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A COMPLETE CLASSIFICATION OF DYNAMICAL SYMMETRIES IN CLASSICAL MECHANICS

GEOFF PRINCE

This paper deals with the interaction between the invariance group of a second order differential equation and its variational formulation. In particular I construct equivalent Lagrangians from all such group actions, thereby successfully completing an earlier attempt of mine which dealt with some traditionally important classes of actions.

0. Introduction

In a recent paper in this journal (Prince [5]) I attempted a classification of one parameter group actions which permute the integral curves of the Euler-Lagrange field of a mechanical system. The approach was to determine whether the corresponding Lie derivative of the Cartan 2-form was also a Cartan 2-form. I showed that this was the case for those actions satisfying a certain simple symmetry condition and indeed that those dynamical symmetries distinguished by other important criteria satisfied the condition. In this paper I will complete the classification by showing that those actions which fail the symmetry condition do in fact produce Cartan 2-forms for the mechanics but not in a global sense. Specifically, the image of a regular Lagrangian for the system under such actions depends on the arbitrary choice of a family of integral curves of the Euler-Lagrange field, and once this choice is made the image is a Lagrangian for the corresponding orbits but, in general, for no others.

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1. Review

I will use as far as possible the notation of Prince [5], Crampin, Prince and Thompson [4] and Crampin and Prince [3]. The orbits of the mechanical system are curves on the n-dimensional configuration space M (taken as a C^∞ manifold) with local co-ordinates (x^a). To obtain a corresponding global vector field one needs to go to evolution space, $E = \mathbb{R} \times TM$, with local co-ordinates (t, x^a, u^a). This vector field is the so-called Euler-Lagrange field $\Gamma \in \mathcal{X}(E)$, a second order vector field the projection under $\tau : E \to M$ of whose integral curves are the orbits of the system. Locally it is

$$\Gamma = \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial x^a} + \Lambda^a \frac{\partial}{\partial u^a} .$$

As the name implies Γ represents the Euler-Lagrange equations, for a regular Lagrangian $L \in F(E)$ ($F(E)$ denotes the ring of smooth functions on E):

$$\frac{\partial^2 L}{\partial u^a \, \partial u^b} \Lambda^b = \frac{\partial L}{\partial x^a} - \frac{\partial^2 L}{\partial u^a \, \partial x^b} u^b - \frac{\partial^2 L}{\partial u^a \, \partial t} ;$$

Λ^a are the accelerations.

These equations can be replaced by

$$\langle \Gamma , dt \rangle = 1 \qquad \Gamma \lrcorner \, d\theta_L = 0$$

where $\theta_L \in \mathcal{X}^*(E)$ is the Cartan 1-form:

$$\theta_L + L \, dt + \frac{\partial L}{\partial u^a} (dx^a - u^a dt) .$$

$d\theta_L$ is called the Cartan 2-form for Γ .

Now I describe the various symmetries of the system. A one parameter group action $\{\phi_s\}$ on $\mathbb{R} \times M$ which permutes the parametrised orbits is called a Lie symmetry of the system. It $\{\phi_s\}$ is generated by a vector field $X \in \mathcal{X}(\mathbb{R} \times M)$ then the corresponding infinitesimal condition is

$$L_{X^{(1)}} \Gamma = h \Gamma \ , \quad h \in F(E) \ ;$$

here $X^{(1)} \in \mathcal{X}(E)$ is the first prolongation of X, see for example Crampin, Prince and Thompson [4]. A one parameter group action $\{\psi_s\}$

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on E which permutes the integral curves of Γ satisfies

$$L_Z\Gamma = f\Gamma \ , \qquad f \in F(E) \ ,$$

where $Z \in X(E)$ generates $\{\psi_s\}$. Such group actions are called dynamical symmetries. The Lie symmetries form a finite dimensional Lie group which is a subgroup of the infinite dimensional Lie group of dynamical symmetries.

Now the Noether-Cartan Theorem relates the first integrals of the system to vector fields on E via $d\theta_L$: if $F \in F(E)$ is a first integral, that is $\Gamma(F) = 0$, then there exists a unique dynamical symmetry Z, with $L_Z\Gamma = 0$, such that

$$dF = Z \lrcorner \, d\theta_L \ .$$

This property of Z means that

$$L_Z \, d\theta_L = 0.$$

Indeed $Z + g\Gamma$ satisfies both these last conditions for any $g \in F(E)$ and conversely any such class of dynamical symmetries Z with $L_Z \, d\theta_L = 0$ produces a unique first integral via $dF = Z \lrcorner \, d\theta_L$. These dynamical symmetries are called Cartan symmetries and the finite dimensional subgroup which are also Lie symmetries are called Noether symmetries.

