Bull. Aust. Math. Soc. **89** (2014), 360–372
doi:10.1017/S0004972713000737

CONGRUENCES MODULO POWERS OF 2 FOR FU'S 5 DOTS BRACELET PARTITIONS

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(Received 20 April 2013; accepted 29 June 2013; first published online 22 August 2013)

Abstract

In 2007, Andrews and Paule introduced a new class of combinatorial objects called broken k-diamond partitions. Recently, Shishuo Fu generalised the notion of broken k-diamond partitions to combinatorial objects which he termed k dots bracelet partitions. Fu denoted the number of k dots bracelet partitions of n by $\mathfrak{B}_k(n)$ and proved several congruences modulo primes and modulo powers of 2. More recently, Radu and Sellers extended the set of congruences proven by Fu by proving three congruences modulo squares of primes for $\mathfrak{B}_5(n)$, $\mathfrak{B}_7(n)$ and $\mathfrak{B}_{11}(n)$. In this note, we prove some congruences modulo powers of 2 for $\mathfrak{B}_5(n)$. For example, we find that for all integers $n \geq 0$, $\mathfrak{B}_5(16n+7) \equiv 0 \pmod{2^5}$.

2010 *Mathematics subject classification*: primary 11P83; secondary 05A17.

Keywords and phrases: congruences, 2-dissections, partitions, k dots bracelet partitions.

1. Introduction

This paper is concerned with congruences modulo powers of 2 for the number of 5 dots bracelet partitions. We find several arithmetic progressions $An + B$ such that for any integer $n \geq 0$,

$$\mathfrak{B}_5(An+B) \equiv 0 \pmod{2^k},$$

where $\mathfrak{B}_5(n)$ is the number of 5 dots bracelet partitions of n and k is a positive integer.

A combinatorial study guided by MacMahon's partition analysis led Andrews and Paule [1] to the construction of a new class of directed graphs called broken k-diamond partitions. A broken k-diamond partition $\pi = (a_1, a_2, \dots; b_2, b_3, b_4, \dots)$ is a plane partition satisfying the relations illustrated in Figure 1, where here and in the rest of the figures, a_i, b_i are nonnegative integers and $a_i \to a_j$ is interpreted as $a_i \geq a_j$. More precisely, each building block in Figure 1, except for the broken block $(b_2, b_3, \dots, b_{2k+2})$, has the same order structure as shown in Figure 2. Each block is called a k-elongated partition diamond of length 1, or a k-elongated diamond, for short.

This work was supported by the National Natural Science Foundation of China (11201188 and 11226300).

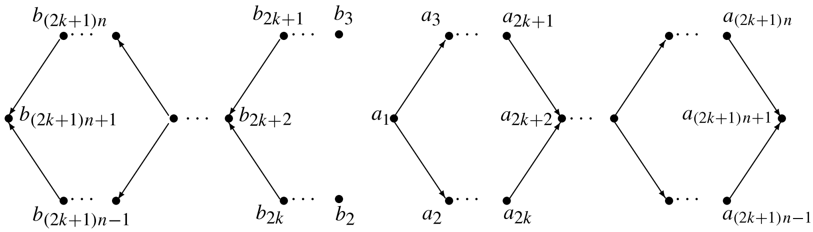

Figure 1. A broken k-diamond of length $2n$.

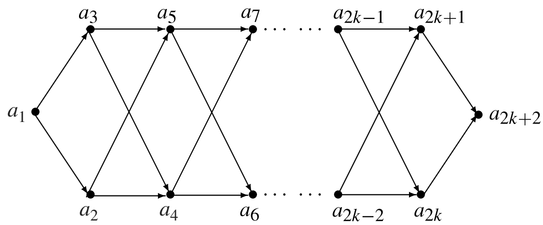

Figure 2. A k-elongated diamond.

