_j is given by

$$L_j = (2_s^*-2)^{-1}((2^{-1}2_s^*)^{j+1}(2_s^*-p-1)+p-1). \tag{3-19}$$

Now,

$$\begin{aligned}\lambda_j &= \frac{2_s^*}{2}\log\Big[\frac{A}{(2_s^*-2)}((2^{-1}2_s^*)^{j+1}(2_s^*-p-1)+p-1)\Big]\\ &= \frac{2_s^*}{2}\Big[\log(A(2_s^*-2)) + \log((2^{-1}2_s^*)^{j+1}(2_s^*-p-1)+p-1)\Big].\end{aligned}$$

Therefore, we can find some C^* such that

$$\lambda_j \le C^*(j+1).$$

We now define a sequence $\{\gamma_j\}$ by

$$\gamma_0 = \eta_0 \quad\text{and}\quad \gamma_{j+1} = \frac{2_s^*}{2}\gamma_j + C^*(j+1) \tag{3-20}$$

for $j \ge 1$. Clearly, $\eta_j \le \gamma_j$ for all $j \ge 0$. Moreover, since

$$\gamma_j = \Big(\frac{2_s^*}{2}\Big)^j(\eta_0 + 2C^*2_s^*(2_s^*-2)^{-2}) - 2C^*(2_s^*-2)^{-1}(j+2_s^*(2_s^*-2)),$$

in view of (3-19), there exists $m > 0$ such that $\gamma_j \le mL_{j-1}$. Hence, $\log(M_j) \le mL_{j-1}$ and we obtain (3-17). Observe that m depends only on η_0, 2_s^* and C^*, whereas C^* depends only on 2_s^*, p and A. This completes the proof. □

REMARK 3.5. It is known that if $u \in \mathcal{L}_s(\mathbb{R}^n) \cap C^{2s+\epsilon}(\Omega)$, when $0 < s < \frac{1}{2}$, $2s+\epsilon < 1$ or $u \in \mathcal{L}_s(\mathbb{R}^n) \cap C^{1,2s+\epsilon-1}(\Omega)$, when $\frac{1}{2} \le s < 1$, $2s+\epsilon-1 < 1$, one can compute $(-\Delta)^s u(x)$ pointwise for all x in Ω. In fact, one can write

$$(-\Delta)^s u(x) = c_{n,s}\,\mathrm{PV}\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy.$$

DEFINITION 3.6. We call $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ a classical solution of (1-1) if it satisfies the following:

(1) $u \in \mathcal{L}_s(\mathbb{R}^n) \cap C^{2s+\epsilon}(\Omega)$, when $0 < s < \frac{1}{2}, 2s + \epsilon < 1$ or $u \in \mathcal{L}_s(\mathbb{R}^n) \cap C^{1,2s+\epsilon-1}(\Omega)$, when $\frac{1}{2} \leq s < 1, 2s + \epsilon - 1 < 1$;
(2) $\mathcal{N}_s u(x) = 0, \quad x \in \mathbb{R}^n \setminus \Omega$;
(3) $d(-\Delta)^s u(x) + u(x) = |u(x)|^{p-1} u(x)$ pointwise for all $x \in \Omega$.

We make similar remarks as in [6], which offers a relation between the weak and classical solutions of (1-1).

REMARK 3.7. Let u_d be the least energy solution of (1-1) in H^s_Ω. Then, by Lemma 2.3, Theorem 1.2 and Lemma 3.4:

(1) for $0 < s < \frac{1}{2}$, $u_d \in \mathcal{L}_s(\mathbb{R}^n) \cap C^2(\Omega)$ if $p > 3 - 2s$ and $u_d \in \mathcal{L}_s(\mathbb{R}^n) \cap C^{1,p-2+2s}(\Omega)$ if $2 < p \leq 3 - 2s$;
(2) for $\frac{1}{2} \leq s < 1$, $u_d \in \mathcal{L}_s(\mathbb{R}^n) \cap C^2(\Omega)$.

Now, using the nonlocal integration by parts formulae given in [14], one can easily check that

$$d(-\Delta)^s u_d(x) + u_d(x) = |u_d(x)|^{p-1} u_d(x)$$

holds pointwise in Ω. This implies that u_d is a classical solution of (1-1). Conversely, if u_d is a classical solution of (1-1) satisfying $u_d \in H^s_\Omega$, then u_d is a weak solution of (1-1).

The following lemma shows that the maximum of the least energy solution is always greater than unity.

LEMMA 3.8. *Let u_d be the least energy solution of (1-1). Let*

$$M_d = \sup_{x \in \overline{\Omega}} u_d(x).$$

Then, $M_d > 1$.

PROOF. Since u_d is a weak solution of (1-1),

$$d\frac{c_{n,s}}{2} \int_{T(\Omega)} \frac{(u_d(x) - u_d(y))(w(x) - w(y))}{|x - y|^{n+2s}}\, dx\, dy + \int_\Omega u_d w\, dx = \int_\Omega u_d^p w\, dx$$

holds for all $w \in H^s_\Omega$. Taking $w = 1$ in the above equation,

$$\int_\Omega u_d(x)\, dx = \int_\Omega u_d^p(x)\, dx.$$

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This implies that

$$\int_\Omega u_d(x)(1 - u_d^{p-1}(x))\,dx = 0.$$

Now, if $u_d(x) \le 1$ for all $x \in \overline{\Omega}$, then

$$1 - u_d(x) \ge 0 \quad \text{for all } x \in \overline{\Omega}.$$

Thus, from the above equation, we get that $u_d(x) = 1$ *a.e.* in $\overline{\Omega}$. Now, by Lemma 3.4, we can assume that u_d is continuous and hence $u_d \equiv 1$ in $\overline{\Omega}$, which is a contradiction to our assumption that u_d is a nonconstant solution. Therefore, there exists x_0 in $\overline{\Omega}$ such that $u_d(x_0) > 1$. Thus, $M_d > 1$. □

4. L^r-estimates on u_d

Here, we derive an L^r-estimate for u_d. The following results are generalisations of [31, Proposition 2.2 and Lemma 2.3] to the nonlocal case.

PROPOSITION 4.1. *For $d_0 > 0$ fixed, there is a constant K_0 such that*

$$d\frac{c_{n,s}}{2}\int_{T(\Omega)} \frac{(u_d(x) - u_d(y))^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega u_d^2\,dx \ge K_0 d^{n/2s}, \tag{4-1}$$

where u_d is the least energy solution of (1-1) with $0 < d < d_0$.

