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FIELDS OF G_a INVARIANTS ARE RULED

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ABSTRACT. The quotient field of the ring of invariants of a rational G_a action on $\mathbf{C}^n$ is shown to be ruled. As a consequence, all rational G_a actions on $\mathbf{C}^4$ are rationally triangulable. Moreover, if an arbitrary rational G_a action on $\mathbf{C}^n$ is doubled to an action of $G_a \times G_a$ on $\mathbf{C}^{2n}$, then the doubled action is rationally triangulable.

1. **Introduction.** Let $GA_n(\mathbf{C})$ denote the group of automorphisms of $\mathbf{C}[x_1, \ldots, x_n]$, the polynomial ring in n variables over the complex field. Since an automorphism is determined by the images, say F_i, of the x_i, we can describe the affine subgroup $Af_n(\mathbf{C})$ of $GA_n(\mathbf{C})$ as $\{(F_1, \ldots, F_n) : \deg(F_i) \leq 1 \text{ for each } i\}$, and the triangular subgroup as $\{(F_1, \ldots, F_n) : F_i \in \mathbf{C}[x_1, \ldots, x_i]\}$. The "generation gap" question of Bass [1] asks whether $GA_n(\mathbf{C})$ is generated by these two subgroups. The answer if yes for $n \leq 2$, but the question remains open for all larger n. Nagata has suggested a particular automorphism of $\mathbf{C}[x_1, x_2, x_3]$ as a possible counterexample and in [1] Bass was able to embed this example in an action of G_a, the additive group of complex numbers, on $\mathbf{C}[x_1, x_2, x_3]$, and to show that the G_a action is not conjugate to a subgroup of the triangular group.

As an approximation to triangulability, a G_a action is called *rationally triangulable* if there are generators $y_1, \ldots, y_n$ of the field of rational functions so that each of the subfields $\mathbf{C}(y_1, \ldots, y_i)$ is stable under the group of $\mathbf{C}$ automorphisms of the rational function field induced by the G_a action on the polynomial ring. It was asked in [1] whether every rational action of a unipotent group on affine space is rationally triangulable. In [2] the authors showed that a G_a action is rationally triangulable if and only if the quotient field of the ring of G_a invariants of the polynomial ring is a pure transcendental extension of $\mathbf{C}$. It was shown, moreover, that all G_a actions on $\mathbf{C}[x_1, x_2, x_3]$ are rationally triangulable, including, of course, those designed by Bass [1] and Popov [4]. The natural conjecture is that all G_a actions on polynomial rings are rationally triangulable.

As an indication of the importance of this conjecture consider its connection with the following version of the "affine cancellation problem": Given a ring R containing $\mathbf{C}$ and an indeterminate x for which $R[x] \equiv \mathbf{C}[x_1, \ldots, x_{n+1}]$, is R isomorphic to a polynomial ring in n variables? The existence of such stable polynomial rings which aren't polynomial rings is an open problem, while the corresponding problem for fields has a negative answer (*i.e.*, there exist stably rational nonrational field extensions). Given such a stable polynomial ring R, the derivation $D = d/dx$ on $R[x]$ is locally nilpotent and can therefore

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be exponentiated to a G_a action on $\mathbf{C}[x_1, \ldots, x_{n+1}]$: for $t \in G_a$, $p \in \mathbf{C}[x_1, \ldots, x_{n+1}]$

$$tp \equiv \sum_{j=0}^{\infty} \frac{(td)^j}{j!}(p).$$

The ring of invariants for this action is clearly R, therefore the validity of the conjecture would imply that the quotient field of such a stable polynomial ring is purely transcendental.

2. **Ruled invariants.** Every rational G_a action on $\mathbf{C}[x_1, \ldots, x_n]$ is obtained as the exponential of a locally nilpotent derivation and it is clear that the ring of G_a invariants is equal to the ring of constants of the derivation. Let D be a locally nilpotent derivation of $\mathbf{C}[x_1, \ldots, x_n]$, C_0 its ring of constants, and $D(g) = f \in C_0$. D extends to a locally nilpotent derivation of $\mathbf{C}[x_1, \ldots, x_n][1/f]$ with ring of constants equal to $C_0[1/f]$. Since $D(g/f) = 1$, setting $s = g/f$ and applying [7] Proposition 2.1, yields that $\mathbf{C}[x_1, \ldots, x_n][1/f] = C_0[1/f, s]$, and on this ring $D = d/ds$.