Let $\Delta_k(n)$ denote the number of broken k-diamond partitions of n. From [1], we know that for any positive integer k,

$$\sum_{n=0}^{\infty} \Delta_k(n) q^n = \frac{f_2 f_{2k+1}}{f_1^3 f_{4k+2}},$$

where here and throughout this paper, f_k is defined by

$$f_k = \prod_{r=1}^{\infty} (1 - q^{kr}) \quad |q| < 1.$$

Employing generating function manipulations, Andrews and Paule [1] proved that for any integer $n \geq 0$,

$$\Delta_1(2n + 1) \equiv 0 \pmod 3. \tag{1.1}$$

Soon after, Hirschhorn and Sellers [5] found an explicit representation of the generating function for $\Delta_1(2n + 1)$ which implied (1.1). Mortenson [7] reproved (1.1) by developing a statistic on the partitions enumerated by $\Delta_1(2n + 1)$ which naturally breaks these partitions into three subsets of equal size. Recently, Fu [4] presented a combinatorial proof of (1.1). He also employed his combinatorial approach to naturally define a generalisation of broken k-diamond partitions which he termed k dots bracelet partitions. To define k dots bracelet partitions, Fu first gave the definition of the 'infinite bracelet with k dots'. For $k \geq 3$, an 'infinite bracelet with k dots' is the

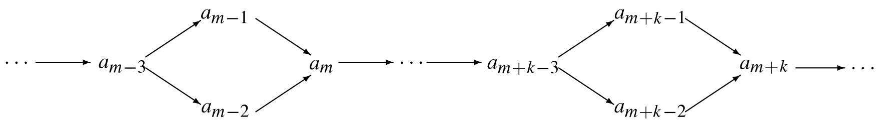

FIGURE 3. Infinite bracelet with k dots.

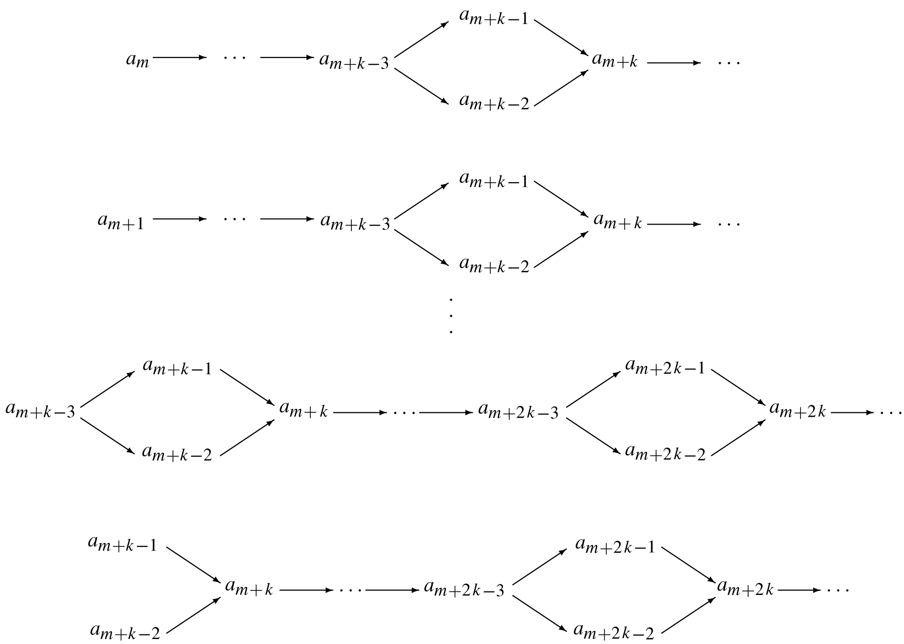

FIGURE 4. $k-1$ different half bracelets.

configuration shown in Figure 3. It is composed of repeating 'diamonds' and 'dots' with $k-2$ 'dots' between two consecutive 'diamonds'.

It is easy to see that there are essentially $k-1$ different ways to cut an infinite bracelet with k dots in half. For each different 'cut', we get a half bracelet (the right half) in different configuration. Graphically, see Figure 4.

For any positive integer k, a 'k dots bracelet partition' is a partition which consists of the $k-1$ different half bracelets (in the sense of different configuration) as shown in Figure 4 above. Fu utilised $\mathfrak{B}_k(n)$ to denote the number of k dots bracelet partitions of n and proved that

$$\sum_{n=0}^{\infty} \mathfrak{B}_k(n)q^n = \frac{f_2 f_k}{f_1^k f_{2k}}. \tag{1.2}$$

Fu also established various congruence properties for $\mathfrak{B}_k(n)$ modulo primes and modulo powers of 2. Very recently, Radu and Sellers [8] extended the set of

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congruences proven by Fu by proving three congruences modulo squares of primes for $\mathfrak{B}_5(n)$, $\mathfrak{B}_7(n)$ and $\mathfrak{B}_{11}(n)$.

