*Bull. Aust. Math. Soc.* **80** (2009), 472–475
doi:10.1017/S0004972709000501

# A NOTE ON CONTINUOUSLY URYSOHN SPACES

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(Received 24 January 2009)

**Abstract**

In this paper, we generalize a result of Bennett and Lutzer and give a condition under which a continuously Urysohn space must have a one-parameter continuous separating family.

2000 *Mathematics subject classification*: primary 54F65; secondary 54C05, 54C30, 54D65.

*Keywords and phrases*: continuously Urysohn spaces, continuous separating family, one-parameter continuous separating family.

## 1. Introduction

In the early 1960s, Arhangel′skii introduced in [1] a certain type of space that could be characterized as the class of preimages of metric spaces under perfect surjections, which he termed paracompact $p$-spaces.

About 30 years after Arhangel′skii's work, Stepanova introduced in [5, 6] a property which is necessary and sufficient for a paracompact $p$-space to be metrizable. It is given in the following definition.

DEFINITION 1.1. A topological space $X$ is *continuously Urysohn* if:

(1) for each pair of distinct points $x, y \in X$ there is a function $f_{x,y} \in C_u(X)$, where $C_u(X)$ is the set of all continuous real-valued functions on $X$, such that $f_{x,y}(x) \neq f_{x,y}(y)$;

(2) the correspondence $\Phi : (x, y) \mapsto f_{x,y}$ is a continuous function from $X^2 \setminus \Delta$ to $C_u(X)$, where $C_u(X)$ carries the topology of uniform convergence and $\Delta = \{(x, x) \mid x \in X\}$.

We call the family $\{f_{x,y} \mid (x, y) \in X^2 \setminus \Delta\}$ a *continuous separating family* for $X$.

Clearly, by defining $f_{x,y}(z) = d(x, z)$, any metric space $(X, d)$ has a continuous separating family. Notice that the continuous separating family really depends on only one of its parameters, namely $x$. As mentioned above, Stepanova showed in [6] that

This work was supported by the NSF of China as project no. 10571081

a paracompact $p$-space is continuously Urysohn if and only if it is metrizable. Since then, the concept of continuously Urysohn spaces has been studied intensively. During his investigations of the matter, David Lutzer observed that he did not know of any continuously Urysohn space $X$ for which one could prove that both parameters are required in describing a continuous separating family for $X$.

This leads to the next definition.

DEFINITION 1.2. If $X$ is a continuously Urysohn space for which the corresponding continuous separating family depends on only one of its parameters, say $x$, then this family is called a *one-parameter continuous separating family* for $X$.

In [3], Halbeisen and Hungerbühler constructed a continuously Urysohn space for which the Urysohn functions $f_{x,y}$ cannot be chosen independently of $y$. This answers the question raised by David Lutzer. It is a natural to ask under what conditions a continuously Urysohn space can have a one-parameter continuous separating family. We answer this question by proving Theorem 2.2, which states that a separable space $X$ has a continuous separating family if and only if $X$ has a one-parameter continuous separating family.

## 2. Results

To obtain our main result, we first prove the following theorem.

THEOREM 2.1. *If $X$ is a separable space, then $X$ has a continuous separating family if and only if $X$ has a weaker metric topology.*

PROOF. If $X$ has a weaker topology induced by a metric on $X$, then the continuous separating family that works for the metric topology also works for the given space $X$.

Conversely, suppose that $X$ has a continuous separating family $\{f_{x,y} \mid (x, y) \in X^2 \setminus \Delta\}$ and is separable. Let $M = \Phi[X^2 \setminus \Delta]$. As separability is finite multiplicative, $X^2$ is separable. Because $X$ is a Hausdorff space, $\Delta$ is closed in $X$ and hence $X^2 \setminus \Delta$ is an open subset of $X^2$. We know that separability is hereditary with respect to open subsets, so $X^2 \setminus \Delta$ is also separable. Therefore, as a continuous image of $X^2 \setminus \Delta$, $M$ is separable, too. It is well-known that $C_u(X)$ is a metric space; so $M$ is a separable metric space and there is a set $\{g_n \mid n \in \omega\} \subseteq M$ that is dense in $M$ with respect to the uniform convergence topology inherited from $C_u(X)$. If $x, y \in X$ are distinct points, then $f_{x,y}$ separates $x$ and $y$ and either some subsequence $\{g_{n_j} \mid j \in \omega\}$ of $\{g_n \mid n \in \omega\}$ converges uniformly to $f_{x,y}$, or $f_{x,y} = g_n$ for some $n$. In either case, there is some $n$ such that $g_n(x) \neq g_n(y)$. Define $G : X \to \mathbb{R}^\omega$ by $G(x) = \langle g_0(x), g_1(x), \ldots\rangle$. Then $G$ is a continuous, injective function from $X$ into $\mathbb{R}^\omega$. Since the topology on $\mathbb{R}^\omega$ is metrizable, the topology on $X$ induced by the metric $d_X(x, y) = d_{\mathbb{R}^\omega}(G(x), G(y))$ is a weaker metric topology on $X$. □