PROOF. In contrast, suppose that there is a sequence $\{d_k\}$ contained in the interval $(0, d_0)$ and a sequence of positive solutions $\{u_k\}$ to (1-1) with $d = d_k$ such that

$$\zeta_k := \frac{1}{d^{n/2s}}\Big(d\frac{c_{n,s}}{2}\int_{T(\Omega)} \frac{(u_k(x) - u_k(y))^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega u_k^2\,dx\Big) \to 0 \quad \text{as } k \to \infty.$$

We are going to follow the same arguments as used in the proof of Lemma 1.2 to prove this proposition. Once again, define the sequences $\{L_k\}$ and $\{M_j\}$ as defined earlier in (3-13) and (3-14), respectively. Instead of C_0, we write ζ_k in the definition of $\{M_j\}$:

$$\begin{aligned} p - 1 + 2L_0 &= 2_s^*, \\ p - 1 + 2L_{j+1} &= 2_s^* L_j, \quad j = 0, 1, 2, \ldots \end{aligned}$$

and

$$\begin{aligned} M_0 &= (A\zeta_k)^{2_s^*/2}, \\ M_{j+1} &= (AL_jM_j)^{2_s^*/2}, \quad j = 0, 1, 2, \ldots. \end{aligned}$$

Further, define the sequences $\{\lambda_j\}$, $\{\eta_j\}$ and $\{\gamma_j\}$ as defined earlier in (3-18) and (3-20). From (3-16),

$$\Big(\int_\Omega u_k^{2_s^* L_{j-1}}\,dx\Big)^{(2_s^* L_{j-1})} \le (M_j d_k^{n/2s})^{1/(2_s^* L_{j-1})}. \tag{4-2}$$

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Since

$$\log(M_j) = \eta_j \le \gamma_j,$$

we have

$$\frac{\log(M_j)}{2_s^* L_{j-1}} \le \frac{\eta_j}{2_s^* L_{j-1}}.$$

Now,

$$\lim_{j\to\infty} \frac{\eta_j}{2_s^* L_{j-1}} = \lim_{j\to\infty} \frac{(\frac{2_s^*}{2})^j[\eta_0 + 2C^* 2_s^*(2_s^* - 2)^{-2}] - 2C^*(2_s^* - 2)^{-1}[j + 2_s^*(2_s^* - 2)]}{\frac{2_s^*}{(2_s^*-2)}\Big[(\frac{2_s^*}{2})^j(2_s^* - p - 1) + p - 1\Big]}$$
$$= \frac{(2_s^* - 2)(\eta_0 + 2C^* 2_s^*(2_s^* - 2)^{-2})}{2_s^*(2_s^* - p - 1)}.$$

Letting $j \to \infty$ in (4-2),

$$\|u_k\|_{L^\infty(\Omega)} \le e^{a_1(\eta_0 + a_2)}, \tag{4-3}$$

with a_1 and a_2 depending only on 2_s^*, p and C^*. Since

$$\eta_0 = \log(M_0) = \frac{2_s^*}{2}\log(A\zeta_k),$$

as $k \to \infty$, $\eta_0 \to -\infty$. Thus, in view of (4-3),

$$\|u_k\|_{L^\infty(\Omega)} \to 0,$$

which leads to a contradiction to Lemma 3.8. □

PROOF OF THEOREM 1.3. First, we show the second part of (1-7).

Case I. $r \ge 2_s^* = 2n/(n-2s)$. Let $\{L_j\}$ be the sequence defined in (3-13). If $r \in \{2_s^* L_j\}$, then the second inequality of (1-7) follows from (3-16). So assume that $2_s^* L_j < r < 2_s^* L_{j+1}$ for some $j \ge 0$. We have

$$r = t2_s^* L_j + (1-t)2_s^* L_{j+1} \text{ for some } t \in (0,1).$$

Using the Hölder inequality and (3-16),

$$\begin{aligned}\int_\Omega u_d^r\,dx &= \int_\Omega u_d^{t2_s^* L_j + (1-t)2_s^* L_{j+1}}\,dx,\\ &\le \Big(\int_\Omega u_d^{2_s^* L_j}\,dx\Big)^t \Big(\int_\Omega u_d^{2_s^* L_{j+1}}\,dx\Big)^{1-t}\\ &\le (M_{j-1} d^{n/2s})^t (M_j d^{n/2s})^{1-t}\\ &= M_{j-1}^t M_j^{1-t} d^{n/2s}.\end{aligned}$$

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Case II. $2 \leq r \leq 2_s^*$. We write

$$r = 2t + (1-t)2_s^*,$$

for some $t \in [0,1]$. Then, using the Hölder inequality, from (1-6) and (3-16) with $j = 0$,

$$\int_\Omega u_d^r\,dx \leq \Big(\int_\Omega u_d^2\,dx\Big)^t \Big(\int_\Omega u_d^{2_s^*}\,dx\Big)^{1-t} \leq C_0^t M_0^{(1-t)} d^{n/2s},$$

where the constant C_0 is independent of d.

Case III. $1 \leq r < p+1$. Integrating both sides of (1-1) and using the condition $\mathcal{N}_s u(x) = 0$ for $x \in C\Omega$,

$$\int_\Omega u_d\,dx = \int_\Omega u_d^p\,dx. \tag{4-4}$$

It is easy to see that

$$p = t + (1-t)(p+1) \quad \text{with } t = \frac{1}{p} \in (0,1).$$

Notice that $p+1 \in (2, 2_s^*)$. Therefore, using the Hölder inequality and (4-4),

$$\int_\Omega u_d^p\,dx \leq \Big(\int_\Omega u_d\,dx\Big)^t \Big(\int_\Omega u_d^{p+1}\,dx\Big)^{(1-t)},$$
$$\int_\Omega u_d^p\,dx \leq \int_\Omega u_d^{p+1}\,dx \leq C_0 d^{n/2s} \quad \text{(by (1-6))},$$

where the constant C_0 depends only upon $p+1$.

Also, in view of (4-4) and (1-6), we observe that the second inequality of (1-7) holds for $r = 1$. Now, repeating the interpolation between 1 and $p+1$, we see that the second inequality of (1-7) holds for all r ≥ 1.

Case IV. Let $0 < r \leq 1$. Taking $F = u_d^r$, $G = 1$, $p = \frac{1}{r}$, $q = \frac{1}{1-r}$ and using the Hölder inequality,

$$\int_\Omega u_d^r\,dx \leq \|F\|_p \|G\|_q = |\Omega|^{1-r}\Big(\int_\Omega u_d\,dx\Big)^r \leq |\Omega|^{1-r} B(1)^r d^{nr/2s}.$$

This proves the second inequality of (1-8).

