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Existence of Multiple Solutions for a p-Laplacian System in $\mathbb{R}^N$ with Sign-changing Weight Functions

Hongxue Song, Caisheng Chen, and Qinglun Yan

Abstract. In this paper, we consider the quasi-linear elliptic problem

$$\begin{aligned} -M\Big(\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p dx\Big)\operatorname{div}\big(|x|^{-ap}|\nabla u|^{p-2}\nabla u\big) &= \frac{\alpha}{\alpha+\beta}H(x)|u|^{\alpha-2}u|v|^{\beta}+\lambda h_1(x)|u|^{q-2}u,\\ -M\Big(\int_{\mathbb{R}^N}|x|^{-ap}|\nabla v|^p dx\Big)\operatorname{div}\big(|x|^{-ap}|\nabla v|^{p-2}\nabla v\big) &= \frac{\beta}{\alpha+\beta}H(x)|v|^{\beta-2}v|u|^{\alpha}+\mu h_2(x)|v|^{q-2}v,\\ u(x)>0,\quad v(x)>0,\quad x\in\mathbb{R}^N, \end{aligned}$$

where $\lambda,\mu>0$, $1<p<N$, $1<q<p<p(\tau+1)<\alpha+\beta<p^*=\frac{Np}{N-p}$, $0\le a<\frac{N-p}{p}$, $a\le b<a+1$, $d=a+1-b>0$, $M(s)=k+ls^{\tau}$, $k>0$, $l,\tau\ge 0$, and the weight $H(x)$, $h_1(x)$, $h_2(x)$ are continuous functions that change sign in $\mathbb{R}^N$. We will prove that the problem has at least two positive solutions by using the Nehari manifold and the fibering maps associated with the Euler functional for this problem.

1 Introduction

By the fibering method, Drabek and Pohozaev [13], Bozhkov and Mitidieri [3] studied respectively the existence of multiple solutions to the following p-Laplacian single equation:

$$-\triangle_p u=\lambda a(x)|u|^{p-2}u+c(x)|u|^{\alpha-1}u\quad\text{in }\Omega,\qquad u=0\quad\text{on }\partial\Omega, \tag{1.1}$$

and system:

$$\begin{aligned} -\triangle_p u &= \lambda a(x)|u|^{p-2}u+(\alpha+1)c(x)|u|^{\alpha-1}u|v|^{\beta+1}, && x\in\Omega,\\ -\triangle_q u &= \mu b(x)|v|^{q-2}v+(\beta+1)c(x)|u|^{\alpha+1}|v|^{\beta-1}v, && x\in\Omega,\\ u(x) &= v(x)=0, && x\in\partial\Omega, \end{aligned}$$

where $p,q>1$, $\triangle_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\Omega\subset\mathbb{R}^N$ is a bounded and connected domain with smooth boundary $\partial\Omega$, λ and μ are positive parameters, α and β are positive numbers, and $a(x), b(x), c(x)\in C(\overline{\Omega})$ are given functions that change sign on $\overline{\Omega}$.

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Recently, Brown and Zhang [5] studied a special case ($p = 2$) of the problem (1.1) by the Nehari manifold [18] and fibering maps. They discuss how the Nehari manifold changed as λ changes and show how existence and non-existence results for positive solutions of this problem are linked to properties of the manifold.

Systems involving quasi-linear operators of p-Laplacian type have been studied by various authors [8, 11, 16, 17, 24]. Among other results, existence and non-existence theorems were obtained. For such purpose, the method of sub-super solutions, the blow-up method, and the Mountain Pass Theorem have been used (see *e.g.*, [11, 17]). For example, Miyagaki and Rodrigues [17] have studied the existence of a positive weak solution to the quasi-linear elliptic system with weights

$$(1.2)\qquad \begin{aligned} -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) &= \lambda|x|^{-(a+1)+c_1}u^{\alpha}v^{\gamma}, && x\in\Omega,\\ -\operatorname{div}(|x|^{-bp}|\nabla v|^{q-2}\nabla v) &= \lambda|x|^{-(b+1)+c_2}u^{\delta}v^{\beta}, && x\in\Omega,\\ u(x) = v(x) &= 0, && x\in\partial\Omega, \end{aligned}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ with

$$\begin{gathered} 0\in\Omega,\quad 1<p,\quad q<N,\quad 0\le a<\tfrac{N-p}{p},\quad 0\le b<\tfrac{N-q}{q},\\ 0\le\alpha<p-1,\quad 0\le\beta<q-1,\quad \delta,\gamma,c_1,c_2>0,\\ \theta=(p-1-\alpha)(q-1-\beta)-\gamma\delta>0. \end{gathered}$$

By the lower and the upper-solution method, they proved that problem (1.2) possesses a positive weak solution $(u,v)\in W_0^{1,p}(\Omega,|x|^{-ap})\times W_0^{1,q}(\Omega,|x|^{-bq})$ for each $\lambda>0$. Similar research can be found in [6, 15, 24] and the references therein. Up until now, much attention has been paid to the existence of solutions for the problems (1.1)–(1.2) in a bounded domain. But for these problems in an unbounded domain Ω or $\mathbb{R}^N$, the existence of a multiplicity of solutions has been a more complicated question.

In this paper, we consider the quasi-linear elliptic problem of the form

$$(1.3)\qquad \begin{cases} -M\Big(\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p dx\Big)\operatorname{div}\big(|x|^{-ap}|\nabla u|^{p-2}\nabla u\big)\\ \qquad\qquad = \dfrac{\alpha}{\alpha+\beta}H(x)|u|^{\alpha-2}u|v|^{\beta}+\lambda h_1(x)|u|^{q-2}u,\\ -M\Big(\int_{\mathbb{R}^N}|x|^{-ap}|\nabla v|^p dx\Big)\operatorname{div}\big(|x|^{-ap}|\nabla v|^{p-2}\nabla v\big)\\ \qquad\qquad = \dfrac{\beta}{\alpha+\beta}H(x)|v|^{\beta-2}v|u|^{\alpha}+\mu h_2(x)|v|^{q-2}v,\\ u(x)>0,\quad v(x)>0,\quad x\in\mathbb{R}^N, \end{cases}$$

where $\lambda,\mu>0$, $1<p<N$, $1<q<p<p(\tau+1)<\alpha+\beta<p^*=\frac{Np}{N-p}$, $0\le a<\frac{N-p}{p}$, $a\le b<a+1$, $d=a+1-b>0$, $M(s)=k+ls^{\tau}$, $k>0$, $l,\tau\ge0$, and the weight $H(x)$, $h_1(x)$, $h_2(x)$ are continuous functions that change sign in $\mathbb{R}^N$.

Problem (1.3) is called nonlocal because of the presence of the term M, which implies that equations in (1.3) are no longer pointwise identities. The problem is analogous to the stationary case of equations that arise in the study of string or membrane vibrations, namely,

$$u_{tt} - \left(k + l\int_{\Omega} |\nabla_x u|^2 dx\right) \triangle_x u = g(x,u),$$

where u denotes the displacement, $g(x,u)$ external force, while l is related to the intrinsic properties of the string (such as Young's modulus). Equations of this type were first proposed by Kirchhoff in 1883 to describe the transversal oscillations of a stretched string, particularly, taking into account the subsequent change in string length caused by oscillations.

The nonlocal effect also finds its applications in biological systems. A parabolic equation of (1.3) can, in theory, be used to describe the growth and movement of a particular species. The movement, modelled by the integral term, is assumed dependent on the energy of the entire system with u being its population density. Alternatively, the movement of a particular species may be subject to the total population density within the domain (for instance, the spreading of bacteria), which gives rise to equations of the type $u_t - l(\int_{\Omega} u dx)\triangle u = g$. Chipot and Lovat [9] and Corrêa [10], for example, studied the existence of solutions and their uniqueness for such nonlocal problems as well as their corresponding elliptic problems.