The aim of this paper is to prove some congruences modulo powers of 2 for $\mathfrak{B}_5(n)$. The proofs mainly rely on some 2-dissections of certain infinite products. Recall that the m-dissection of the power series $P(q) = \sum_{n=0}^{\infty} a_n q^n$ is the presentation of $P(q)$ as

$$P(q) = P_0(q) + P_1(q) + \cdots + P_{m-1}(q),$$

where

$$P_k(q) = \sum_{n=0}^{\infty} a_{mn+k} q^{mn+k}.$$

The main results of this paper can be stated as follows.

THEOREM 1.1. *For any integer $n \geq 0$,*

$$\mathfrak{B}_5(4n+3) \equiv 0 \pmod{2^2}, \tag{1.3}$$
$$\mathfrak{B}_5(8n+7) \equiv 0 \pmod{2^4}, \tag{1.4}$$
$$\mathfrak{B}_5(16n+7) \equiv 0 \pmod{2^5}, \tag{1.5}$$
$$\mathfrak{B}_5(32n+31) \equiv 0 \pmod{2^5}. \tag{1.6}$$

THEOREM 1.2. *Let n be a nonnegative integer. We have*

$$\mathfrak{B}_5(32n+15) \equiv 2^4 \mathfrak{B}_5(8n+4) \pmod{2^5}. \tag{1.7}$$

If $24n+11$ is a prime, then

$$\mathfrak{B}_5(32n+15) \equiv \begin{cases} 2^4 \pmod{2^5} & \text{if } 24n+11 \text{ is of the form } 11x^2+108xy+396y^2 \\ & \quad \text{or } 44x^2+108xy+99y^2, \text{ where } x \text{ and } y \text{ are integers}, \\ 0 \pmod{2^5} & \text{otherwise}. \end{cases} \tag{1.8}$$

THEOREM 1.3. *For all nonnegative integers n and $i = 1, 2, 3, 4$,*

$$\mathfrak{B}_5(32n+19) \equiv 0 \pmod{2^3}, \tag{1.9}$$
$$\mathfrak{B}_5(64n+51) \equiv 0 \pmod{2^3}, \tag{1.10}$$
$$\mathfrak{B}_5(64n+59) \equiv 0 \pmod{2^3}, \tag{1.11}$$
$$\mathfrak{B}_5(128n+91) \equiv 0 \pmod{2^3}, \tag{1.12}$$
$$\mathfrak{B}_5(640n+128i+27) \equiv 0 \pmod{2^3} \tag{1.13}$$

and

$$\mathfrak{B}_5(64n+3) \equiv \mathfrak{B}_5(640n+27) \equiv \begin{cases} 4 \pmod{2^3} & \text{if } n = P_k, \\ 0 \pmod{2^3} & \text{otherwise}, \end{cases} \tag{1.14}$$

where P_k is either of the kth generalised pentagonal numbers $k(3k \pm 1)/2$.

Theorem 1.4. *Let n and k be nonnegative integers. We have*

$$\mathfrak{B}_5\Big(2^{2k+3}n + \frac{2^{2k+3}+1}{3}\Big) \equiv \mathfrak{B}_5(8n+3) \pmod{2^3}. \tag{1.15}$$

Theorem 1.5. *For all nonnegative integers n and* $i = 1, 2, 3, 4$,

$$\mathfrak{B}_5(20n + 4i + 2) \equiv \mathfrak{B}_5(40n + 22) \equiv 0 \pmod 2$$

and

$$\mathfrak{B}_5(40n+2) \equiv \begin{cases} 1 \pmod 2 & \text{if } n = P_k, \\ 0 \pmod 2 & \text{otherwise}, \end{cases}$$

where P_k *is either of the kth generalised pentagonal numbers* $k(3k \pm 1)/2$.