Now, let us prove the first inequality of (1-7) and (1-8). In view of (3-5) and (4-1),

$$\int_\Omega u_d^{p+1} \geq K_0 d^{n/2s}.$$

Since

$$\sup_\Omega u_d(x) \leq C_1 \quad \text{for some constant } C_1 > 0,$$

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we have

$$K_0 d^{n/2s} \le \int_\Omega u_d^{p+1} = \int_\Omega (u_d^{p+1-r})(u_d^r)\,dx \le C_1^{p+1-r} \int_\Omega u_d^r\,dx.$$

This implies that

$$\int_\Omega u_d^r\,dx \ge K_0 C_1^{r-p-1} d^{n/2s}, \ r < p+1.$$

For $r > p+1$, we write $p+1 = 1 + (1-t)r$. Therefore,

$$\begin{aligned} K_0 d^{n/2s} &\le \int_\Omega u_d^{p+1}\,dx \\ &= \int_\Omega u_d^{1+(1-t)r}\,dx \\ &\le (u_d\,dx)^t (u_d^r\,dx)^{1-t} \\ &\le (B(1)d^{n/2s})^t (u_d^r\,dx)^{1-t}. \end{aligned}$$

This yields that

$$\int_\Omega u_d^r\,dx \ge (K_0 B(1)^{-t})^{1/(1-t)} d^{n/2s}.$$

□

5. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. Its proof is intricate and requires some scaling and compactness arguments. We prove the statements of the theorem one by one. Let $z_d \in \overline{\Omega}$ be a point of maximum of u_d. We approximate u_d around z_d by a scaled positive radial solution of (1-5). It gives us an upper bound on c_d, which is closely related to the location of point z_d.

Step I. We prove that there exists a positive constant K_* such that

$$\rho(z_d, \partial\Omega) \le K_* d^{1/2s}. \tag{5-1}$$

If the inequality in (5-1) is not true, then there is a decreasing sequence $d_j \downarrow 0$ such that

$$\rho_j := \frac{\rho(z_j, \partial\Omega)}{d_j^{1/2s}} \to +\infty \quad \text{as } j \to \infty, \tag{5-2}$$

where $z_j := z_{d_j}$ is a point of maximum of u_{d_j} on $\overline{\Omega}$. Define

$$\phi_j(y) := u_{d_j}(y d_j^{1/2s} + z_j) \quad \text{for } y \in \mathbb{R}^n.$$

Since u_d is a classical solution of (1-1),

$$(-\Delta)^s \phi_j + \phi_j = \phi_j^p \quad \text{in } B_{\rho_j}, \tag{5-3}$$

and:

(1) $\phi_j \in C^{0,2s+\epsilon}(B_{\rho_j})$, when $0 < s < \frac{1}{2}$, $2s + \epsilon < 1$;
(2) $\phi_j \in C^{1,2s+\epsilon-1}(B_{\rho_j})$, when $\frac{1}{2} \le s < 1$, $2s + \epsilon - 1 < 1$.

First, we claim that the sequence $\{\phi_j\}$ contains a convergent subsequence. Let $\{R_k\}$ be a monotone increasing sequence of positive numbers with $R_k \to +\infty$ as $k \to \infty$. Therefore, we have for each k, there is a number j_k such that $4R_k < \rho_j$ whenever $j \ge j_k$. Since $u_d \in L^\infty(\mathbb{R}^n) \cap \mathcal{L}_s(\mathbb{R}^n)$, we have $\phi_j \in L^\infty(\mathbb{R}^n) \cap \mathcal{L}_s(\mathbb{R}^n)$ for each $j \ge 1$. Now, we can use [19, Theorem 1.4] to get the following estimates.

For $0 < s < \frac{1}{2}$, $2s + \epsilon < 1$:

(i) let $4s + \epsilon < 1$, then

$$\|\phi_j\|_{C^{0,4s+\epsilon}(B_{2R_k})} \le C(\|\phi_j\|_{L^\infty(\mathbb{R}^n)} + \|\phi_j^p - \phi_j\|_{C^{0,2s+\epsilon}(B_{4R_k})});$$

(ii) let $1 < 4s + \epsilon < 2$, then

$$\|\phi_j\|_{C^{1,4s+\epsilon-1}(B_{2R_k})} \le C(\|\phi_j\|_{L^\infty(\mathbb{R}^n)} + \|\phi_j^p - \phi_j\|_{C^{0,2s+\epsilon}(B_{4R_k})});$$

and for $\frac{1}{2} \le s < 1$, $2s + \epsilon - 1 < 1$:

(iii) let $4s + \epsilon - 1 < 1$, then

$$\|\phi_j\|_{C^{1,4s+\epsilon-1}(B_{2R_k})} \le C(\|\phi_j\|_{L^\infty(\mathbb{R}^n)} + \|\phi_j^p - \phi_j\|_{C^{1,2s+\epsilon-1}(B_{4R_k})});$$

(iv) let $1 < 4s + \epsilon - 1 < 2$, then

$$\|\phi_j\|_{C^{2,4s+\epsilon-1}(B_{2R_k})} \le C(\|\phi_j\|_{L^\infty(\mathbb{R}^n)} + \|\phi_j^p - \phi_j\|_{C^{1,2s+\epsilon-1}(B_{4R_k})}),$$

where the constant $C > 0$ is independent of j.

Let us recall the inequality (1-6) here:

$$d\frac{c_{n,s}}{2}\int_{T(\Omega)} \frac{|u_d(x) - u_d(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega u^2\,dx = \int_\Omega u_d^{p+1} \le C_0 d^{n/2s},$$

where C_0 is independent of d. This yields

$$\int_{B_{\rho_j}} \phi_j^{p+1} \le C_0,$$

and

$$\|\phi_j\|_{H^s(B_{\rho_j})} \le C_0 \quad \text{for all } j \ge 1. \tag{5-4}$$

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Also, by Theorem 1.3,

$$\int_\Omega u_d^r \le B(r)d^{n/2s} \quad \text{for all } r \ge 1,$$

which implies that

$$\int_{B_{\rho_j}} \phi_j^r \le B(r) \quad \text{for all } j \ge 1 \text{ and } r \ge 1. \tag{5-5}$$