Motivated by [3, 5, 8, 10, 13, 17, 24], we are concerned here with the existence of multiple positive weak solutions of problem (1.3). Our purpose is to show how to use an idea and a method similar to those in [5] to investigate the p-Laplacian system (1.3), and then get the existence result for multiple positive weak solutions. Many authors proved the existence of multiple solutions for the quasi-linear elliptic equation involving the concave and convex nonlinear terms by Nehari manifold and the fibering maps; see [1, 2, 7, 21, 22] and the references therein. Since $\Omega = \mathbb{R}^N$ is an unbounded domain, the loss of compactness of the Sobolev embedding renders variational technique more delicate.

In fact, in order to preserve this compactness in our problem (1.3), we introduce a weighted Sobolev space and impose some conditions on the weighted functions $H(x)$, $h_1(x)$, and $h_2(x)$. The following Gagliardo–Nirenberg–Sobolev inequality [6] will be needed. There exists a constant $S = S(N,p) > 0$ such that for every $u \in C_0^\infty(\mathbb{R}^N)$,

$$\left(\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*} dx\right)^{1/p^*} \le S\left(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx\right)^{1/p}, \tag{1.4}$$

where $-\infty < a < \frac{N-p}{p}$, $a \le b < a+1$, $d = a+1-b$, and $p^* = \frac{Np}{N-p}$.

Let X be the completion of the space $C_0^\infty(\mathbb{R}^N)$ endowed with the norm of

$$\|u\|_1 = \left(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx\right)^{1/p}.$$

From the standard approximation argument, it is easy to see that inequality (1.4) holds on X.

Let $L_b^{p^*}(\mathbb{R}^N)$ be the completion of the space $C_0^\infty(\mathbb{R}^N)$ endowed with the norm of

$$\|u\|_{L_b^{p^*}} = \left(\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*} dx\right)^{\frac{1}{p^*}}.$$

Throughout the paper, we assume that $H(x), h_1(x), h_2(x)$ satisfy the following conditions:

(A_1) $h_i(x)|x|^{bq} \in L^\theta(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),\ \theta = p^*/(p^* - q),\ i = 1, 2;$

(A_2) $H(x)|x|^{b(\alpha+\beta)} \in L^\delta(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),\ \delta = p^*/(p^* - \alpha - \beta),\ i = 1, 2;$

We set

$$h_{1\theta} = \Big(\int_{\mathbb{R}^N} (|h_1||x|^{bq}\Big)^\theta dx)^{1/\theta}, \quad h_{2\theta} = \Big(\int_{\mathbb{R}^N} (|h_2||x|^{bq})^\theta dx\Big)^{1/\theta} \tag{1.5}$$

with $\theta = p^*/(p^* - q)$.

The natural functional space to study (1.3) is $E = X \times X$ with respect to the norm

$$\|(u, v)\| = \Big(\int_{\mathbb{R}^N} (|x|^{-ap}|\nabla u|^p + |x|^{-ap}|\nabla v|^p)dx\Big)^{1/p}.$$

Then E is a reflexive Banach space endowed with the norm $\|(u, v)\|$.

Definition 1.1 A pair of functions $(u, v) \in E$ is said to be a weak solution of problem (1.3) if for any $(\varphi, \psi) \in E$, there holds

$$\begin{aligned} &M(\|u\|_1) \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^{p-2}\nabla u \nabla\varphi dx \\ &+ M(\|v\|_1) \int_{\mathbb{R}^N} |x|^{-ap}|\nabla v|^{p-2}\nabla v \nabla\psi dx \\ &- \frac{\alpha}{\alpha+\beta} \int_{\mathbb{R}^N} H|u|^{\alpha-2}u|v|^\beta \varphi dx - \frac{\beta}{\alpha+\beta} \int_{\mathbb{R}^N} H|v|^{\beta-2}v|u|^\alpha \psi dx \\ &- \lambda \int_{\mathbb{R}^N} h_1|u|^{q-2}u\varphi dx - \mu \int_{\mathbb{R}^N} h_2|v|^{q-2}v\psi dx = 0. \end{aligned} \tag{1.6}$$

By the assumptions (A_1) and (A_2), all the integrals in (1.6) are well defined and convergent.

Our main result is the following theorem.

Theorem 1.2 *Assume* (A_1) *and* (A_2) *are fulfilled. There exists* $\Lambda_0 > 0$ *such that if the parameters* $\lambda, \mu > 0$ *satisfy*

$$0 < \lambda h_{1\theta} + \mu h_{2\theta} < \Lambda_0,$$

then problem (1.3) *has at least two positive solutions, where* $h_{1\theta}$, $h_{2\theta}$ *are given by* (1.5).

This paper is organized as follows. In Section 2, we give some properties of the Nehari manifold and set up the variational framework of problem (1.3). In Section 3, we prove Theorem 1.2.

2 Preliminaries

It is clear that problem (1.3) has a variational structure. Let $J(u,v)\colon E \to \mathbb{R}^1$ be the corresponding Euler functional of problem (1.3), which is defined by

$$J(u,v) = \frac{k}{p}\|(u,v)\|^p + \frac{l}{\sigma}\|(u,v)\|^\sigma - \frac{1}{m}\int_{\mathbb{R}^N} H|u|^\alpha|v|^\beta dx - \frac{1}{q}F(u,v),$$

where $\sigma = p(\tau+1)$, $m = \alpha+\beta$, and

$$F(u,v) = \lambda\int_{\mathbb{R}^N} h_1|u|^q dx + \mu\int_{\mathbb{R}^N} h_2|v|^q dx.$$

Then we see that the functional $J \in C^1(E,\mathbb{R}^1)$ and for for all $(\varphi,\psi) \in E$, there holds

$$\begin{aligned}(2.1)\quad \langle J'(u,v),(\varphi,\psi)\rangle = {} & M(\|u\|_1)\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^{p-2}\nabla u\nabla\varphi dx \\ & + M(\|v\|_1)\int_{\mathbb{R}^N} |x|^{-ap}|\nabla v|^{p-2}\nabla v\nabla\psi dx \\ & - \frac{\alpha}{m}\int_{\mathbb{R}^N} H|u|^{\alpha-2}u|v|^\beta\varphi dx - \frac{\beta}{m}\int_{\mathbb{R}^N} H|v|^{\beta-2}v|u|^\alpha\psi dx \\ & - \lambda\int_{\mathbb{R}^N} h_1|u|^{q-2}u\varphi dx - \mu\int_{\mathbb{R}^N} h_2|v|^{q-2}v\psi dx,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual duality. In particular, it follows from (2.1) that

$$\Big\langle J'(u,v),(u,v)\Big\rangle = k\|u,v\|^p + l\|u,v\|^\sigma - \int_{\mathbb{R}^N} H|u|^\alpha|v|^\beta dx - F(u,v).$$

It is well known that the weak solution of problem (1.3) is the critical point of the Euler functional J(see [20]). As J is not bounded below on E, it is useful to consider the functional J on the Nehari manifold

$$\mathcal{N} = \Big\{(u,v) \in E\backslash(0,0) \mid \langle J'(u,v),(u,v)\rangle = 0\Big\}.$$

Thus, $(u,v) \in \mathcal{N}$ if and only if

$$k\|(u,v)\|^p + l\|(u,v)\|^\sigma - \int_{\mathbb{R}^N} H|u|^\alpha|v|^\beta dx - F(u,v) = 0.$$