Theorem 1.6. *For all nonnegative integers n,*

$$\mathfrak{B}_5(4n+1) \equiv b_5(n) \pmod 2,$$

where $b_5(n)$ *is the number of 5-regular partitions of n.*

By Theorem 1.6 and some results proved in [3, 6], we can obtain the following corollaries.

Corollary 1.7. *For all nonnegative integers n,* $\mathfrak{B}_5(8n+1)$ *is odd if and only if* $12n+1$ *is a perfect square and* $\mathfrak{B}_5(16n+5)$ *is even unless* $24n+7 = 2x^2+5y^2$ *for some integers x and y.*

Corollary 1.8. *For all nonnegative integers n,* $\mathfrak{B}_5(80n+21)$, $\mathfrak{B}_5(80n+53)$, *and* $\mathfrak{B}_5(4624n+261)$ *are even numbers.*

Corollary 1.9. *Suppose that p is any prime greater than 3 such that* -10 *is a quadratic nonresidue modulo p, u is the reciprocal of 24 modulo* p^2, *and* $p \nmid r$. *Then, for all nonnegative integers n,* $\mathfrak{B}_5(16p^2n + 16u(pr-7)+5)$ *is even.*

2. Two lemmas

In this section, to prove our main results, we first present two lemmas.

Lemma 2.1. *The following 2-dissection holds*

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{2.1}$$

Equation (2.1) follows from Entry 25 (v) and (vi) in [2, page 40]. The authors also presented another proof of (2.1) in [11].

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LEMMA 2.2. *The following* 2*-dissections hold*

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}} \tag{2.2}$$

and

$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_4^2 f_{40}}{f_8 f_{10}^2}. \tag{2.3}$$

Equation (2.2) was proved by Hirschhorn and Sellers in [6]; see also [10, 11]. It is easy to check that for any odd integer $k \geq 1$,

$$f_k(-q) = \frac{f_{2k}^3}{f_k f_{4k}}. \tag{2.4}$$

Replacing q by $-q$ in (2.2) and employing (2.4),

$$\frac{f_1 f_4 f_{10}^3}{f_2^3 f_5 f_{20}} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} - q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}},$$

which yields (2.3).

3. Proofs of the main results

In this section, we provide proofs of Theorems 1.1–1.6 and Corollaries 1.7–1.9 by employing Lemmas 2.1 and 2.2.

Setting $k = 5$ in (1.2),

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(n) q^n = \frac{f_2 f_5}{f_1^5 f_{10}}. \tag{3.1}$$

From (2.1), (2.2) and (3.1), we see that

$$\begin{aligned}\sum_{n=0}^{\infty} \mathfrak{B}_5(n) q^n &= \frac{f_2}{f_{10}} \frac{f_5}{f_1} \frac{1}{f_1^4} = \frac{f_2}{f_{10}} \left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}} \right)\left(\frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}} \right) \\ &= \frac{f_4^{14} f_{20}^2}{f_2^{15} f_8^3 f_{10} f_{40}} + q\frac{f_4^{17} f_{40}}{f_2^{16} f_8^5 f_{20}} + 4q\frac{f_4^2 f_8^5 f_{20}^2}{f_2^{11} f_{10} f_{40}} + 4q^2\frac{f_4^5 f_8^3 f_{40}}{f_2^{12} f_{20}},\end{aligned}$$

which yields

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(2n) q^n = \frac{f_2^{14} f_{10}^2}{f_1^{15} f_4^3 f_5 f_{20}} + 4q\frac{f_2^5 f_4^3 f_{20}}{f_1^{12} f_{10}} \tag{3.2}$$

and

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(2n+1) q^n = \frac{f_2^{17} f_{20}}{f_1^{16} f_4^5 f_{10}} + 4\frac{f_2^2 f_4^5 f_{10}^2}{f_1^{11} f_5 f_{20}}. \tag{3.3}$$