By Lemma 3.4 and Theorem 1.2,

$$\|u_d\|_{L^\infty(\mathbb{R}^n)} \le C_1, \tag{5-6}$$

where the constant C_1 is independent of the diffusion constant d. So, (5-5), (5-6) and [19, Theorem 1.3] imply that

$$\|\phi_j\|_{X_s(\overline{B}_{R_k})} < C_2 \quad \text{for all } j \ge j_k,$$

where the constant $C_2 > 0$ is independent of j and the space $X_s(\overline{B}_{R_k})$ is identified with one of the spaces $C^{0,4s+\epsilon}(\overline{B}_{R_k})$, $C^{1,4s+\epsilon-1}(\overline{B}_{R_k})$ or $C^{2,4s+\epsilon-1}(\overline{B}_{R_k})$ with the same assumptions on s and ϵ as above. Therefore, $\{\phi_j\}$ is a relatively compact set in $X_s(\overline{B}_{R_k})$, and hence by the standard diagonal process, we can extract a convergent subsequence of $\{\phi_j\}$, which we continue to denote by $\{\phi_j\}$ itself such that

$$\phi_j \to v \quad \text{in } C^{0,2s+\epsilon}_{loc}(\mathbb{R}^n) \quad \text{when } 0 < s < \tfrac{1}{2}, 2s + \epsilon < 1$$

or

$$\phi_j \to v \quad \text{in } C^{1,2s+\epsilon-1}_{loc}(\mathbb{R}^n) \quad \text{when } \tfrac{1}{2} < s < 1, 2s + \epsilon - 1 < 1$$

for some v. The limit $v \in C^{0,2s+\epsilon}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$ when $0 < s < \frac{1}{2}, 2s + \epsilon < 1$ or $v \in C^{1,2s+\epsilon-1}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$ when $\frac{1}{2} < s < 1, 2s + \epsilon - 1 < 1$ follows from (5-4). Consequently,

$$\lim_{|x|\to\infty} v(x) = 0.$$

Using [17, Theorem 1.1], we have $(-\Delta)^s\phi_j(x)$ converges to $(-\Delta)^s v(x)$ point-wise in $\mathbb{R}^n$. Consequently, we see that the limit v satisfies the equation

$$(-\Delta)^s v + v = v^p \quad \text{in } \mathbb{R}^n.$$

Clearly, $v \ge 0$ because each $\phi_j \ge 0$. Since by Lemma 3.8 we have $\phi_j(0) = u_{d_j}(z_j) > 1$ for each $j \ge 1$, one can see that $v \not\equiv 0$.

Using Theorem 2.9, one can see that v is radially symmetric and decreasing about some point in $\mathbb{R}^n$. Since

$$\nabla v(0) = \lim_{j\to\infty} \nabla\phi_j(0) = 0,$$

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v is radially symmetric about the origin. Additionally, by Theorem 2.8, v has a power type of decay at infinity, that is,

$$v(r) \leq \frac{C_2}{r^{n+2s}}, \quad r \geq 1.$$

Now we derive a lower bound on the critical value c_{d_j}. Let us define

$$\delta_R := \frac{C_2}{R^{n+2s}}, \tag{5-7}$$

where $R > 0$ is an arbitrarily large real number. Then, there exists a positive integer j_R such that if $j \geq j_R$, then $\rho_j \geq 2R$ and

$$\|\phi_j - v\|_{C^2(\overline{B}_{2R})} \leq \delta_R. \tag{5-8}$$

By Lemma 3.3,

$$c_{d_j} = M[u_{d_j}] = J_{d_j}(u_{d_j}).$$

Using this fact and (3-6),

$$\begin{aligned} c_{d_j} &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big) \int_\Omega u_{d_j}^{p+1}\, dx \\ &\geq \Big(\frac{1}{2} - \frac{1}{p+1}\Big) \int_{|x-z_j|<d_j^{1/2s}R} u_{d_j}^{p+1}\, dx \\ &= d_j^{n/2s}\Big(\frac{1}{2} - \frac{1}{p+1}\Big) \int_{|y|<R} \phi_j^{p+1}\, dy. \end{aligned}$$

Now,

$$c_{d_j} = d_j^{n/2s}\bigg(\Big(\frac{1}{2} - \frac{1}{p+1}\Big) \int_{B_R} v^{p+1}\, dy + F_j\bigg), \tag{5-9}$$

where

$$F_j := \Big(\frac{1}{2} - \frac{1}{p+1}\Big) \int_{B_R} \Big(\phi_j^{p+1} - v^{p+1}\Big)\, dy.$$

By Equation (5-8), we have for all $y \in B_R$, $j \geq j_R$,

$$|\phi_j^{p+1} - v^{p+1}| \leq C|\phi_j - v| \leq \delta_R,$$

where $C > 0$ is some constant. This implies that

$$|F_j| \leq \Big(\frac{1}{2} - \frac{1}{p+1}\Big) C|B_R|\delta_R = C_3 R^n \delta_R,$$

where

$$C_3 = \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\frac{w_n}{n}C$$

and w_n denotes the surface area of the unit sphere in $\mathbb{R}^n$. Consequently, (5-9) becomes

$$c_{d_j} \geq d_j^{n/2s}\Big[\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_{B_R} v^{p+1}\,dy - C_3 R^n \delta_R\Big]. \tag{5-10}$$

Now, it is easy to see that

$$\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_{B_R} v^{p+1}\,dy = F(v) - \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_{|y|>R} v^{p+1}\,dy,$$

where $F(v)$ is defined earlier in (2-2). Simplifying the second term on right-hand side,

$$\begin{aligned}\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_{|y|>R} v^{p+1}\,dy &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_R^\infty \frac{r^{n-1}w_n}{r^{(n+2s)(p+1)}}\,dr\\ &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\frac{w_n}{(n+2s)p+2s}\frac{1}{R^{(n+2s)p+2s}} = \frac{C_4}{R^{(n+2s)p+2s}}.\end{aligned}$$

Therefore, one can write

$$\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_{B_R} v^{p+1}\,dy = F(v) - \frac{C_4}{R^{(n+2s)p+2s}}. \tag{5-11}$$

On combining (5-7), (5-10) and (5-11), we get for $j \geq j_R$,

$$\begin{aligned}c_{d_j} &\geq d_j^{n/2s}\Big(F(v) - \frac{C_4}{R^{(n+2s)p+2s}} - \frac{C_2C_3}{R^{2s}}\Big)\\ &\geq d_j^{n/2s}\Big(F(v) - \frac{C_5}{R^{2s}}\Big),\end{aligned} \tag{5-12}$$

where C_5 is independent of j and R.