In particular, on $\mathcal{N}$ we have

$$\begin{aligned}(2.2)\quad J(u,v) &= k\Big(\frac{1}{p}-\frac{1}{q}\Big)\|(u,v)\|^p + l\Big(\frac{1}{\sigma}-\frac{1}{q}\Big)\|(u,v)\|^\sigma \\ &\quad - \Big(\frac{1}{m}-\frac{1}{q}\Big)\int_{\mathbb{R}^N} H|u|^\alpha|v|^\beta dx \\ &= k\Big(\frac{1}{p}-\frac{1}{m}\Big)\|(u,v)\|^p + l\Big(\frac{1}{\sigma}-\frac{1}{m}\Big)\|(u,v)\|^\sigma - \Big(\frac{1}{q}-\frac{1}{m}\Big)F(u,v).\end{aligned}$$

Furthermore, we define

$$\Phi(u,v) = \Big\langle J'(u,v),(u,v)\Big\rangle, \quad \forall (u,v) \in E.$$

Then for any $(u,v) \in \mathcal{N}$, we have

$$
\begin{aligned}
(2.3)\quad & \langle \Phi'(u,v),(u,v)\rangle \\
&= kp\|(u,v)\|^p + l\sigma\|(u,v)\|^\sigma - m\int_{\mathbb{R}^N} H|u|^\alpha|v|^\beta dx - qF(u,v) \\
&= k(p-q)\|(u,v)\|^p + l(\sigma-q)\|(u,v)\|^\sigma - (m-q)\int_{\mathbb{R}^N} H|u|^\alpha|v|^\beta dx \\
&= k(p-m)|(u,v)\|^p + l(\sigma-m)\|(u,v)\|^\sigma - (q-m)F(u,v).
\end{aligned}
$$

It is natural to split $\mathcal{N}$ into three parts:

$$
\begin{aligned}
(2.4)\qquad \mathcal{N}^+ &= \{(u,v)\in\mathcal{N}, \mid \langle\Phi'(u,v),(u,v)\rangle > 0\}, \\
\mathcal{N}^0 &= \{(u,v)\in\mathcal{N}, \mid \langle\Phi'(u,v),(u,v)\rangle = 0\}, \\
\mathcal{N}^- &= \{(u,v)\in\mathcal{N}, \mid \langle\Phi'(u,v),(u,v)\rangle < 0\}.
\end{aligned}
$$

We now derive some properties of $\mathcal{N}$.

Lemma 2.1 *J is coercive and bounded below on $\mathcal{N}$.*

Proof Since $h_i(x)|x|^{bq} \in L^\theta(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, $\theta = p^*/(p^*-q)$, $(i=1,2)$, we obtain from the Hölder and Caffarelli–Kohn–Nirenberg inequalities that

$$
\begin{aligned}
\int_{\mathbb{R}^N} h_1|u|^q dx &\le \Big(\int_{\mathbb{R}^N} (|h_1||x|^{bq})^\theta dx)^{1/\theta}\Big(\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*}dx\Big)^{q/p^*} \\
&\le h_{1\theta}S^q\Big(\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p dx\Big)^{q/p} \le h_{1\theta}S^q\|(u,v)\|^q.
\end{aligned}
$$

Similarly, we have $\int_{\mathbb{R}^N} h_2|u|^q dx \le h_{2\theta}S^q\|(u,v)\|^q$. Then

$$
(2.5)\qquad F(u,v) \le (\lambda h_{1\theta} + \mu h_{2\theta})S^q\|(u,v)\|^q.
$$

It follows from (2.2) and (2.5) that

$$
\begin{aligned}
(2.6)\quad J(u,v) \ge k\Big(\frac{1}{p}-\frac{1}{m}\Big)\|(u,v)\|^p + l\Big(\frac{1}{\sigma}-\frac{1}{m}\Big)\|(u,v)\|^\sigma \\
- \Big(\frac{1}{q}-\frac{1}{m}\Big)(\lambda h_{1\theta}+\mu h_{2\theta})S^q\|(u,v)\|^q.
\end{aligned}
$$

Since $q < p \le \sigma < m$, inequality (2.6) shows that J is coercive and bounded below on $\mathcal{N}$. Thus, the proof is completed. ∎

Lemma 2.2 *There exists $\Lambda_1 > 0$ such that $\mathcal{N}^0 = \varnothing$ for all λ, μ, which satisfy $0 < \lambda h_{1\theta} + \mu h_{2\theta} < \Lambda_1$, where $h_{1\theta}$ and $h_{2\theta}$ are given by (1.5).*

Proof In fact, we let

$$
(2.7)\qquad \Lambda_1 = \frac{k(m-p)}{(m-q)S^q}\Big(\frac{k(p-q)}{(m-q)H_\delta S^m}\Big)^{(p-q)/(m-p)},
$$

where $\delta = p^*/(p^* - m)$ and

$$H_\delta = \Big(\int_{\mathbb{R}^N} (|H||x|^{bm}\Big)^\delta dx)^{1/\delta} < +\infty.$$

Suppose otherwise; thus, there exist λ and μ that satisfy $0 < \lambda h_{1\theta} + \mu h_{2\theta} < \Lambda_1$, such that $\mathcal{N}^0 \neq \varnothing$; that is, there exists $(u, v) \in \mathcal{N}^0$. Then, it follows from (2.3)–(2.5) that

$$\|(u, v)\| \leq \Big(\frac{(m-q)(\lambda h_{1\theta} + \mu h_{2\theta}) S^q}{k(m-p)} \Big)^{1/(p-q)}. \tag{2.8}$$

By (A_2) and the Hölder inequality we have

$$\begin{aligned} \int_{\mathbb{R}^N} H|u|^m dx &\leq \Big(\int_{\mathbb{R}^N} \big(|H||x|^{bm}\big)^\delta dx \Big)^{1/\delta} \Big(\int_{\mathbb{R}^N} |x|^{-bp^*} |u|^{p^*} dx \Big)^{m/p^*} \\ &\leq H_\delta S^m \|(u, v)\|^m, \end{aligned}$$

Similarly, we have $\int_{\mathbb{R}^N} H|v|^m dx \leq H_\delta S^m \|(u, v)\|^m$. Hence,

$$\int_{\mathbb{R}^N} H|u|^\alpha |v|^\beta dx \leq H_\delta S^m \|(u, v)\|^m. \tag{2.9}$$

Therefore, from (2.3)–(2.4) and (2.9) we have

$$\|(u, v)\| \geq \Big(\frac{k(p-q)}{(m-q) H_\delta S^m} \Big)^{1/(m-p)}. \tag{2.10}$$

Relations (2.8) and (2.10) give that $\lambda h_{1\theta} + \mu h_{2\theta} \geq \Lambda_1$, which is a contradiction. This completes the proof. ∎

By Lemma 2.2, we write $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$ for $0 < \lambda h_{1\theta} + \mu h_{2\theta} < \Lambda_1$ and define

$$\delta^+ = \inf_{(u,v) \in \mathcal{N}^+} J(u, v), \quad \delta^- = \inf_{(u,v) \in \mathcal{N}^-} J(u, v).$$

Also, as proved in Binding, Drabek, and Huang [5] or in Brown and Zhang [2], we have the following lemma.

Lemma 2.3 *For $0 < \lambda h_{1\theta} + \mu h_{2\theta} < \Lambda_1$. Suppose (u_0, v_0) is a local minimizer for J on $\mathcal{N}$. Then if $(u_0, v_0) \notin \mathcal{N}^0$, then (u_0, v_0) is a critical point of J.*

Lemma 2.4 *If λ and μ satisfy $0 < \lambda h_{1\theta} + \mu h_{2\theta} < \frac{q}{p} \Lambda_1$, then*

(i) $\delta^+ < 0$,
(ii) $\exists \gamma_0 > 0$ *such that* $\delta^- > \gamma_0$.