THEOREM 2.2. *If $X$ is a separable space, then $X$ has a continuous separating family if and only if $X$ has a one-parameter continuous separating family.*

PROOF. If $X$ has a one-parameter continuous separating family, then it has a continuous separating family.

Conversely, suppose $X$ has a continuous separating family and is separable. Then, applying Theorem 2.1, we know that $X$ has a weaker metric topology. Using [3, Proposition 1.2], we conclude that $X$ has a one-parameter continuous separating family. □

REMARK 2.3. Actually, Theorem 2.1 is a generalization of a result proved by Bennett and Lutzer in [2]. They proved it only for generalized ordered spaces.

As an application, we use Theorem 2.2 to show that certain topological spaces are not separable.

EXAMPLE 2.4. There is a nonseparable topological space that has a continuous separating family but does not have a one-parameter continuous separating family.

PROOF. The space $\boldsymbol{S}$ that we need is constructed in [3, Section 2] as follows. For an ordinal number $\alpha$, let $2^\alpha$ be the set of all functions $\mu : \alpha \to \{0, 1\}$. Let

$$S = \{\mu \mid \mu \in 2^\alpha \text{ for some } \alpha < \omega_1\} \quad \text{and} \quad \bar{S} = \{\bar{\mu} \mid \bar{\mu} \in 2^{\omega_1}\}.$$

For $\mu \in S$, let

$$O_\mu = \{\bar{\mu} \in \bar{S} \mid \mu = \bar{\mu} \restriction \mathrm{dom}(\mu)\},$$

where $\bar{\mu} \restriction \alpha$ is the restriction of the function $\bar{\mu}$ to the set $\alpha$. On the set $S$ we define a partial order as follows: $\nu \preccurlyeq \mu$ if and only if $\mathrm{dom}(\nu) \le \mathrm{dom}(\mu)$ and $\bar{\mu} \restriction \mathrm{dom}(\nu) = \nu$. We write $\nu \prec \mu$ if $\nu \preccurlyeq \mu$ and $\nu \ne \mu$. Further, we use $\{O_\mu \mid \mu \in S\}$ as the base for a topology $\tau$ on the set $\bar{S}$ and define $\boldsymbol{S}= (\bar{S}, \tau)$. It is easy to see that $\boldsymbol{S}$ is a topological space which does not contain isolated points.

In [3], Halbeisen and Hungerbühler showed that $\boldsymbol{S}$ is a continuously Urysohn space which is paracompact and does not have a one-parameter continuous separating family. It follows from Theorem 2.2 that $\boldsymbol{S}$ is not a separable space. □

REMARK 2.5. A topological space with a one-parameter continuous separating family is not necessarily separable. To see this, choose any nonseparable metric space; then the space has a one-parameter continuous separating family. Conversely, a separable topological space does not necessarily have a one-parameter continuous separating family, or even a continuous separating family. To see this, we consider the following example.

EXAMPLE 2.6. There is a separable topological space that does not have a continuous separating family.

PROOF. Let $X$ be the Sorgenfrey line and let $\tilde{X} = \mathbb{R} \times \{0, -1\}$. Regard $\tilde{X}$ as the subset of $\mathbb{R} \times \mathbb{Z}$ with the lexicographical ordering. Then $\tilde{X}$ with the order topology is a linearly ordered extension of $X$. It is well-known that $X$ is dense in $\tilde{X}$. Because $\mathbb{Q}$ is dense in $X$, it is also dense in $\tilde{X}$. Thus, $\tilde{X}$ is separable. However, in [4], Shi and

https://doi.org/10.1017/S0004972709000501 Published online by Cambridge University Press

Gao showed that any linearly ordered extension of the Sorgenfrey line does not admit a continuous separating family. So $\tilde{X}$ does not have a continuous separating family. □

## Acknowledgement

The authors would like to thank the referee for valuable comments.

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https://doi.org/10.1017/S0004972709000501 Published online by Cambridge University Press