Now, we derive an upper bound on the critical value c_{d_j}. Without loss of generality, we may assume that the domain Ω is a subset of $\mathbb{R}^n_+$ and $0 \in \partial\Omega$. Given Definition 2.13, let w be the ground state solution of (1-5). Define

$$\Omega_d := \Big\{\frac{x}{d^{1/2s}} \mid x \in \Omega\Big\},$$
$$w_d(x) := w\Big(\frac{x}{d^{1/2s}}\Big), \quad \text{for } x \in \mathbb{R}^n.$$

Since $w \geq 0$, this implies that $w_d \geq 0$. Define

$$g_d(t) := J_d(tw_d), \quad t \geq 0.$$

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Then, by Lemma 3.3, there exists a unique $t_0 = t_0(d) > 0$ at which g_d attains a maximum. It is easy to see that $t_0(d) \to 1$ as $d \downarrow 0$. Hence,

$$\begin{aligned}
M[w_d] &= J_d\Big(t_0 w_d\Big) \\
&= \frac{t_0^2}{2}\Big[\frac{dc_{n,s}}{2}\int_{T(\Omega)} \frac{|w_d(x) - w_d(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega w_d^2\,dx\Big] - \frac{t_0^{p+1}}{p+1}\int_\Omega w_d^{p+1}\,dx \\
&= \frac{t_0^2}{2}\Big[\frac{dc_{n,s}}{2}\int_{T(\Omega)} \frac{\Big|w\Big(\frac{x}{d^{1/2s}}\Big) - w\Big(\frac{y}{d^{1/2s}}\Big)\Big|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega w^2\Big(\frac{x}{d^{1/2s}}\Big)\,dx\Big] \\
&\quad - \frac{t_0^{p+1}}{p+1}\int_\Omega w^{p+1}\Big(\frac{x}{d^{1/2s}}\Big)\,dx.
\end{aligned}$$

The change of variables

$$\frac{x}{d^{1/2s}} = a,\ \frac{y}{d^{1/2s}} = b,$$

gives us

$$\begin{aligned}
M[w_d] &= d^{n/2s}\Big(\frac{t_0^2}{2}\Big[\frac{c_{n,s}}{2}\int_{T(\Omega_d)} \frac{|w(a) - w(b)|^2}{|a-b|^{n+2s}}\,da\,db + \int_{\Omega_d} w^2\,da\Big] - \frac{t_0^{p+1}}{p+1}\int_{\Omega_d} w^{p+1}\,da\Big) \\
&= d^{n/2s} I_d
\end{aligned}$$

where I_d is the expression

$$\begin{aligned}
&\frac{t_0^2}{2}\Big[\frac{c_{n,s}}{2}\int_{\Omega_d}\int_{\Omega_d} \frac{|w(a) - w(b)|^2}{|a-b|^{n+2s}}\,da\,db + 2c_{n,s}\int_{C\Omega_d}\int_{\Omega_d} \frac{|w(a) - w(b)|^2}{|a-b|^{n+2s}}\,da\,db + \int_{\Omega_d} w^2\,da\Big] \\
&\quad - \frac{t_0^{p+1}}{p+1}\int_{\Omega_d} w^{p+1}\,da.
\end{aligned}$$

Since

$$t_0(d) \to 1 \quad \text{as } d \downarrow 0,$$

we get for I_d

$$\begin{aligned}
&\frac{1}{2}\Big[\frac{c_{n,s}}{2}\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \frac{|w(a) - w(b)|^2}{|a-b|^{n+2s}}\,da\,db + 2c_{n,s}\int_{C\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \frac{|w(a) - w(b)|^2}{|a-b|^{n+2s}}\,da\,db + \int_{\mathbb{R}^n_+} w^2\,da\Big] \\
&\quad - \frac{1}{p+1}\int_{\mathbb{R}^n_+} w^{p+1}\,da + o(1)
\end{aligned}$$

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as $d \downarrow 0$. Further, w being nonnegative and radially symmetric implies that

$$\int_{\mathbb{R}^n_+} w^2\,da = \frac{1}{2}\int_{\mathbb{R}^n} w^2\,da, \quad \int_{\mathbb{R}^n_+} w^{p+1}\,da = \frac{1}{2}\int_{\mathbb{R}^n} w^{p+1}\,da,$$

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \frac{|w(a)-w(b)|^2}{|a-b|^{n+2s}}\,da\,db = \frac{1}{4}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|w(a)-w(b)|^2}{|a-b|^{n+2s}}\,da\,db,$$

$$\int_{\mathcal{C}\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \frac{|w(a)-w(b)|^2}{|a-b|^{n+2s}}\,da\,db = \frac{1}{4}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|w(a)-w(b)|^2}{|a-b|^{n+2s}}\,da\,db.$$

Using these estimates,

$$\begin{aligned} I_d &< \frac{1}{2}\Big(\frac{1}{2}\Big[\frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|w(a)-w(b)|^2}{|a-b|^{n+2s}}\,da\,db + \int_{\mathbb{R}^n} w^2\,da\Big] - \frac{1}{p+1}\int_{\mathbb{R}^n} w^{p+1}\,da\Big) + o(1) \\ &= \frac{1}{2}F(w) + o(1), \end{aligned}$$

as $d \downarrow 0$. Thus,

$$M[w_d] = d^{n/2s} I_d < \frac{d^{n/2s}}{2}F(w) + o(1),$$

as $d \downarrow 0$. Using part (c) of Theorem 2.12, we have $0 < F(w) \le F(v)$ for any nonnegative nonzero classical solution v of (1-5) and by Lemma 3.3,

$$c_{d_j} \le M[w_{d_j}] < \frac{d_j^{n/2s}}{2}F(v)$$

for d_j sufficiently small. By letting R be sufficiently large in (5-12), we reach a contradiction. This proves (5-1).

REMARK 5.1. In the classical case [35], the authors have defined diffeomorphisms, which straighten a boundary portion near $Q \in \partial\Omega$. Further, using scaling and translations of the least energy solutions u_d of (1-3), the classical problem (1-3) gets transferred into a new elliptic equation. Due to the nonlocal nature of the fractional Laplacian and of the boundary condition in our problem, it seems almost impossible to introduce such scaling and translation arguments.