Proof (i) Let $(u, v) \in \mathcal{N}^+$. It follows from (2.3) and (2.4) that

$$\int_{\mathbb{R}^N} H|u|^\alpha |v|^\beta dx < \frac{k(p-q)}{m-q} \|(u, v)\|^p + \frac{l(\sigma - q)}{m-q} \|(u, v)\|^\sigma. \tag{2.11}$$

Then by (2.2) and (2.11), we have that

$$J(u, v) < -\frac{k(p-q)(m-p)}{pqm} \|(u, v)\|^p - \frac{l(m-p)(\sigma - q)}{p\sigma m} \|(u, v)\|^\sigma < 0,$$

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which gives

$$\delta^+ = \inf_{(u,v)\in\mathcal{N}^+} J(u,v) < 0.$$

(ii) Let $(u,v) \in \mathcal{N}^-$. From (2.2) and (2.5) we have

$$\begin{aligned}J(u,v) &\geq k\frac{m-p}{mp}\|(u,v)\|^p - \frac{m-q}{mq}(\lambda h_{1\theta} + \mu h_{2\theta})S^q\|(u,v)\|^q \\ &= \|(u,v)\|^q\Big(k\frac{m-p}{mp}\|(u,v)\|^{p-q} - \frac{m-q}{mq}(\lambda h_{1\theta}+\mu h_{2\theta})S^q\Big).\end{aligned} \tag{2.12}$$

Thus, it follows from (2.10) and (2.12) that

$$\begin{aligned}J(u,v) &\geq \Big(\frac{k(p-q)}{(m-q)H_\delta S^m}\Big)^{\frac{q}{m-p}} \\ &\quad\times\Big(k\frac{m-p}{mp}\Big(\frac{k(p-q)}{(m-q)H_\delta S^m}\Big)^{\frac{p-q}{m-p}} - \frac{m-q}{mq}(\lambda h_{1\theta}+\mu h_{2\theta})S^q\Big).\end{aligned}$$

If $0 < \lambda h_{1\theta} + \mu h_{2\theta} < \frac{q}{p}\Lambda_1$, then there exists $\gamma_0(p,q,\alpha,\beta,H_\delta,S) > 0$ such that $\delta^- > \gamma_0$. Thus, the proof of Lemma 2.4 is completed. ■

For each $(u,v) \in E$ with $\int_{\mathbb{R}^N} H|u|^\alpha|v|^\beta dx > 0$, we set

$$z(t) = kt^{p-q}\|(u,v)\|^p + lt^{\sigma-q}\|(u,v)\|^\sigma - t^{m-q}\int_{\mathbb{R}^N} H|u|^\alpha|v|^\beta dx.$$

Then $z'(t) = t^{p-q-1}E(t)$, where

$$E(t) = k(p-q)\|(u,v)\|^p + l(\sigma-q)t^{p\tau}\|(u,v)\|^\sigma - (m-q)t^{m-p}\int_{\mathbb{R}^N} H|u|^\alpha|v|^\beta dx.$$

Set

$$t^* = \Big(\frac{l(\sigma-q)p\tau\|(u,v)\|^\sigma}{(m-q)(m-p)\int_{\mathbb{R}^N} H|u|^\alpha|v|^\beta dx}\Big)^{1/(m-\sigma)}.$$

Then it is easy to see that $E(t)$ achieves its maximum at t^*, increasing for $t \in [0,t^*)$ and decreasing for $t \in (t^*,\infty)$. Since $E(0) > 0$ and $E(t) \to -\infty$ as $t \to \infty$, $E(t^*) > 0$ and there exists a unique $0 < t^* < t_l$ such that $E(t_l) = 0$ and $z(t)$ achieves its maximum at t_l, increasing for $t \in [0,t_l)$ and decreasing for $t \in (t_l,\infty)$. In particular, for $l = 0$, we have

$$t_0 = \Big(\frac{k(p-q)\|(u,v)\|^p}{(m-q)\int_{\mathbb{R}^N} H|u|^\alpha|v|^\beta dx}\Big)^{1/(m-p)} \tag{2.13}$$

and $E(t_0) = E(t_l) = 0$ implies $t_0 \leq t_l$ for $l \geq 0$. Thus,

$$z(t_l) \geq k\frac{m-p}{m-q}t_l^{p-q}\|(u,v)\|^p \geq k\frac{m-p}{m-q}t_0^{p-q}\|(u,v)\|^p = z(t_0). \tag{2.14}$$

Lemma 2.5 *Assume $\int_{\mathbb{R}^N} H|u|^\alpha|v|^\beta dx > 0$ and $0 < \lambda h_{1\theta} + \mu h_{2\theta} < \Lambda_1$.*

(i) *If $F(u,v) \leq 0$, there exists unique $t^- > t_l$ such that $(t^-u, t^-v) \in \mathcal{N}^-$ and*

$$J(t^-u, t^-v) = \sup_{t\geq 0} J(tu,tv).$$

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(ii) *If $F(u,v) > 0$, there exist $0 < t^+ < t_l < t^-$ such that $(t^+u, t^+v) \in \mathcal{N}^+$, $(t^-u, t^-v) \in \mathcal{N}^-$ and*

$$J(t^+u, t^+v) = \inf_{0 \le t \le t_l} J(tu, tv), \quad J(t^-u, t^-v) = \sup_{t \ge 0} J(tu, tv).$$

Proof Set

$$\begin{aligned}\Psi_0(t) &= \Phi(tu, tv) = \langle J'(tu, tv), (tu, tv)\rangle \\ &= kt^p\|(u,v)\|^p + lt^\sigma\|(u,v)\|^\sigma - t^m\int_{\mathbb{R}^N} H|u|^\alpha|v|^\beta dx - t^q F(u,v), \\ \Psi_1(t) &= \langle \Phi'(tu, tv), (tu, tv)\rangle \\ &= kpt^p\|(u,v)\|^p + l\sigma t^\sigma\|(u,v)\|^\sigma - mt^m\int_{\mathbb{R}^N} H|u|^\alpha|v|^\beta dx - qt^q F(u,v), \\ \Psi_2(t) &= J(tu, tv) \\ &= \frac{kt^p}{p}\|(u,v)\|^p + \frac{lt^\sigma}{\sigma}\|(u,v)\|^\sigma - \frac{t^m}{m}\int_{\mathbb{R}^N} H|u|^\alpha|v|^\beta dx - \frac{t^q}{q}F(u,v).\end{aligned}$$

Then

$$\Psi_0(t) = t^q\big(z(t) - F(u,v)\big). \tag{2.15}$$

(i) $F(u,v) \le 0$: There exists a unique $t^- > t_l$ such that $z(t^-) = F(u,v)$. It follows from (2.15) that $\Psi_0(t^-) = 0$ and $(t^-u, t^-v) \in \mathcal{N}$. Then, $\Psi_1(t^-) = (t^-)^{q+1}z'(t^-) < 0$, which implies that $(t^-u, t^-v) \in \mathcal{N}^-$. By simple calculation, we obtain that $\Psi_2'(t) = t^{q-1}(z(t) - F(u,v))$. Furthermore, $\Psi_2'(t) > 0$ for $t \in [0, t^-)$, $\Psi_2'(t) < 0$ for $t \in [t^-, +\infty)$. Then $\Psi_2(t)$ gets its maximum at t^-; that is,

$$J(t^-u, t^-v) = \sup_{t \ge 0} J(tu, tv).$$

(ii) $F(u,v) > 0$: Since $0 < \lambda h_{1\theta} + \mu h_{2\theta} < \Lambda_1$, by (2.7) and (2.13)–(2.14), we get that

$$0 < F(u,v) \le (\lambda h_{1\theta} + \mu h_{2\theta})S^q\|(u,v)\|^q \le z(t_0) \le z(t_l).$$