LEMMA 2.1. *Define* $F\colon \mathbf{C}[x_1, \ldots, x_n][1/f] \longrightarrow C_0[1/f]$ *by*

$$F(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} s^i D^i(x).$$

Then the expression of any $x \in \mathbf{C}[x_1, \ldots, x_n][1/f]$, *as a polynomial in* s *is, with* $D^0(x) = x$,

$$x = \sum_{i=0}^{\infty} \frac{s^i}{i!} F\big(D^i(x)\big).$$

PROOF. Apply D to $F(x)$ to see that $F(x)$ is a constant. To verify that $x \in \mathbf{C}[x_1, \ldots, x_n][1/f]$ has the stated form, argue by induction on the least power of D that annihilates x. The assertion is clear if x is a constant. For general x, the induction hypothesis yields $D(x) = F\big(D(x)\big) + sF\big(D^2(x)\big) + s^2/2!\, F\big(D^3(x)\big) + \cdots$. Note that $x = F(x) + sD(x) - \frac{s^2}{2!}D^2(x) + \cdots$ so that the constant term of x with respect to s is $F(x)$. Now integration with respect to s gives the desired expression.

Denote by $\mathrm{qf}(C_0)$ the quotient field of C_0.

COROLLARY 2.2. *Let* $h \in \mathbf{C}[x_1, \ldots, x_n]$ *satisfy* $D(h) \neq 0$. *Then the least power of* D *which annihilates* h *is exactly* $[\mathbf{C}(x_1, \ldots, x_n) : \mathrm{qf}(C_0)(h)] - 1$.

PROOF. Since s generates $\mathbf{C}(x_1, \ldots, x_n)$ as a simple transcendental extension of $\mathrm{qf}(C_0)$, the expression of h as a polynomial in s yields the result.

Define an element w of $\mathbf{C}[x_1, \ldots, x_n]$ to be a variable if there exist

$$w_2, \ldots, w_n \in \mathbf{C}[x_1, \ldots, x_n]$$

for which

$$\mathbf{C}[x_1, \ldots, x_n] = \mathbf{C}[w, w_2, \ldots, w_n].$$

THEOREM 2.3. *If* $\mathbf{C}[x_1, \ldots, x_n]$ *has a variable* w *for which* $D(w) \neq 0 = D^2(w)$, *then the* G_a *action associated to* D *is rationally triangulable.*

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PROOF. By Corollary 2.2, $\mathbf{C}(x_1,\ldots,x_n) = \mathrm{qf}(C_0)(w)$. Since w is a variable $\mathbf{C}(x_1,\ldots,x_n) = \mathbf{C}(w_2,\ldots,w_n)(w)$. Although the general cancellation problem for function fields is false, in the present context we have the same variable w in the equality $\mathbf{C}(w_2,\ldots,w_n)(w) = \mathrm{qf}\big(C_0\big)(w)$. A result of Samuel [5] shows that $\mathrm{qf}(C_0) \equiv \mathbf{C}(w_2,\ldots,x_n)$ and thus that the G_a action is rationally triangulable by [2] Theorem 3.1.

COROLLARY 2.4. *For any nonconstant variable w,*

$$\mathbf{C}(x_1,\ldots,x_n) = \mathrm{qf}(C_0)\big(w, D(w)\big).$$

PROOF. Let $n \geq 2$ be the least power of D which annihilates w. If $n = 2$, then w itself generates by the remarks preceding Lemma 2.1. If $n > 2$, Corollary 2.2 yields

$$[\mathbf{C}(x_1,\ldots,x_n) : \mathrm{qf}(C_0)(w)] = n - 1$$

while

$$\big[\mathbf{C}(x_1,\ldots,x_n) : \mathrm{qf}(C_0)\big(D(w)\big)\big] = n - 2.$$

Since $\big[\mathbf{C}(x_1,\ldots,x_n) : \mathrm{qf}(C_0)\big(w, D(w)\big)\big]$ divides both $n - 1$ and $n - 2$, this number is equal to 1.

A field extension K of $\mathbf{C}$ is called *ruled* provided $K = F(Z)$ where $\mathbf{C} \subset F$ and Z is transcendental over F.