Step II. Now, we claim that $z_d \in \partial\Omega$. Suppose that there is a decreasing sequence $d_j \downarrow 0$ such that $z_{d_j} := z_j \in \Omega$. We have from Lemma 1.4 that the sequence $\{z_j\}$ converges to some $z \in \partial\Omega$. Without loss of generality, let us assume that $z = 0$. Define

$$\widehat{u_j}(x) := \begin{cases} u_{d_j}(x) & \text{in } \mathbb{R}^n_+, \\ u_{d_j}(x', -x_n) & \text{in } \mathbb{R}^n_-, \end{cases}$$

where

$$x' = (x_1, x_2, \ldots, x_{n-1}), \quad \mathbb{R}^n_+ = \{(x', x_n) \mid x_n \ge 0\},\ \mathbb{R}^n_- = \{(x', x_n) \mid x_n \le 0\}.$$

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Also, define a scaled function

$$\psi_j(y) := \widehat{u}_j(yd_j^{1/2s} + z_j) \quad \text{for } y \in \mathbb{R}^n. \tag{5-13}$$

Now, for $z_j = (z'_j, z_{jn})$, we can write $z_{jn} = \alpha_j d_j^{1/2s}$ for some $\alpha_j > 0$. The sequence $\{\alpha_j\}$ is bounded, which follows from Lemma 1.4. Let

$$\rho_j := \frac{\rho(z_j, \partial\Omega)}{d_j^{1/2s}},$$

where $\rho(z_j, \partial\Omega)$ denotes the distance between z_j and $\partial\Omega$. One can see easily that the function ψ_j satisfies the equation

$$(-\Delta)^s \psi_j(y) + \psi_j(y) = \psi_j(y)^p + d_j h(y) \quad \text{in } B_{\rho_j}$$

for some function h of y. To see this, let $y \in B_{\rho_j}$, so

$$\begin{aligned}
(-\Delta)^s \psi_j(y) &= c_{n,s}\, \mathrm{PV} \int_{\mathbb{R}^n} \frac{\psi_j(y) - \psi_j(x)}{|y-x|^{n+2s}}\, dx = c_{n,s} \lim_{\epsilon \to 0} \int_{CB_\epsilon(y)} \frac{\psi_j(y) - \psi_j(x)}{|y-x|^{n+2s}}\, dx \\
&= c_{n,s} \lim_{\epsilon \to 0} \Big[\int_{CB_\epsilon(y)} \frac{\widehat{u}_j(yd_j^{1/2s} + z_j) - \widehat{u}_j(xd_j^{1/2s} + z_j)}{|y-x|^{n+2s}}\, dx \Big] \\
&= c_{n,s} \lim_{\epsilon \to 0} \int_{\{x_n \geq -\alpha_j\} \bigcap CB_\epsilon(y)} \frac{\widehat{u}_j(yd_j^{1/2s} + z_j) - \widehat{u}_j(xd_j^{1/2s} + z_j)}{|y-x|^{n+2s}}\, dx \\
&\quad + c_{n,s} \lim_{\epsilon \to 0} \int_{\{x_n \leq -\alpha_j\} \bigcap CB_\epsilon(y)} \frac{\widehat{u}_j(yd_j^{1/2s} + z_j) - \widehat{u}_j(xd_j^{1/2s} + z_j)}{|y-x|^{n+2s}}\, dx.
\end{aligned} \tag{5-14}$$

For $y_n \geq -\alpha_j$,

$$\begin{aligned}
&(-\Delta)^s \psi_j(y) \\
&= c_{n,s} \lim_{\epsilon \to 0} \int_{\{x_n \geq -\alpha_j\} \bigcap CB_\epsilon(y)} \frac{u_j(yd_j^{1/2s} + z_j) - u_j(xd_j^{1/2s} + z_j)}{|y-x|^{n+2s}}\, dx \\
&\quad + c_{n,s} \lim_{\epsilon \to 0} \int_{\{x_n \leq -\alpha_j\} \bigcap CB_\epsilon(y)} \frac{u_j(yd_j^{1/2s} + z_j) - u_j(x' d_j^{1/2s} + z'_j, -(x_n + \alpha_j d_j^{1/2s}))}{|y-x|^{n+2s}}\, dx \\
&= c_{n,s} \lim_{\epsilon \to 0} \int_{\{x_n \geq -\alpha_j\} \bigcap CB_\epsilon(y)} \frac{u_j(yd_j^{1/2s} + z_j) - u_j(xd_j^{1/2s} + z_j)}{|y-x|^{n+2s}}\, dx \\
&\quad + c_{n,s} \lim_{\epsilon \to 0} \int_{\{x_n \leq -\alpha_j\} \bigcap CB_\epsilon(y)} \frac{u_j(yd_j^{1/2s} + z_j) - u_j(x' d_j^{1/2s} + z'_j, -(x_n + \alpha_j d_j^{1/2s}))}{|y-x|^{n+2s}}\, dx
\end{aligned}$$

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$$
\begin{aligned}
&= c_{n,s} \lim_{\epsilon\to 0} \int_{CB_\epsilon(y)} \frac{u_j(yd_j^{1/2s}+z_j) - u_j(xd_j^{1/2s}+z_j)}{|y-x|^{n+2s}}\,dx \\
&\quad + c_{n,s} \lim_{\epsilon\to 0} \int_{\{x_n\le -\alpha_j\}\bigcap CB_\epsilon(y)} \frac{u_j(xd_j^{1/2s}+z_j) - u_j(x'd_j^{1/2s}+z_j', -(x_n+\alpha_j d_j^{1/2s}))}{|y-x|^{n+2s}}\,dx \\
&= c_{n,s} \lim_{\epsilon\to 0} \int_{CB_\epsilon(y)} \frac{u_j(yd_j^{1/2s}+z_j) - u_j(xd_j^{1/2s}+z_j)}{|y-x|^{n+2s}}\,dx \\
&\quad + c_{n,s} \lim_{\epsilon\to 0} \int_{\{x_n\le -\alpha_j\}\bigcap CB_\epsilon(y)} \frac{u_j(xd_j^{1/2s}+z_j) - \widehat{u_j}(xd_j^{1/2s}+z_j)}{|y-x|^{n+2s}}\,dx.
\end{aligned}
$$

Making the change of variables

$$
yd_j^{1/2s}+z_j = a \text{ and } xd_j^{1/2s}+z_j = b,
$$

we get

$$
\begin{aligned}
(-\Delta)^s\psi_j(y) &= d_j(-\Delta)^s u_j(a) + d_j c_{n,s} \lim_{\eta\to 0} \int_{\{b_n\le 0\}\bigcap CB_\eta(a)} \frac{u_j(b)-\widehat{u_j}(b)}{|a-b|^{n+2s}}\,db \\
&= d_j(-\Delta)^s u_j(a) + d_j h(a), \qquad (5\text{-}15)
\end{aligned}
$$

where

$$
\eta = \epsilon d_j^{1/2s}
$$

and

$$
h(a) = c_{n,s} \lim_{\eta\to 0} \int_{\{b_n\le 0\}\bigcap CB_\eta(a)} \frac{u_j(b)-\widehat{u_j}(b)}{|a-b|^{n+2s}}\,db.
$$

Note that $a \in \Omega$.