Then there exist t^+ and t^- such that $0 < t^+ < t_l < t^-$ and $z(t^+) = z(t^-) = F(u,v)$. Similar to the argument in (i), we have $(t^+u, t^+v) \in \mathcal{N}^+$ and $(t^-u, t^-v) \in \mathcal{N}^-$. Since $\Psi_2'(t) < 0$ for $t \in [0, t^+)$ and $\Psi_2'(t) > 0$ for $t \in [t^+, t_l)$, $J(t^+u, t^+v) = \inf_{0 \le t \le t_l} J(tu, tv)$. Furthermore, it is easy to find that $\Psi_2'(t) > 0$ for $t \in [t^+, t^-)$, $\Psi_2'(t) < 0$ for $t \in [t^-, +\infty)$ and $\Psi_2(t) \le 0$ for $t \in [0, t^+]$. Since $(t^-u, t^-v) \in \mathcal{N}^-$, by Lemma 2.4(ii), we have $\Psi_2(t^-) > 0$. Then $J(t^-u, t^-v) = \sup_{t \ge 0} J(tu, tv)$. This completes the proof of Lemma 2.5. ■

For each $(u,v) \in E$ with $F(u,v) > 0$, we set

$$\eta(t) = kt^{p-m}\|(u,v)\|^p + lt^{\sigma-m}\|(u,v)\|^\sigma - t^{q-m}F(u,v), \quad t > 0.$$

Then it is easy to check that $\eta(t) \to -\infty$ as $t \to 0^+$, $\eta(t) \to 0$ as $t \to +\infty$, and $\eta(t)$ achieves its maximum at some $t = T_l$.

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Lemma 2.6 *For each $(u,v) \in E$ with $F(u,v) > 0$ and $0 < \lambda h_{1\theta} + \mu h_{2\theta} < \Lambda_1$, the following hold:*

(i) *If $\int_{\mathbb{R}^N} H|u|^\alpha |v|^\beta dx \leq 0$, then there exists unique $t^+ < T_l$ such that $(t^+u, t^+v) \in \mathcal{N}^+$ and*

$$J(t^+u, t^+v) = \inf_{0 \leq t \leq T_l} J(tu, tv).$$

(ii) *If $\int_{\mathbb{R}^N} H|u|^\alpha |v|^\beta dx > 0$, there exist $0 < t^+ < T_l < t^-$ such that $(t^+u, t^+v) \in \mathcal{N}^+$, $(t^-u, t^-v) \in \mathcal{N}^-$ and*

$$J(t^+u, t^+v) = \inf_{0 \leq t \leq T_l} J(tu, tv), \quad J(t^-u, t^-v) = \sup_{t \geq 0} J(tu, tv).$$

Proof Note that $\Psi_2'(t) = t^{m-1}(\eta(t) - \int_{\mathbb{R}^N} H|u|^\alpha |v|^\beta dx)$, similar to the argument in the proof of Lemma 2.5, we can obtain the results of Lemma 2.6. ■

As proved in [23], we have the following lemma.

Lemma 2.7 *If $u_n \rightharpoonup u_0$, $v_n \rightharpoonup v_0$ weakly in E, then there exists a subsequence of $\{(u_n, v_n)\}$, still denoted by $\{(u_n, v_n)\}$, such that*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} h_1|u_n|^q dx = \int_{\mathbb{R}^N} h_1|u_0|^q dx, \quad \lim_{n\to+\infty} \int_{\mathbb{R}^N} h_2|v_n|^q dx = \int_{\mathbb{R}^N} h_2|v_0|^q dx,$$
$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} H|u_n|^\alpha |v_n|^\beta dx = \int_{\mathbb{R}^N} H|u_0|^\alpha |v_0|^\beta dx,$$

3 Existence of Positive Solutions

First, we will use the idea of Ni–Takagi [19] to get the following results.

Lemma 3.1 *For each $(u,v) \in \mathcal{N}^+$, there exists $\epsilon > 0$ and a differential function $t\colon B_\epsilon(0,0) \subset E \to \mathbb{R}^1$ such that $t(0,0) = 1$, the function $t(\nu,\omega)(u-\nu, v-\omega) \in \mathcal{N}^+$ for $(\nu,\omega) \in B_\epsilon(0,0)$, and*

(3.1)
$$\begin{aligned}
&\langle (t'(0,0), (\varphi,\psi) \rangle \\
&= -[k(p-q)\|(u,v)\|^p + l(\sigma-q)\|(u,v)\|^\sigma - (m-q)\int_{\mathbb{R}^N} H|u|^\alpha |v|^\beta dx]^{-1} \\
&\quad \cdot [(kp + l\sigma\|(u,v)\|^{p\tau}) \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^{p-2} \nabla u \nabla v dx - \int_{\mathbb{R}^N} \alpha H|u|^{\alpha-2} u|v|^\beta \varphi dx \\
&\quad - \beta \int_{\mathbb{R}^N} H|u|^\alpha |v|^{\beta-2} v\psi dx - \lambda \int_{\mathbb{R}^N} h_1|u|^{q-2} u\nu dx - \mu \int_{\mathbb{R}^N} h_2|v|^{q-2} v\omega dx]
\end{aligned}$$

for any $(\varphi,\psi) \in E$.

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Proof For $(u,v) \in \mathcal{N}^+$, we define $G_{(u,v)}(t,(\nu,\omega))\colon \mathbb{R}^+ \times E \to \mathbb{R}^1$ by

$$\begin{aligned} G_{(u,v)}\big(t,(\nu,\omega)\big) &= \Big\langle J'\big(t(u-\nu),t(v-\omega)\big),\big(t(u-\nu),t(v-\omega)\big)\Big\rangle \\ &= \big(k+l\|(t(u-\nu),t(v-\omega)\big)\|^{p\tau}\big)\|\big(t(u-\nu),t(v-\omega)\big)\|^p \\ &\quad -\int_{\mathbb{R}^N} H|t(u-\nu)|^\alpha|t(v-\omega)|^\beta dx - F\big(t(u-\nu),t(v-\omega)\big). \end{aligned}$$

Then $G_{(u,v)}(1,(0,0)) = \langle J'(u,v),(u,v)\rangle = 0$ and by (2.3)–(2.4),

$$\begin{aligned} &\frac{\partial}{\partial t}\big[G_{(u,v)}(t,(0,0))\big]|_{t=1} \\ &\quad = \big(kp + l\sigma\|(u,v)\|^{p\tau}\big)\|(u,v)\|^p - m\int_{\mathbb{R}^N} H|u|^\alpha|v|^\beta dx - qF(u,v) \\ &\quad = k(p-q)\|(u,v)\|^p + l(\sigma-q)\|(u,v)\|^\sigma - (m-q)\int_{\mathbb{R}^N} H|u|^\alpha|v|^\beta dx > 0. \end{aligned}$$

According to the implicit function theorem, there exist $\epsilon > 0$ and a differential function $t\colon B_\epsilon(0,0)\subset E \to \mathbb{R}^1$ such that $t(0,0)=1$ and

$$\begin{aligned} &\langle(t'(0,0),(\varphi,\psi)\rangle \\ &\quad = -[k(p-q)\|(u,v)\|^p + l(\sigma-q)\|(u,v)\|^\sigma - (m-q)\int_{\mathbb{R}^N} H|u|^\alpha|v|^\beta dx]^{-1} \\ &\quad\cdot [kp + l\sigma\|(u,v)\|^{p\tau})\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^{p-2}\nabla u\nabla\nu dx - \alpha\int_{\mathbb{R}^N} H|u|^{\alpha-2}u|v|^\beta\varphi dx \\ &\quad - \beta\int_{\mathbb{R}^N} H|u|^\alpha|v|^{\beta-2}v\psi dx - \lambda\int_{\mathbb{R}^N} h_1|u|^{q-2}u\nu dx - \mu\int_{\mathbb{R}^N} h_2|v|^{q-2}v\omega dx] \end{aligned}$$

for any $(\varphi,\psi)\in E$. Additionally, for any $(\nu,\omega)\in B_\epsilon(0,0)$,

$$G_{(u,v)}\big(t(\nu,\omega),(\nu,\omega)\big) = 0, \quad \text{for any } (\nu,\omega)\in B_\epsilon(0,0),$$