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In view of (2.1) and (2.3), we deduce that, modulo 32,

$$\begin{aligned}
\frac{f_2^{17}f_{20}}{f_1^{16}f_4^5f_{10}}+4\frac{f_2^2f_4^5f_{10}^2}{f_1^{11}f_5f_{20}} &= \frac{f_2^{17}f_{20}}{f_4^5f_{10}}\frac{1}{f_1^{16}}+4\frac{f_2^2f_4^5f_{10}^2}{f_{20}}\frac{1}{f_1^{12}}\frac{f_1}{f_5}\\
&= \frac{f_2^{17}f_{20}}{f_4^5f_{10}}\Big(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\Big)^4+4\frac{f_2^2f_4^5f_{10}^2}{f_{20}}\Big(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\Big)^3\Big(\frac{f_2f_8f_{20}^3}{f_4f_{10}^3f_{40}}-q\frac{f_4^2f_{40}}{f_8f_{10}^2}\Big)\\
&\equiv \frac{f_2^{17}f_{20}}{f_4^5f_{10}}\Big(\Big(\frac{f_4^{14}}{f_2^{14}f_8^4}\Big)^4+16q\Big(\frac{f_4^{14}}{f_2^{14}f_8^4}\Big)^3\frac{f_4^2f_8^4}{f_2^{10}}\Big)\\
&\quad + 4\frac{f_2^2f_4^5f_{10}^2}{f_{20}}\Big(\Big(\frac{f_4^{14}}{f_2^{14}f_8^4}\Big)^3+12q\Big(\frac{f_4^{14}}{f_2^{14}f_8^4}\Big)^2\frac{f_4^2f_8^4}{f_2^{10}}\Big)\Big(\frac{f_2f_8f_{20}^3}{f_4f_{10}^3f_{40}}-q\frac{f_4^2f_{40}}{f_8f_{10}^2}\Big)\\
&\equiv \frac{f_4^{51}f_{20}}{f_2^{39}f_8^{16}f_{10}}+4\frac{f_4^{46}f_{20}^2}{f_2^{39}f_8^{11}f_{10}f_{40}}+16q\frac{f_4^{39}f_{20}}{f_2^{35}f_8^8f_{10}}\\
&\quad - 4q\frac{f_4^{49}f_{40}}{f_2^{40}f_8^{13}f_{20}}+16q\frac{f_4^{34}f_{20}^2}{f_2^{35}f_8^3f_{10}f_{40}}-16q^2\frac{f_4^{37}f_{40}}{f_2^{36}f_8^5f_{20}}.
\end{aligned}\tag{3.4}$$

Combining (3.3) and (3.4), we find that, modulo 32,

$$\sum_{n=0}^{\infty}\mathfrak{B}_5(4n+3)q^n\equiv 16\frac{f_2^{39}f_{10}}{f_1^{35}f_4^8f_5}-4\frac{f_2^{49}f_{20}}{f_1^{40}f_4^{13}f_{10}}+16\frac{f_2^{34}f_{10}^2}{f_1^{35}f_4^3f_5f_{20}}.\tag{3.5}$$

Congruence (1.3) follows from (3.5).

Employing (2.1) and (2.3), we have, modulo 8,

$$\begin{aligned}
&4\frac{f_2^{39}f_{10}}{f_4^8}\frac{1}{f_1^{36}}\frac{f_1}{f_5}-\frac{f_2^{49}f_{20}}{f_4^{13}f_{10}}\frac{1}{f_1^{40}}+4\frac{f_2^{34}f_{10}^2}{f_4^3f_{20}}\frac{1}{f_1^{36}}\frac{f_1}{f_5}\\
&= 4\frac{f_2^{39}f_{10}}{f_4^8}\Big(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\Big)^9\Big(\frac{f_2f_8f_{20}^3}{f_4f_{10}^3f_{40}}-q\frac{f_4^2f_{40}}{f_8f_{10}^2}\Big)-\frac{f_2^{49}f_{20}}{f_4^{13}f_{10}}\Big(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\Big)^{10}\\
&\quad +4\frac{f_2^{34}f_{10}^2}{f_4^3f_{20}}\Big(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\Big)^9\Big(\frac{f_2f_8f_{20}^3}{f_4f_{10}^3f_{40}}-q\frac{f_4^2f_{40}}{f_8f_{10}^2}\Big)\\
&\equiv 4\frac{f_2^{39}f_{10}}{f_4^8}\Big(\frac{f_4^{14}}{f_2^{14}f_8^4}\Big)^9\Big(\frac{f_2f_8f_{20}^3}{f_4f_{10}^3f_{40}}-q\frac{f_4^2f_{40}}{f_8f_{10}^2}\Big)-\frac{f_2^{49}f_{20}}{f_4^{13}f_{10}}\Big(\frac{f_4^{14}}{f_2^{14}f_8^4}\Big)^{10}\\
&\quad + 4\frac{f_2^{34}f_{10}^2}{f_4^3f_{20}}\Big(\frac{f_4^{14}}{f_2^{14}f_8^4}\Big)^9\Big(\frac{f_2f_8f_{20}^3}{f_4f_{10}^3f_{40}}-q\frac{f_4^2f_{40}}{f_8f_{10}^2}\Big)\\
&\equiv 4\frac{f_4^{117}f_{20}^3}{f_2^{86}f_8^{35}f_{10}^2f_{40}}-4q\frac{f_4^{120}f_{40}}{f_2^{87}f_8^{37}f_{10}}-\frac{f_4^{127}f_{20}}{f_2^{91}f_8^{40}f_{10}}+4\frac{f_4^{122}f_{20}^2}{f_2^{91}f_8^{35}f_{10}f_{40}}-4q\frac{f_4^{125}f_{40}}{f_2^{92}f_8^{37}f_{20}}.
\end{aligned}\tag{3.6}$$