THEOREM 2.5. *Let D be a locally nilpotent derivation of* $\mathbf{C}[x_1,\ldots,x_n]$ *with ring of constants C_0. Then* $\mathrm{qf}(C_0)$ *is ruled.*

PROOF. If $D = 0$ the result is clear. Assume then that $D(x_1) \neq 0$ and let $D(s) = 1$ with s as in Lemma 2.1. The residue field of the place associated to the $1/s$-adic valuation on $\mathbf{C}(x_1,\ldots,x_n)$ is $\mathrm{qf}(C_0)$ (see again the remarks preceding Lemma 2.1). Observe that as a polynomial of positive degree in s, x_1 has strictly negative value, so that $1/x_1$ is in the maximal ideal of the valuation ring. This holds as well for the restriction of the place to $\mathbf{C}(x_1,\ldots,x_{n-1})$, so that the residue field has transcendence degree $n-2$ over $\mathbf{C}$. In particular, $\mathrm{qf}(C_0)$ is not algebraic over the residue field. An application of the ruled residue theorem [3] to $\mathbf{C}(x_1,\ldots,x_{n-1})(x_n)$ yields that $\mathrm{qf}(C_0)$ is ruled.

COROLLARY 2.6. *Every rational G_a action on* $\mathbf{C}[x_1,\ldots,x_n]$ *is rationally triangulable for $n \leq 4$.*

PROOF. By Theorem 2.5, $\mathrm{qf}(C_0) = F(Z)$ where F is a unirational extension of $\mathbf{C}$ of transcendence degree $n - 2$. If $n \leq 4$, the transcendence degree of F is at most 2, so

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Castelnuovo's Criterion [6] implies that F, hence also $\mathrm{qf}(C_0)$, is rational.

COROLLARY 2.7. *Given any G_a action on $\mathbf{C}[x_1, \ldots, x_n]$ $\mathrm{qf}(C_0 \otimes_{\mathbf{C}} C_0)$ is a pure transcendental extension of $\mathbf{C}$.*

PROOF. Since $\mathrm{qf}(C_0)(s) = \mathbf{C}(x_1, \ldots, x_n)$ for some (actually any) $s \in \mathbf{C}(x_1, \ldots, x_n)$ with $D(s) = 1$, $\mathrm{qf}(C_0)$ is stably rational. From Theorem 2.5, we have $\mathrm{qf}(C_0) = F(Z)$ so that

$$\begin{aligned}\mathrm{qf}(C_0 \bigotimes{}_{\mathbf{C}} C_0) &= \mathrm{qf}\Big(F(Z) \bigotimes{}_{\mathbf{C}} C_0\Big) \\ &= \mathrm{qf}\Big(F \bigotimes{}_{\mathbf{C}} \mathrm{qf}\big(C_0(Z)\big)\Big) \\ &= \mathrm{qf}\Big(F \bigotimes{}_{\mathbf{C}} \mathbf{C}(x_1, \ldots, x_n)\Big) \\ &= \mathrm{qf}\Big(F(x_1, x_2) \bigotimes{}_{\mathbf{C}} \mathbf{C}(x_3, \ldots, x_n)\Big) \\ &= \mathbf{C}(y_1, \ldots, y_n) \bigotimes{}_{\mathbf{C}} \mathbf{C}(x_3, \ldots, x_n) \\ &= \mathbf{C}(y_1, \ldots, y_n, x_3, \ldots, x_n)\end{aligned}$$

for certain algebraically independent elements $y_1, \ldots, y_n, x_3, \ldots, x_n$.

COROLLARY 2.8. *Let G_a act rationally on $\mathbf{C}^n$ with $\sigma\colon G_a \times \mathbf{C}^n \longrightarrow \mathbf{C}^n$ denoting the action. Then the "doubled" action $\sigma^2\colon (G_a \times G_a) \times \mathbf{C}^{2n} \longrightarrow \mathbf{C}^{2n}$ given by $\sigma^2\colon (\bar{a}, \bar{b}) \mapsto \big(\sigma(\bar{a}), \sigma(\bar{b})\big)$, $\bar{a}, \bar{b} \in \mathbf{C}^n$, is rationally triangulable.*

PROOF. The associated $G_a \times G_a$ action on the coordinate ring

$$\mathbf{C}[y_1, \ldots, y_n, x_1, \ldots, x_n]$$

has $C_0 \otimes_{\mathbf{C}} C_0'$ as its ring of invariants, where $C_0 \cong C_0'$ is the ring of invariants of the action on $\mathbf{C}[y_1, \ldots, y_n]$. In particular, the quotient field of the ring of the $G_a \times G_a$ invariants is a pure transcendental extension of $\mathbf{C}$ by Corollary 2.7. Finally, $\mathbf{C}(y_1, \ldots, y_n) = C_0(y)$ where the action on y is translation by complex numbers (resp. $\mathbf{C}(x_1, \ldots, x_n) = C_0'(x)$) so that the action is rationally triangulable.

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