which is equivalent to

$$\Big\langle J'(t(\nu,\omega)(u-\nu,v-\omega)),t(\nu,\omega)(u-\nu,v-\omega)\Big\rangle = 0;$$

that is,

$$t(\nu,\omega)(u-\nu,v-\omega)\in\mathcal{N}.$$

Since

$$\begin{aligned} \langle\Phi'(u,v),(u,v)\rangle = k(p-q)\|(u,v)\|^p + l(\sigma-q)\|(u,v)\|^\sigma \\ - (m-q)\int_{\mathbb{R}^N} H|u|^\alpha|v|^\beta dx > 0, \end{aligned}$$

by the continuity of function $\Phi'(\nu,\omega)$, $t(\nu,\omega)$ and $t(0,0)=1$, we have that for any $(\nu,\omega)\in B_\epsilon(0,0)$,

$$\begin{aligned} &\langle\Phi'(t(\nu,\omega)(u-\nu,v-\omega)),t(\nu,\omega)(u-\nu,v-\omega)\rangle \\ &\quad = k(p-q)\|t(\nu,\omega)(u-\nu,v-\omega)\|^p + l(\sigma-q)\|t(\nu,\omega)(u-\nu,v-\omega)\|^\sigma \\ &\quad - (m-q)\int_{\mathbb{R}^N} H|t(\nu,\omega)(u-\nu)|^\alpha|t(\nu,\omega)(v-\omega)|^\beta dx > 0 \end{aligned}$$

if $\epsilon > 0$ is sufficiently small. This implies that $t(\nu,\omega)(u-\nu, v-\omega) \in \mathcal{N}^+$ for any $(\nu,\omega) \in B_\epsilon(0,0)$. This completes the proof of Lemma 3.1. ■

Lemma 3.2 *Assume* (A_1) *and* (A_2). *Let* $\Lambda_0 = \min\{\frac{q}{p}\Lambda_1, \Lambda_2\}$, *where*

$$\Lambda_2 = \Big(\frac{k(m-p)}{m-q}\Big)^{q/p} \frac{1}{S^q} \Big(\frac{m-\sigma}{\sigma-q} \Big(\frac{k(p-q)}{m-q} \Big)^{m/(m-p)} \frac{1}{(H_\delta S^m)^{m/(m-p)}} \Big)^{(p-q)/p}.$$

Then for $0 < \lambda h_{1\theta} + \mu h_{2\theta} < \Lambda_0$, *there exists a minimizing sequence* $\{(u_n, v_n)\} \subset \mathcal{N}^+$ *such that,*

$$J(u_n, v_n) \to \delta^+, \quad J'(u_n, v_n) \to 0 \quad \textit{in } E^* \textit{ as } n \to \infty.$$

Proof By the Ekeland variational principle [14], there exists a minimizing sequence $\{(u_n, v_n)\} \subset \mathcal{N}^+$ such that

$$J(u_n, v_n) < \delta^+ + \frac{1}{n}$$

$$J(u_n, v_n) < J(\nu,\omega) + \frac{1}{n}\|(u_n - \nu, v_n - \omega)\| \text{ for each } (\nu,\omega) \in \mathcal{N}^+. \tag{3.2}$$

By taking large n, from Lemma 2.4(i), we have

$$\begin{aligned} J(u_n, v_n) &= k\Big(\frac{1}{p} - \frac{1}{m}\Big) \big\|(u_n, v_n)\big\|^p \\ &\quad + l\Big(\frac{1}{\sigma} - \frac{1}{m}\Big)\big\|(u_n, v_n)\big\|^\sigma - \Big(\frac{1}{q} - \frac{1}{m}\Big) F(u_n, v_n) \\ &< \delta^+ + \frac{1}{n} < \frac{\delta^+}{2}. \end{aligned} \tag{3.3}$$

This implies

$$-\frac{mq}{m-q}\frac{\delta^+}{2} < F(u_n, v_n) \le (\lambda h_{1\theta} + \mu h_{2\theta}) S^q \|(u_n, v_n)\|^q. \tag{3.4}$$

Consequently, $(u_n, v_n) \neq (0,0)$ and putting together (3.3), (3.4), and the Hölder inequality, we obtain

$$\|(u_n, v_n)\| > \Big(-\frac{mq}{m-q}\frac{\delta^+}{2}\frac{1}{(\lambda h_{1\theta} + \mu h_{2\theta})S^q}\Big)^{1/q}, \tag{3.5}$$

$$\|(u_n, v_n)\| < \Big(\frac{p(m-q)}{kq(m-p)}(\lambda h_{1\theta} + \mu h_{2\theta})S^q\Big)^{1/(p-q)}. \tag{3.6}$$

Now, we will show that $\|J'(u_n, v_n)\|_{E^*} \to 0$ as $n \to \infty$. Applying Lemma 3.1 with (u_n, v_n) to obtain the functions $t_n \colon B_{\epsilon_n}(0,0) \subset E \to \mathbb{R}^1$ for some $\epsilon_n > 0$, such that $t_n(\nu,\omega)(u_n - \nu, v_n - \omega) \in \mathcal{N}^+$ for any $(\nu,\omega) \in B_{\epsilon_n}(0,0)$. Fixed $n \in \mathbb{N}$, we choose $0 < \rho < \epsilon_n$. Let $(u,v) \in E\backslash\{(0,0)\}$ and $(\nu_\rho, \omega_\rho) = \frac{\rho(u,v)}{\|(u,v)\|}$; then $(\nu_\rho, \omega_\rho) \in B_{\epsilon_n}(0,0)$ and

$$(\varphi_\rho, \psi_\rho) = t_n(\nu_\rho, \omega_\rho)(u_n - \nu_\rho, v_n - \omega_\rho) \in \mathcal{N}^+.$$

Thus, we deduce from (3.2) that

$$J(\varphi_\rho, \psi_\rho) - J(u_n, v_n) > -\frac{1}{n}\|(\varphi_\rho - u_n, \psi_\rho - v_n)\|$$

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and by the mean value theorem, we have

$$\Big\langle J'(u_n,v_n),(\varphi_\rho-u_n,\psi_\rho-v_n)\Big\rangle \geq -\frac{1}{n}\big\|(\varphi_\rho-u_n,\psi_\rho-v_n)\big\|+o\big(\|(\varphi_\rho-u_n,\psi_\rho-v_n)\|\big).$$

Therefore,

$$\text{(3.7)}\quad \langle J'(u_n,v_n),-(\nu_\rho,\omega_\rho)\rangle+(t_n(\nu_\rho,\omega_\rho)-1)\langle J'(u_n,v_n),(u_n-\nu_\rho,v_n-\omega_\rho)\rangle \geq \\ -\frac{1}{n}\|(\varphi_\rho-u_n,\psi_\rho-v_n)\|+o(\|(\varphi_\rho-u_n,\psi_\rho-v_n)\|).$$

It follows from $t_n(\nu,\omega)(u_n-\nu,v_n-\omega)\in\mathcal{N}$ and (3.7) that

$$\begin{aligned}
&-\rho\Big\langle J'(u_n,v_n),\frac{(u,v)}{\|(u,v)\|}\Big\rangle \\
&\qquad+(t_n(\nu_\rho,\omega_\rho)-1)\Big\langle J'(u_n,v_n)-J'(\varphi_\rho,\psi_\rho),(u_n-\nu_\rho,v_n-\omega_\rho)\Big\rangle \\
&\qquad\qquad\geq -\frac{1}{n}\|(\varphi_\rho-u_n,\psi_\rho-v_n)\|+o\big(\|(\varphi_\rho-u_n,\psi_\rho-v_n)\|\big).
\end{aligned}$$