In the earlier paper I calculated $L_Y \, d\theta_L$ for dynamical symmetries and showed that

$$L_Y d\theta_L = d\theta_{L^1} \quad ,$$

where L' is a Lagrangian for the same system, if and only if

$$\frac{\partial^2 L}{\partial u^a \, \partial u^b} \frac{\partial}{\partial u^c} (\xi^b - u^b\sigma) = \frac{\partial^2 L}{\partial u^c \, \partial u^b} \frac{\partial}{\partial u^a} (\xi^b - u^b\sigma).$$

Here

$$Y = \sigma \frac{\partial}{\partial t} + \xi^a \frac{\partial}{\partial x^a} + \eta^a \frac{\partial}{\partial u^a}$$

with $\sigma, \ \xi^a \ , \ \eta^a \in F(E)$.

Crampin [2] has indicated the geometric content of this condition and Crampin and Prince [3] have given a manifestly geometric proof of the result. The condition essentially states that $d\theta_L$ must vanish on the images under the symmetry of the fibres of $\mathbb{R} \times TM$. We also gave the

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details of the relation between $L_Y\, d\theta_L$ and $\psi_s^*\, d\theta_L$ where Y generates $\{\psi_s\}$.

Briefly

$$\psi_s^*\, d\theta_L = d\theta_L + sL_Y d\theta_L + \frac{s^2}{2!} L_Y^2 d\theta_L + \ldots$$

by exponentiation of the vector field, so that $\psi_s^*\, d\theta_L$ is a Cartan 2-form for a particular Lagrangian if and only if

$$L_Y^m\, d\theta_L = d\theta_{L_m}\ , \quad m = 1,2,\ldots$$

is a Cartan 2-form, possibly degenerate. On the hand $\psi_s^*\, d\theta_L$ is a Cartan 2-form for all Lagrangians L if and only if $L_Y d\theta_L$ is a Cartan 2-form for all Lagrangians L(then $L_Y^m\, d\theta_L$ is always such a 2-form). Lie symmetries are clearly in this last category.

2. Completing the Classification

The object here is to take a dynamical symmetry, Z say, which fails the symmetry condition and construct from it one which does satisfy it. This is done in the following way: take an integral curve ρ of Γ and its images under the group action of Z and project these integral curves down to M along with the corresponding restriction of Z. Then this vector field on M acts as a Lie symmetry for the given family and also automatically satisfies the symmetry condition (its components are independent of u^a). The action of this Lie symmetry on $d\theta_L$ thus produces another Cartan 2-form on the images of ρ.

Now to the formalities, beginning with some prefatory definitions and lemmas to the main theorem. In what follows the horizontal component of Γ (see Crampin, Prince and Thompson [4]) is nowhere zero on the integral curve ρ or its images under the group action. This ensures that the corresponding congruence of orbits have nowhere zero tangent field. Further the images of ρ are assumed to have distinct projections, that is the group action does not locally entail a mere change of parametrisation of an orbit.

By analogy with the definition of Jacobi fields in Riemannian geometry, consider an orbit $\gamma : t \to \gamma(t) \in M$ of the mechanical system, then:

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DEFINITION. A vector field X along an orbit γ is called a *Lie field for* γ if it satisfies the equation

$$L_\Gamma X^{(1)} = 0$$

along the integral curve $\rho(t) = (t, \gamma(t), \dot\gamma(t))$ of Γ. □

In co-ordinates with $X = \xi^a \left.\frac{\partial}{\partial x^a}\right|_{x^a = \gamma^a}$

this is just

$$\ddot\xi^a - \dot\xi^b \left.\frac{\partial f^a}{\partial u^b}\right|_{u^b = \dot\gamma^b} - \xi^b \left.\frac{\partial f}{\partial x^b}\right|_{x^b = \gamma^b} = 0$$

where

$$X^{(1)} = \xi^a \left.\frac{\partial}{\partial x^a}\right|_{x^a = \gamma^a} + \dot\xi^b \left.\frac{\partial}{\partial u^b}\right|_{u^b = \dot\gamma^b}$$

is the prolongation of X from a vector field on an orbit on M to a vector field on the corresponding integral curve on E. It is important to realise that X need only be defined on the orbit, not necessarily on some open region of M. This means, for example, what while $X^{(1)}(g)$ is defined for $g \in F(E)$, $L_{X^{(1)}}\Gamma$ is not.