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Combining (3.5) and (3.6), we see that, modulo 32,

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(8n+7)q^n \equiv 16\frac{f_2^{120}f_{20}}{f_1^{87}f_4^{37}f_5} + 16\frac{f_2^{125}f_{20}}{f_1^{92}f_4^{37}f_{10}}. \tag{3.7}$$

Congruence (1.4) follows from (3.7).

By (2.1) and (2.3), we have, modulo 2,

$$\begin{aligned}
&\frac{f_2^{120}f_{20}}{f_4^{37}}\frac{1}{f_1^{88}}\frac{f_1}{f_5} + \frac{f_2^{125}f_{20}}{f_1^{92}f_4^{37}f_{10}} \\
&\quad = \frac{f_2^{120}f_{20}}{f_4^{37}}\Big(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\Big)^{22}\Big(\frac{f_2f_8f_{20}^3}{f_4f_{10}^3f_{40}} - q\frac{f_4^2f_{40}}{f_8f_{10}^2}\Big) \\
&\qquad + \frac{f_2^{125}f_{20}}{f_4^{37}f_{10}}\Big(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\Big)^{23} \\
&\quad \equiv \frac{f_2^{120}f_{20}}{f_4^{37}}\Big(\frac{f_4^{14}}{f_2^{14}f_8^4}\Big)^{22}\Big(\frac{f_2f_8f_{20}^3}{f_4f_{10}^3f_{40}} - q\frac{f_4^2f_{40}}{f_8f_{10}^2}\Big) + \frac{f_2^{125}f_{20}}{f_4^{37}f_{10}}\Big(\frac{f_4^{14}}{f_2^{14}f_8^4}\Big)^{23} \\
&\quad \equiv \frac{f_4^{270}f_{20}^4}{f_2^{187}f_8^{87}f_{10}^3f_{40}} - q\frac{f_4^{273}f_{20}f_{40}}{f_2^{188}f_8^{89}f_{10}^2} + \frac{f_4^{285}f_{20}}{f_2^{197}f_8^{92}f_{10}}.
\end{aligned} \tag{3.8}$$

It follows from (3.7) and (3.8) that, modulo 32,

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(16n+7)q^n \equiv 16\frac{f_2^{270}f_{10}^4}{f_1^{187}f_4^{87}f_5^3f_{20}} + 16\frac{f_2^{285}f_{10}}{f_1^{197}f_4^{92}f_5} \tag{3.9}$$

and

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(16n+15)q^n \equiv 16\frac{f_2^{273}f_{10}f_{20}}{f_1^{188}f_4^{89}f_5^2}. \tag{3.10}$$