Now, consider the case $y_n \le -\alpha_j$. Equation (5-14) becomes

$$
(-\Delta)^s\psi_j(y) = I_1 + I_2
$$

where

$$
I_1 = \int_{\{x_n\ge -\alpha_j\}\bigcap CB_\epsilon(y)} \frac{u_j(y'd_j^{1/2s}+z_j', -(y_n d_j^{1/2s}+\alpha_j d_j^{1/2s})) - u_j(xd_j^{1/2s}+z_j)}{|y-x|^{n+2s}}\,dx
$$

and

$$
I_2 = \int_{\{x_n\le -\alpha_j\}\bigcap CB_\epsilon(y)} \frac{u_j(y'd_j^{1/2s}+z_j', -(y_n d_j^{1/2s}+\alpha_j d_j^{1/2s})) - u_j(x'd_j^{1/2s}+z_j', -(x_n d_j^{1/2s}+\alpha_j d_j^{1/2s}))}{|y-x|^{n+2s}}\,dx \qquad (5\text{-}16)
$$

Let us introduce some notation. We write $\widehat{x} = (x', -x_n)$, $\widetilde{x} = (\widehat{x'}, \widehat{x_n})$ and $\widehat{x_n} = -x_n$ for $x = (x', x_n) \in \mathbb{R}^n, n > 1$. Using these, let us compute

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$$I_2 = c_{n,s} \lim_{\epsilon\to 0}\Big[\int_{\{\widetilde{x}_n\ge\alpha_j\}\bigcap CB_\epsilon(\widetilde{y})} \frac{u_j(\widetilde{y}' d_j^{1/2s} + \hat{z}_j', \widetilde{y}_n d_j^{1/2s} + \widetilde{\alpha}_j d_j^{1/2s}) - u_j(\widetilde{x}' d_j^{1/2s} + \widetilde{z}_j', \widetilde{x}_n d_j^{1/2s} + \widetilde{\alpha}_j d_j^{1/2s})}{|\widetilde{y}-\widetilde{x}|^{n+2s}} d\widetilde{x}\Big]$$
$$= c_{n,s} \lim_{\epsilon\to 0}\Big[\int_{\{\widetilde{x}_n\ge\alpha_j\}\bigcap CB_\epsilon(\widetilde{y})} \frac{u_j(\widetilde{y} d_j^{1/2s} + \widetilde{z}_j) - u_j(\widetilde{x} d_j^{1/2s} + \widetilde{z}_j)}{|\widetilde{y}-\widetilde{x}|^{n+2s}} d\widetilde{x}\Big]$$
$$= c_{n,s} \lim_{\epsilon\to 0}\Big[\int_{\{\widetilde{x}_n\ge\alpha_j\}\bigcap CB_\epsilon(\widetilde{y})} \frac{u_j(\widetilde{y} d_j^{1/2s} + \widetilde{z}_j) - u_j(\widetilde{x} d_j^{1/2s} + \widetilde{z}_j)}{|\widetilde{y}-\widetilde{x}|^{n+2s}} d\widetilde{x}\Big].$$

Now, we simplify I_1:

$$I_1 = c_{n,s} \lim_{\epsilon\to 0}\Big[\int_{\{x_n\ge-\alpha_j\}\bigcap CB_\epsilon(y)} \frac{u_j(y' d_j^{1/2s} + z_j', -(y_n d_j^{1/2s} + \alpha_j d_j^{1/2s})) - u_j(x' d_j^{1/2s} + z_j', -(x_n d_j^{1/2s} + \alpha_j d_j^{1/2s}))}{|y-x|^{n+2s}}$$
$$+ \int_{\{x_n\ge-\alpha_j\}\cap CB_\epsilon(y)} \frac{u_j(w' d_j^{1/2s} + z_j', -(x_n d_j^{1/2s} + \alpha_j d_j^{1/2s})) - u_j(x d_j^{1/2s} + z_j)}{|y-x|^{n+2s}}\, dx\Big]$$
$$= c_{n,s} \lim_{\epsilon\to 0}\Big[\int_{\{\widetilde{x}_n\le\alpha_j\}\bigcap CB_\epsilon(\widetilde{y})} \frac{u_j(\widetilde{y} d_j^{1/2s} + \widetilde{z}_j) - u_j(\widetilde{x} d_j^{1/2s} + \widetilde{z}_j)}{|\widetilde{y}-\widetilde{x}|^{n+2s}} d\widetilde{x}$$
$$+ \int_{\{x_n\ge-\alpha_j\}\cap CB_\epsilon(\widetilde{y})} \frac{u_j(\widetilde{x} d_j^{1/2s} + \widetilde{z}_j) - \widehat{u}_j(\widetilde{x} d_j^{1/2s} + \widetilde{z}_j)}{|\widetilde{y}-\widetilde{x}|^{n+2s}} d\widetilde{x}\Big].$$

Using these estimates for I_1 and I_2 in (5-16),

$$(-\Delta)^s\psi_j(y) = c_{n,s}\ \mathrm{PV}\int_{\mathbb{R}^n} \frac{u_j(\widetilde{y} d_j^{1/2s} + \widetilde{z}_j) - u_j(\widetilde{x} d_j^{1/2s} + \widetilde{z}_j)}{|\widetilde{y}-\widetilde{x}|^{n+2s}} d\widetilde{x}$$
$$+ c_{n,s} \lim_{\epsilon\to 0}\int_{\{x_n\ge-\alpha_j\}\bigcap CB_\epsilon(\widetilde{y})} \frac{u_j(\widetilde{x} d_j^{1/2s} + \widetilde{z}_j) - \widehat{u}_j(\widetilde{x} d_j^{1/2s} + \widetilde{z}_j)}{|\widetilde{y}-\widetilde{x}|^{n+2s}} d\widetilde{x}.$$

By the change of variables

$$\widetilde{y} d_j^{1/2s} + \widetilde{z}_j = e \quad\text{and}\quad \widetilde{w} d_j^{1/2s} + \widetilde{z}_j = f,$$

we get

$$(-\Delta)^s\psi_j(y) = d_j(-\Delta)^s u_j(e)$$
$$+ d_j c_{n,s} \lim_{\eta\to 0}\int_{\{f_n\le 0\}\bigcap CB_\eta(e)} \frac{u_j(f) - \widehat{u}_j(f)}{|e-f|^{n+2s}}\, df, \text{ where } f_n \text{ is the } n\text{th coordinate of } f$$
$$= d_j(-\Delta)^s u_j(e) + d_j h(e). \tag{5-17}$$