LEMMA. *Suppose* ρ *is an integral curve of* Γ *and that* Z *is a vector field defined along* ρ *such that* $L_\Gamma Z = 0$. *Then the vector field* $\tau_* Z$ *along the orbit* $\gamma = \tau \circ \rho$ *is a Lie field for* γ.

Proof. Firstly choose Z with no Γ component in the basis $\left\{\Gamma, \frac{\partial}{\partial x^a}, \frac{\partial}{\partial u^a}\right\}$: clearly $Z \to Z - \langle Z, dt\rangle\, \Gamma$ does not affect $L_\Gamma Z = 0$. Now $(\tau_* Z)^{(1)} = Z$ because the condition $L_\Gamma Z = 0$ forces Z to be of the form

$$Z = \zeta^a \left.\frac{\partial}{\partial x^a}\right|_{x^a = \gamma^a} + \dot\zeta^a \left.\frac{\partial}{\partial u^a}\right|_{u^a = \dot\gamma^a}$$

Thus $\tau_* Z$ is a Lie field for $\gamma = \tau \circ \rho$. (I refer the reader to Prince and Crampin [6] for a similar and extended approach to Jacobi fields.) □

Now one takes a dynamical symmetry Z of Γ (with $L_\Gamma Z = 0$ and its corresponding one-parameter group action $\{\psi_s\}$ on E. The action of $\{\psi_s\}$ on an integral curve ρ of Γ generates a submanifold S of co-dimension $2n$.

The previous lemma then gives

LEMMA. *The projection of the restriction of* Z *to* S *is a Lie field* X *for the orbits* $\{\gamma_s = \tau \circ (\psi_s \circ \rho)\}$. *Moreover* $L_{X^{(1)}}\Gamma = 0$ *on the submanifold* S.

Proof. The earlier lemma guarantees that the vector field $X_s = \tau_*(Z|_{\psi_s \circ \rho})$ is a Lie field for γ_s. The union X over $\{\gamma_s\}$ of these fields is a vector field on the projection, S^+ , of S and consequently $X^{(1)}$ exists in the usual way on $\mathbb{R} \times TS^+$ and so $L_{X^{(1)}}\Gamma = L_\Gamma X^{(1)} = 0$. $\square$

One now has a field $X \in \mathcal{X}(S^+)$ and a prolongation $X^{(1)}$ on $\mathbb{R} \times TS^+$ which agrees with Z on S (and is tangent to S as is Γ). $X^{(1)}$ clearly satisfies the symmetry condition and, since $L_{X^{(1)}}\Gamma = 0$ on S, the 2-form on $\mathbb{R} \times TS^+$, $L_{X^{(1)}} d\theta_L$, is a Cartan 2-form on S but not in general elsewhere.

This constitutes the proof of the main theorem:

THEOREM. *The 2-form* $L_{X^{(1)}} d\theta_L$ *is a Cartan 2-form for* Γ *on the integral curves* $\psi_s \circ \rho$. $\square$

REMARKS. (i) It is evident from the derivations of the symmetry conditions in Prince [5] or Crampin and Prince [3] that

$$L_{X^{(1)}} d\theta_L = d\theta_K$$

for some $K \in F(E)$, however in general $\Gamma \lrcorner\, d\theta_K = 0$ only on S. Thus $d\theta_K$ will be a Cartan 2-form on $\mathbb{R} \times TS^+$, possibly degenerate, for some mechanical system $\tilde{\Gamma}$ which coincides with Γ on S, that is, both systems share the orbits $\{\gamma_s\}$.

(ii) The foregoing construction does not rely on the failure of the symmetry condition by Z. If Z does fail the symmetry condition it is because it fails to map the fibres of TM to Lagrange subspaces of $d\theta_L$. Consequently $L_Z\, d\theta_L$ is not a Cartan 2-form for Γ on S because $L_Z\, d\theta_L$ depends on the components of $d\theta_L$ in E of S. $X^{(1)}$ has the same effect as Z on the orbits on S and moreover satisfies the symmetry condition trivially by preserving the fibres of TM.

(iii) By virtue of the last remark in the previous section $\phi_s^*\, d\theta_L$ is a Cartan 2-form for Γ on S for all Lagrangians where $X^{(1)}$ generates $\{\phi_s\}$. Indeed for a dynamical symmetry satisfying the symmetry condition but for which $\psi_s^*\, d\theta_L$ may neither be a Cartan 2-form for a particular Lagrangian nor a generic one, the construction presented guarantees such Cartan 2-forms at least on a one-parameter family of orbits.