Hence,

(3.8)

$$\begin{aligned}
\Big\langle J'(u_n,v_n),\frac{(u,v)}{\|(u,v)\|}\Big\rangle &\leq \frac{1}{n\rho}\|(\varphi_\rho-u_n,\psi_\rho-v_n)\|+\frac{o(\|(\varphi_\rho-u_n,\psi_\rho-v_n)\|)}{\rho} \\
&\quad+\frac{(t_n(\nu_\rho,\omega_\rho)-1)}{\rho}\Big\langle J'(u_n,v_n)-J'(\varphi_\rho,\psi_\rho),(u_n-\nu_\rho,v_n-\omega_\rho)\Big\rangle.
\end{aligned}$$

Since

$$\begin{aligned}
&\|\varphi_\rho-u_n,\psi_\rho-v_n)\|\leq\rho|t_n(\nu_\rho,\omega_\rho)|+|(t_n(\nu_\rho,\omega_\rho)-1)|\,\|(u_n,v_n)\|,\\
&\lim_{\rho\to0}\frac{|(t_n(\nu_\rho,\omega_\rho)-1)|}{\rho}\leq\|t_n'(0,0)\|,
\end{aligned}$$

if we let $\rho\to0$ in (3.8), then by (3.6) we can find a constant $C_2>0$, independent of ρ, such that

$$\Big\langle J'(u_n,v_n),\frac{(u,v)}{\|(u,v)\|}\Big\rangle\leq\frac{C_2}{n}\big(1+\|t_n'(0,0)\|\big).$$

We are done once we show that $\|t_n'(0,0)\|$ is uniformly bounded with respect to n. By (3.1), (3.6), and the Hölder inequality, we know that there exists some constant $C_3>0$, independent of n, such that

$$\big|\langle t_n'(0,0),(\varphi,\psi)\rangle\big|\leq\frac{C_3\|(\varphi,\psi)\|}{|B_n|},\quad\forall(\varphi,\psi)\in E$$

where

$$B_n=k(p-q)\|(u_n,v_n)\|^p+l(\sigma-q)\|(u_n,v_n)\|^\sigma-(m-q)\int_{\mathbb{R}^N}H|u_n|^\alpha|v_n|^\beta dx.$$

For our claim, it is sufficient to show that $|B_n|\geq C_4$ for some $C_4>0$ and n large enough. Suppose otherwise; thus, there exists a subsequence, again denoted

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by $\{(u_n, v_n)\}$, satisfying $B_n \to 0$ as $n \to \infty$; that is,

$$A_n = k(p-q)\|(u_n, v_n)\|^p + l(\sigma - q)\|(u_n, v_n)\|^\sigma = (m-q)\int_{\mathbb{R}^N} H|u_n|^\alpha |v_n|^\beta dx + o_n(1). \tag{3.9}$$

Combining (3.5) and (3.9), we can find a suitable constant $C_4 > 0$ such that

$$\int_{\mathbb{R}^N} H|u_n|^\alpha |v_n|^\beta dx \geq C_4 \tag{3.10}$$

for sufficiently large n. In addition, (3.9) and the fact that $(u_n, v_n) \in \mathcal{N}$ give that

$$\begin{aligned} F(u_n, v_n) &= k\|(u_n, v_n)\|^p + l\big\|(u_n, v_n)\big\|^\sigma - \int_{\mathbb{R}^N} H|u_n|^\alpha |v_n|^\beta dx \\ &= k\frac{m-p}{m-q}\|(u_n, v_n)\|^p + l\frac{m-\sigma}{m-q}\big\|(u_n, v_n)\big\|^\sigma + o_n(1) \end{aligned}$$

and

$$\|(u_n, v_n)\| \leq \Big(\frac{m-q}{k(m-p)}(\lambda h_{1\theta} + \mu h_{2\theta})S^q\Big)^{1/(p-q)} + o_n(1). \tag{3.11}$$

If we denote

$$D = \frac{m-\sigma}{(\sigma-q)(m-q)^{m/(m-p)}},$$

then for large n,

$$\begin{aligned} S_n &= \frac{DA_n^{m/(m-p)}}{(\int_{\mathbb{R}^N} H|u_n|^\alpha |v_n|^\beta dx)^{p/(m-p)}} - F(u_n, v_n) \\ &= D(m-q)^{p/(m-p)}A_n - \Big(k\frac{m-p}{m-q}\|(u_n, v_n)\|^p + l\frac{m-\sigma}{m-q}\|(u_n, v_n)\|^\sigma\Big) + o_n(1) \\ &= -\frac{kp\tau}{\sigma-q}\|(u_n, v_n)\|^p + o_n(1) < 0. \end{aligned} \tag{3.12}$$

However, by (3.10), (3.11), and $\lambda h_{1\theta} + \mu h_{2\theta} < \Lambda_0$, we have

$$\begin{aligned} S_n &\geq \frac{D(k(p-q)\|(u_n, v_n)\|^p)^{m/(m-p)}}{(H_\delta S^m\|(u_n, v_n)\|^m)^{p/(m-p)}} - (\lambda h_{1\theta} + \mu h_{2\theta})S^q\|(u_n, v_n)\|^q \\ &\geq \|(u_n, v_n)\|^q\Big[D(k(p-q))^{\frac{m}{m-p}}(H_\delta S^m)^{\frac{-p}{m-p}}\Big(\frac{m-q}{k(m-p)}(\lambda h_{1\theta} + \mu h_{2\theta})S^q\Big)^{\frac{-q}{p-q}} \\ &\qquad - (\lambda h_{1\theta} + \mu h_{2\theta})S^q\Big] > 0. \end{aligned}$$

This contradicts (3.12). Thus, we get

$$\Big\langle J'(u_n, v_n), \frac{(u, v)}{\|(u, v)\|}\Big\rangle \leq \frac{C_5}{n}.$$

This completes the proof of Lemma 3.2. ∎

Similar to Lemmas 3.1 and 3.2, we can get the following lemma.

Lemma 3.3 *Assume* (A_1) *and* (A_2)*. Then for* $0 < \lambda h_{1\theta} + \mu h_{2\theta} < \Lambda_0$*, there exists a sequence* $\{(u_n, v_n)\} \subset \mathcal{N}^-$ *such that, as* $n \to \infty$,

$$J(u_n, v_n) \to \delta^-, \quad J'(u_n, v_n) \to 0 \text{ in } E^*.$$

Now, we establish the existence of a local minimum for J on $\mathcal{N}^+$.

Theorem 3.4 *Assume* (A_1) *and* (A_2)*. Then for* $0 < \lambda h_{1\theta} + \mu h_{2\theta} < \Lambda_0$*, then functional* J *has a minimizer* (u_0^+, v_0^+) *in* $\mathcal{N}^+$ *and it satisfies*

(i) $J(u_0^+, v_0^+) = \delta^+$;

(ii) (u_0^+, v_0^+) *is a positive solution of problem* (1.3).