Note that $e \in \Omega$. Further, for $y \in B_{\rho_j}$,

$$\psi_j(y) = \widehat{u}_j(y d_j^{1/2s} + z_j) = \begin{cases} u_{d_j}(y d_j^{1/2s} + z_j) & \text{if } y_n \ge -\alpha_j,\\ u_{d_j}(y' d_j^{1/2s} + z_j', -(y_n d_j^{1/2s} + \alpha_j d_j^{1/2s})) & \text{if } y_n \le -\alpha_j.\end{cases}$$

We can write

$$(y' d_j^{1/2s} + z_j', -(y_n d_j^{1/2s} + \alpha_j d_j^{1/2s})) = \widetilde{y} d_j^{1/2s} + \widetilde{z}_j.$$

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Again re-naming the variables $yd_j^{1/2s} + z_j$ and $\widetilde{y}d_j^{1/2s} + \widetilde{z}_j$ by a and e, respectively,

$$\psi_j(y) = \begin{cases} u_{d_j}(a) & \text{if } y_n \geq -\alpha_j, \\ u_{d_j}(e) & \text{if } y_n \leq -\alpha_j. \end{cases}$$

We know that u_j satisfies (1-1) in the point-wise sense as well. Therefore, combining above equation with (5-15), (5-17), we have for $y \in B_{\rho_j}$,

$$(-\Delta)^s \psi_j(y) + \psi_j(y) = \psi_j(y)^p + d_j h(y).$$

Now, arguing as in the proof of Step I with minor modifications, one can obtain a convergent subsequence of $\{\psi_j\}$, which we denote again by $\{\psi_j\}$ such that $\psi_j \to v$ in $C^2_{loc}(\mathbb{R}^n)$. Therefore, as $d_j \downarrow 0$,

$$(-\Delta)^s v + v = v^p \quad \text{in } \mathbb{R}^n.$$

Since $v \in H^s(\mathbb{R}^n)$ and v is radially decreasing, v is spherically symmetric to $y = 0$. Moreover, v has power-type decay at infinity, which follows from Theorem 2.8, that is,

$$v(r) \leq \frac{C_2}{r^{n+2s}}, \quad r \geq 1,$$

for some constant $C_2 > 0$. Let us define δ_R as in (5-7), that is,

$$\delta_R := \frac{C_2}{R^{n+2s}}$$

for R sufficiently large to be defined later. Then, there exists an integer j_R such that for $j \geq j_R$,

$$\|\psi_j - v\|_{C^2(\overline{B_{4R}})} \leq \delta_R. \tag{5-18}$$

We choose R sufficiently large that $R > \alpha_j$ for all j, where the α_j terms are the same as defined earlier right after (5-13). We can choose such an R because $\{\alpha_j\}$ is a bounded sequence. The following lemma is very useful to prove our claim that $z_d \in \partial\Omega$.

LEMMA 5.2 (see [35, Lemma 4.2]). *Let $f \in C^2(\overline{B_t})$ be a radial function. Assume that f satisfies $f'(0) = 0$ and $f''(r) < 0$ for $0 \leq r \leq t$. Then, there exists a $\eta > 0$ such that if $g \in C^2(\overline{B_t})$ satisfies:*

(1) $\nabla g(0) = 0$;
(2) $\|f - g\|_{C^2(\overline{B_t})} < \eta$,

then $\nabla g \neq 0$ for $x \neq 0$.

Now, we use this lemma to show that ψ_j has only one local maximum point in B_R. For this, we choose two numbers k, l $(0 < k < l)$ such that $v''(r) < 0$ for $0 \leq r \leq k$. Further, we see that $v''(0) < 0$ and $v(k) < 1$. Let us define

$$\theta = \min\{|v'(r)| \mid k \leq r \leq l\}.$$

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It is easy to observe that $\theta > 0$ because $v' < 0$ for $r > 0$. Then for $\delta_R < \theta$, we have by (5-18) that

$$0 < \theta - \delta_R \le |\nabla v(y)| - |\nabla\psi_j(y) - \nabla v(y)| \le |\nabla\psi_j(y)| \text{ for } k \le |y| \le l.$$

Applying Lemma 5.2 in the ball $\overline{B}_k$, we conclude that $y = 0$ is the only local maximum point of ψ_j in B_l. If y_j is a maximum point of ψ_j in B_R, then by Lemma 3.8, we have $\psi_j \ge 1$. Choose $R > 0$ sufficiently large so that $\delta_R < 1 - v(l)$. Therefore,

$$\psi_j(y) \le v(y) + \delta_R \le v(l) + \delta_R < 1.$$

Hence, $y_j \in B_l$ and therefore $y_j = 0$.

If $\alpha_j > 0$, then by the definition of $\widehat{u_j}$, $z_R^* = (z_j', -\alpha_j d_j^{1/2s})$ is also a maximum point of $\widehat{u_j}$. This implies that $(0, -\alpha_j)$ is another maximum point of ψ_j in B_R, which is a contradiction. This proves our claim and hence completes the proof of Theorem 1.4.

Appendix A

PROOF OF (3-8). For real numbers $x, y \ge 0$ and $k \ge 1$, we show that

$$\frac{1}{k}(x^k - y^k)^2 \le (x - y)(x^{2k-1} - y^{2k-1}).$$

Clearly, the inequality holds when either x or y or both are zero or $x = y$. Thus, without loss of generality, we may assume that $x > y > 0$. Now, our claim is reduced to showing that

$$\frac{1}{k}\left(1 - \left(\frac{y}{x}\right)^k\right)^2 \le \left(1 - \frac{y}{x}\right)\left(1 - \left(\frac{y}{x}\right)^{2k-1}\right),$$

that is, to show that

$$(1 - a^k)^2 \le k(1 - a)(1 - a^{2k-1}),$$

where $0 < a := \frac{y}{x} < 1$. Consider

$$\begin{aligned} f(a) &:= k(1 - a)(1 - a^{2k-1}) - (1 - a^k)^2 \\ &\ge (1 - a^k)(k(1 - a) - (1 - a^k)) \\ &\ge (1 - a^k)(1 - a)(k - (1 + a + a^2 + \cdots + a^{k-1})) \\ &\ge (1 - a^k)(1 - a)(k - k) = 0. \end{aligned}$$

This proves the inequality. □

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