EXAMPLE. As an illustration of the features described above I consider the two dimensional free particle for which

$$L(t\ ,\ \underset{\sim}{x}\ ,\ \underset{\sim}{u}) = \tfrac{1}{2}\ \delta_{ab}u^a u^b\ ,$$

$$\theta_L = \tfrac{1}{2}\ \delta_{ab}u^a u^b dt + \delta_{ab}u^a(dx^b - u^b dt)\ ,$$

$$d\theta_L = \delta_{ab}du^a \wedge (dx^b - u^b dt)\ ,$$

and

$$\Gamma = \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial x^a}\ .$$

It is simple to check that

$$Z = u^2(x^1 - u^1 t)^2 \frac{\partial}{\partial x^1} \in X(E)$$

is a dynamical symmetry of Γ with $L_Z\Gamma = 0$ which fails the symmetry condition.

The generic integral curve ρ of Γ is

$$\rho(t) = (t,\ at + b,\ ct + d,\ a,\ b).$$

The one parameter group $\{\psi_s\}$ generated by Z can be shown to give

$$\psi_s(\rho(t)) = \left(t,\ \frac{b}{1 - cbs} + at,\ ct + d,\ a,\ b\right);$$

taking c, b non-zero S^+ is just a half plane depending on the constants a, b, c and d(only the branch of $f(s) = \dfrac{b}{1 - cbs}$ continuously connected to $s = 0$ is allowed).

Then

$$X = \tau_*(Z|_{\psi_s \circ \rho}) = c(x^1 - at)^2 \frac{\partial}{\partial x^1}$$

and

$$X^{(1)} = c(x^1 - at)^2 \frac{\partial}{\partial x^1} + 2c(x^1 - at)(u^1 - a)\frac{\partial}{\partial u^1}$$

which co-incides with Z on S. Also

$$L_{X^{(1)}}\Gamma = -2c(u^1 - a)^2 \frac{\partial}{\partial u^1}$$

which vanishes on the submanifold $u^1 = a$ containing S.

A little straightforward computation gives

$$L_{X^{(1)}} d\theta_L = d\theta_K = 4c(x^1 - at)(du^1 - \Lambda^1 dt) \wedge (dx^1 - u^1 dt)$$

where

$$K(t, \underset{\sim}{x}, \underset{\sim}{u}) = 2c(x^1 - at)u^1(u^1 - a)$$

is a degenerate Lagrangian for mechanical systems whose x^1 component of acceleration is Λ^1. Clearly the restriction of $L_{X^{(1)}} d\theta_L$ to S is just a multiple of $d\theta_L$ (the multiple is a first integral on S), however the degenerate Lagrangian K leads to the equation of motion

$$2(x^1 - at)\ddot{x}^1 = -(\dot{x}^1 - a)^2$$

which has orbits with the two parameter x^1 component on S^+ of

$$(x^1 - at) = (A + Bt)^{2/3}.$$

The orbits $\tau \circ (\psi_s \circ \rho)$ are obviously included in the family.

4. Final Remarks

In demonstrating this construction of alternative Lagrangians for any dynamical symmetry and in showing that the symmetry condition of my earlier paper is satisfies when this can be performed without reference to a particular integral curve, I have answered the remaining open question posed in that paper. The other two, namely "does a pair of equivalent Lagrangians lead to a dynamical symmetry satisfying the condition?" and "does a pair of Lagrangians so related lead to a closed form first integral?" where resolved in the affirmative by Sarlet [7] and Crampin [1] respectively. Along with the geometric interpretation and proof of the condition in Crampin and Prince [3] this present paper provides a clearer picture of the interaction of symmetries of the second order differential equation field with the variational aspects of the mechanics. This may be of some use in unravelling the Helmholtz condition for the inverse problem in Lagrangian mechanics. These are the conditions that must be satisfied by a second order ordinary differential equation admitting a variational formulation. They can be regarded as equations whose solutions are equivalent Lagrangians for the mechanics, and as such their relation to the results here is obviously close but unclear. Indeed, in Crampin and Prince [3] we derived the symmetry condition by invariance considerations on these equations, however the actual business of using dynamical symmetries to solve the Helmholtz conditions is not yet established. I hope this will be the subject of subsequent papers.

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Department of Mathematics,
Royal Melbourne Institute of Technology,
G.P.O. Box 2476V,
Melbourne,
Victoria 3001,
Australia.

Current address:

Department of Mathematics,
La Trobe University,
Bundoora,
Victoria 3083,
Australia.

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