It is easy to see that for any positive integer k, modulo 2,

$$f_k^2 \equiv f_{2k}. \tag{3.11}$$

Employing (3.11), we deduce that, modulo 2,

$$\frac{f_2^{270}f_{10}^4}{f_1^{187}f_4^{87}f_5^3f_{20}} \equiv f_1^5f_5, \tag{3.12}$$

$$\frac{f_2^{285}f_{10}}{f_1^{197}f_4^{92}f_5} \equiv f_1^5f_5 \tag{3.13}$$

and

$$\frac{f_2^{273}f_{10}f_{20}}{f_1^{188}f_4^{89}f_5^2} \equiv f_2f_{20}. \tag{3.14}$$

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From (3.9), (3.12) and (3.13), we see that, modulo 32,

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(16n+7)q^n \equiv 0. \tag{3.15}$$

Congruence (1.5) follows from (3.15).

In view of (3.10) and (3.14), we obtain, modulo 32,

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(32n+15)q^n \equiv 16 f_1 f_{10} \tag{3.16}$$

and

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(32n+31)q^n \equiv 0,$$

which yields (1.6).

Let $c(n)$ be defined by

$$f_1 f_{10} = 1 + \sum_{n=1}^{\infty} c(n)q^n. \tag{3.17}$$

In [9], Sun proved that if $24n+11$ is a prime, then

$$c(n) = \begin{cases} 1 & \text{if } 24n+11 \text{ is of the form } 11x^2+108xy+396y^2, \\ -1 & \text{if } 24n+11 \text{ is of the form } 44x^2+108xy+99y^2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.18}$$

Congruence (1.8) follows from (3.16), (3.17) and (3.18).

By (3.5) and (3.11), we find that, modulo 8,

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(4n+3)q^n \equiv 4\frac{f_2^{49} f_{20}}{f_1^{40} f_4^{13} f_{10}} \equiv 4\frac{f_8 f_{10}}{f_2},$$

which yields

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(8n+3)q^n \equiv 4\frac{f_4 f_5}{f_1}. \tag{3.19}$$

It follows from (2.2) and (3.11) that, modulo 2,

$$\begin{aligned} \frac{f_4 f_5}{f_1} &= f_4\left(\frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}\right) = \frac{f_4 f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^4 f_{10} f_{40}}{f_2^3 f_8 f_{20}} \\ &\equiv f_8 + q\frac{f_2 f_{40}}{f_{10}}. \end{aligned} \tag{3.20}$$

By (3.19) and (3.20), we deduce that, modulo 8,

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(16n+3)q^n \equiv 4 f_4 \tag{3.21}$$

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and

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(16n+11)q^n \equiv 4\frac{f_1 f_{20}}{f_5}. \tag{3.22}$$

Congruences (1.9) and (1.10) follow from (3.21).

In view of (2.3) and (3.11), we have, modulo 2,

$$\begin{aligned} \frac{f_1 f_{20}}{f_5} &= f_{20}\left(\frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_4^2 f_{40}}{f_8 f_{10}^2}\right) = \frac{f_2 f_8 f_{20}^4}{f_4 f_{10}^3 f_{40}} - q\frac{f_4^2 f_{20} f_{40}}{f_8 f_{10}^2} \\ &\equiv \frac{f_8 f_{10}}{f_2} - q f_{40}. \end{aligned} \tag{3.23}$$

Employing (3.22) and (3.23), we find that, modulo 8,

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(32n+11)q^n \equiv 4\frac{f_4 f_5}{f_1} \tag{3.24}$$

and

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(32n+27)q^n \equiv 4 f_{20}. \tag{3.25}$$

Congruences (1.11), (1.12) and (1.13) follow from (3.25).

In view of (3.21) and (3.25), we have, modulo 8,

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(64n+3)q^n \equiv \sum_{n=0}^{\infty} \mathfrak{B}_5(640n+27)q^n \equiv 4 f_1. \tag{3.26}$$

By Euler's pentagonal number theorem,

$$f_1 = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2} = 1 + \sum_{k=1}^{\infty} (-1)^k (q^{k(3k-1)/2} + q^{k(3k+1)/2}). \tag{3.27}$$

Combining (3.26) and (3.27), we deduce that, modulo 8,

$$\begin{aligned} \sum_{n=0}^{\infty} \mathfrak{B}_5(64n+3)q^n \equiv \sum_{n=0}^{\infty} \mathfrak{B}_5(640n+27)q^n &\equiv 4 + 4\sum_{k=1}^{\infty} (-1)^k (q^{k(3k-1)/2} + q^{k(3k+1)/2}) \\ &\equiv 4 + 4\sum_{k=1}^{\infty} (q^{k(3k-1)/2} + q^{k(3k+1)/2}), \end{aligned}$$

which yields (1.14).