Proof Let $\{u_n, v_n\} \subset \mathcal{N}^+$ be a minimizing sequence for J on $\mathcal{N}^+$ such that

$$J(u_n, v_n) \longrightarrow \inf_{(u,v)\in\mathcal{N}^+} J(u, v), \quad J'(u_n, v_n) \longrightarrow 0 \text{ in } E^*.$$

Since $J(u, v)$ is coercive, $\{(u_n, v_n)\}$ is bounded on E. Thus, we may assume, without loss of generality, that $u_n \rightharpoonup u_0^+$, $v_n \rightharpoonup v_0^+$ in X. By Lemma 2.4 and 2.7, we have

$$\lim_{n\to+\infty} J(u_n, v_n) = \delta^+ < 0, \quad \lim_{n\to+\infty} F(u_n, v_n) = F(u_0^+, v_0^+).$$

It follows from (2.2) that

$$J(u_n, v_n) = k\Big(\frac{1}{p} - \frac{1}{m}\Big) \left\| (u_n, v_n)\right\|^p + l\Big(\frac{1}{\sigma} - \frac{1}{m}\Big) \left\| (u_n, v_n)\right\|^\sigma - \Big(\frac{1}{q} - \frac{1}{m}\Big)F(u_n, v_n)\Big). \tag{3.13}$$

Letting $n \to +\infty$ in (3.13), we see that $F(u_0^+, v_0^+) > 0$. Moreover, by Lemma 2.6, there is a unique $t_0^+ < T_l$ such that $(t_0^+ u_0^+, t_0^+ v_0^+) \in \mathcal{N}^+$ and

$$\Psi_0(t_0^+) = \big\langle J'(t_0^+ u_0^+, t_0^+ v_0^+), (t_0^+ u_0^+, t_0^+ v_0^+)\big\rangle = 0.$$

Now we show that $u_n \to u_0^+$, $v_n \to v_0^+$ in X. Suppose otherwise; then

$$\|u_0^+\|_X < \liminf_{n\to+\infty} \|u_n\|_X \quad \text{or} \quad \|v_0^+\|_X < \liminf_{n\to+\infty} \|v_n\|_X.$$

Thus, as

$$\begin{aligned}\big\langle J'(tu_n, tv_n), (tu_n, tv_n)\big\rangle &= kt^p \|(u_n, v_n)\|^p + lt^\sigma \|(u_n, v_n)\|^\sigma \\ &\quad - t^m \int_{\mathbb{R}^N} H|u_n|^\alpha |v_n|^\beta dx - t^q F(u_n, v_n),\\ \big\langle J'(tu_0^+, tv_0^+), (tu_0^+, tv_0^+)\big\rangle &= kt^p \|(u_0^+, v_0^+)\|^p + lt^\sigma \|(u_0^+, v_0^+)\|^\sigma \\ &\quad - t^m \int_{\mathbb{R}^N} H|u_0^+|^\alpha |v_0^+|^\beta dx - t^q F(u_0^+, v_0^+),\end{aligned}$$

it follows that $\langle J'(t_0^+ u_n, t_0^+ v_n), (t_0^+ u_n, t_0^+ v_n)\rangle > 0$ for n sufficiently large. Since $\{(u_n, v_n)\} \subseteq \mathcal{N}^+$, it is easy to see that $\langle J'(u_n, v_n), (u_n, v_n)\rangle = 0$, and for $0 < t < 1$, $\langle J'(tu_n, tv_n), (tu_n, tv_n)\rangle < 0$. So we derive $t_0^+ > 1$. But $(t_0^+ u_0^+, t_0^+ v_0^+) \in \mathcal{N}^+$, and so

$$J(t_0^+ u_0^+, t_0^+ v_0^+) < J(u_0^+, v_0^+) < \liminf_{n\to+\infty} J(u_n, v_n) = \delta^+,$$

which is a contradiction. Hence, $u_n \to u_0^+$, $v_n \to v_0^+$ in X and so

$$J(u_0^+, v_0^+) = \lim_{n\to+\infty} J(u_n, v_n) = \delta^+.$$

Thus, (u_0^+, v_0^+) is minimizer for J on $\mathcal{N}^+$. Since $J(u_0^+, v_0^+) = J(|u_0^+|, |v_0^+|)$ and $(|u_0^+|, |v_0^+|) \in \mathcal{N}^+$, by Lemma 2.3, we can assume that (u_0^+, v_0^+) is a nonnegative solution of problem (1.3). Furthermore, we obtain that $u_0^+ > 0$, $v_0^+ > 0$ by the maximum principle; see [4, 12]. This concludes the proof. ■

Theorem 3.5 *Assume (A_1) and (A_2); then for $0 < \lambda h_{1\theta} + \mu h_{2\theta} < \Lambda_0$, then functional J has a minimizer (u_0^-, v_0^-) in $\mathcal{N}^-$ and it satisfies*

(i) $J(u_0^-, v_0^-) = \delta^-$;
(ii) (u_0^-, v_0^-) *is a positive solution of problem* (1.3).

Proof By Lemma 3.3, there exists a minimizing sequence for J on $\mathcal{N}^-$ such that

$$J(u_n, v_n) \longrightarrow \inf_{(u,v)\in\mathcal{N}^-} J(u, v), \qquad J'(u_n, v_n) \longrightarrow 0 \text{ in } E^*.$$

Since $J(u, v)$ is coercive, $\{(u_n, v_n)\}$ is bounded on E. Thus, we can assume, without loss of generality, that $u_n \rightharpoonup u_0^-$, $v_n \rightharpoonup v_0^-$ in X. By Lemma 2.4 and 2.7, we have

$$\lim_{n\to+\infty} J(u_n, v_n) = \delta^- > 0, \qquad \lim_{n\to+\infty} \int H|u_n|^\alpha |v_n|^\beta dx = \int H|u_0^-|^\alpha |v_0^-|^\beta dx.$$

Furthermore, (2.2) gives that

$$J(u_n, v_n) = k\Big(\frac{1}{p} - \frac{1}{q}\Big) \big\| (u_n, v_n)\big\|^p + l\Big(\frac{1}{\sigma} - \frac{1}{q}\Big) \big\| (u_n, v_n)\big\|^\sigma - \Big(\frac{1}{m} - \frac{1}{q}\Big) \int_{\mathbb{R}^N} H|u_n|^\alpha |v_n|^\beta dx. \tag{3.14}$$

Letting $n \to +\infty$ in (3.14), we obtain that $\int_{\mathbb{R}^N} H|u_0^-|^\alpha |v_0^-|^\beta dx > 0$. Thus, by Lemma 2.5, there is a unique t_0^- such that $(t_0^- u_0^-, t_0^- v_0^-) \in \mathcal{N}^-$.

We now show that $u_n \to u_0^-$, $v_n \to v_0^-$ in X. Suppose otherwise; then

$$\|u_0^-\|_X < \liminf_{n\to+\infty} \|u_n\|_X \quad \text{or} \quad \|v_0^-\|_X < \liminf_{n\to+\infty} \|v_n\|_X.$$

Since $(u_n, v_n) \in \mathcal{N}^-$, Lemma 2.5 and a simple transformation imply that $J(u_n, v_n) \geq J(tu_n, tv_n)$ for all $t \geq 0$. Then we have

$$J(t_0^- u_0^-, t_0^- v_0^-) < \liminf_{n\to+\infty} J(t_0^- u_n, t_0^- v_n) \leq \lim_{n\to+\infty} J(u_n, v_n) = \delta^-,$$

which is a contradiction. Hence $u_n \to u_0^-$, $v_n \to v_0^-$ in X and so

$$J(u_0^-, v_0^-) = \lim_{n\to+\infty} J(u_n, v_n) = \delta^-.$$

Thus, (u_0^-, v_0^-) is minimizer for J on $\mathcal{N}^-$. Since $J(u_0^-, v_0^-) = J(|u_0^-|, |v_0^-|)$ and $(|u_0^-|, |v_0^-|) \in \mathcal{N}^-$, similar to the argument in Theorem 3.4, we can also get that (u_0^-, v_0^-) is a positive solution of problem (1.3). ■

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Proof of Theorem 1.2 By Theorems 3.4 and 3.5, we obtain that problem (1.3) has two positive solutions $(u_0^+, v_0^+) \in \mathcal{N}^+$ and $(u_0^-, v_0^-) \in \mathcal{N}^-$. Since $\mathcal{N}^+ \cap \mathcal{N}^- = \varnothing$, the solutions (u_0^+, v_0^+) and (u_0^-, v_0^-) are distinct. This concludes the proof. ■

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(Song, Chen) *College of Science, Hohai University, Nanjing 210098, P. R. China*
e-mail: songhx@njupt.edu.cn cshengchen@hhu.edu.cn

(Song, Yan) *College of Science, Nanjing University of Posts and Telecommunications,, Nanjing 210023, P. R. China*
e-mail: yanqinglun@njupt.edu.cn