From (3.19) and (3.24), we see that, modulo 8,

$$\mathfrak{B}_5(32n+11) \equiv \mathfrak{B}_5(8n+3). \tag{3.28}$$

By (3.28) and mathematical induction, we find that (1.15) is true.

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By (2.3), (3.2) and (3.11), we see that, modulo 2,

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(2n)q^n \equiv \frac{f_1}{f_5} \equiv \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} + q\frac{f_4^2 f_{40}}{f_8 f_{10}^2} \equiv \frac{f_2 f_4}{f_{10}} + q f_{20},$$

which implies that, modulo 2,

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(4n)q^n \equiv \frac{f_1 f_2}{f_5}$$

and

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(4n+2)q^n \equiv f_{10}. \tag{3.29}$$

Theorem 1.5 follows from (3.27) and (3.29).

Employing (2.3) and (3.11), we obtain that, modulo 2,

$$\begin{aligned}\frac{f_1 f_2}{f_5} &= f_2\Big(\frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_4^2 f_{40}}{f_8 f_{10}^2}\Big)\\ &= \frac{f_2^2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_2 f_4^2 f_{40}}{f_8 f_{10}^2} \equiv \frac{f_8}{f_{10}} + q f_2 f_{20},\end{aligned}$$

which implies that, modulo 2,

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(8n+4)q^n \equiv f_1 f_{10}. \tag{3.30}$$

Congruence (1.7) follows from (3.16) and (3.30).

By (3.3), (3.4) and (3.11), we find that, modulo 2,

$$\sum_{n=0}^{\infty} \mathfrak{B}_5(4n+1)q^n \equiv \frac{f_2^{51} f_{10}}{f_1^{39} f_4^{16} f_5} \equiv \frac{f_5}{f_1}. \tag{3.31}$$

Let $b_5(n)$ denote the number of 5-regular partitions of n. Adopting the convention that $b_5(0) = 1$, the generating function for $b_5(n)$ is then

$$\sum_{n=0}^{\infty} b_5(n)q^n = \frac{f_5}{f_1}. \tag{3.32}$$

Theorem 1.6 follows from (3.31) and (3.32).

Using the theory of modular forms, Calkin *et al.* [3] proved a result equivalent to the following: for all integers $n \geq 0$, $b_5(2n)$ is odd if and only if $12n + 1$ is a perfect square. Employing nothing more than Jacobi's triple product identity, Hirschhorn and Sellers [6] proved the following result: for all integers $n \geq 0$, $b_5(4n + 1)$ is even unless $24n + 7 = 2x^2 + 5y^2$ for some integers x and y.

Corollary 1.7 follows from these two results and Theorem 1.6.

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In [3], Calkin *et al.* also proved that for all integers $n \geq 0$, modulo 2,

$$b_5(20n + 5) \equiv 0 \tag{3.33}$$

and

$$b_5(20n + 13) \equiv 0. \tag{3.34}$$

Hirschhorn and Sellers [6] discovered that, modulo 2,

$$b_5(1156n + 65) \equiv 0. \tag{3.35}$$

Corollary 1.8 follows from (3.33), (3.34), (3.35) and Theorem 1.6.

Hirschhorn and Sellers [6] also proved the following result: Suppose that p is any prime greater than 3 such that -10 is a quadratic nonresidue modulo p, u is the reciprocal of 24 modulo p^2, and $p \nmid r$. Then, for all integers n, modulo 2,

$$b_5(4p^2n + 4u(pr - 7) + 1) \equiv 0.$$

Corollary 1.9 follows from the above result and Theorem 1.6. This completes the proof.

Acknowledgements

The authors are grateful to the editor and the anonymous referee for their valuable suggestions, corrections and comments which resulted in a great improvement of the original manuscript.

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https://doi.org/10.1017/S0004972713000737 Published online by Cambridge University Press

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