COMPOSITIO MATHEMATICA

K-stability of birationally superrigid Fano varieties

Charlie Stibitz and Ziquan Zhuang

Compositio Math. **155** (2019), 1845–1852.

doi:10.1112/S0010437X19007498

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY EST. 1865

Compositio Math. 155 (2019) 1845–1852
doi:10.1112/S0010437X19007498

K-stability of birationally superrigid Fano varieties

Charlie Stibitz and Ziquan Zhuang

Abstract

We prove that every birationally superrigid Fano variety whose alpha invariant is greater than (respectively no smaller than) $\frac{1}{2}$ is K-stable (respectively K-semistable). We also prove that the alpha invariant of a birationally superrigid Fano variety of dimension n is at least $1/(n+1)$ (under mild assumptions) and that the moduli space (if it exists) of birationally superrigid Fano varieties is separated.

1. Introduction

The notion of birational superrigidity was introduced as a generalization of Iskovskih and Manin's work [IM71] on the non-rationality of quartic threefolds; on the other hand, the concept of K-stability emerges in the study of Kähler–Einstein metrics on Fano manifolds. While the two notions have different nature of origin, they seem to resemble each other in the following sense: it is well known that a Fano variety X of Picard number one is birationally superrigid if and only if (X, M) has canonical singularities for every movable boundary $M \sim_{\mathbb{Q}} -K_X$; on the other hand, by the recent work of [FO18, BJ17], the K-(semi)stability of X is (roughly speaking) characterized by the log canonicity of basis type divisors, which is the average of a basis of some pluri-anticanonical system. In other words, both notions are tied to the singularities of certain anticanonical $\mathbb{Q}$-divisors and so it is very natural to expect some relation between them. Indeed, the slope stability (a weaker notion of K-stability) of birationally superrigid Fano manifolds has been established by [OO13] under some mild assumptions, and it is conjectured [OO13, KOW18] that birationally rigid Fano varieties are always K-stable.

In this note, we give a partial solution to this conjecture. Here is our main result.

Definition 1.1. The alpha invariant $\alpha(X)$ of a $\mathbb{Q}$-Fano variety X (i.e. X has Kawamata log terminal (klt) singularities and $-K_X$ is ample) is defined as the supremum of all $t > 0$ such that (X, tD) is log canonical (lc) for every effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$.

Theorem 1.2. *Let X be a $\mathbb{Q}$-Fano variety of Picard number one. If X is birationally superrigid (or, more generally, (X, M) is log canonical for every movable boundary $M \sim_{\mathbb{Q}} -K_X$) and $\alpha(X) \geqslant \frac{1}{2}$ (respectively $> \frac{1}{2}$), then X is K-semistable (respectively K-stable).*

It is well known that smooth Fano hypersurfaces of index one and dimension $n \geqslant 3$ are birationally superrigid [IM71, dFEM03, dF16] and their alpha invariants are at least $n/(n+1)$ [Che01]; hence, we have the following immediate corollary, reproving the K-stability of Fano hypersurfaces of index one.

Received 10 August 2018, accepted in final form 8 April 2019, published online 7 August 2019.
2010 Mathematics Subject Classification 14J45, 14E05, 32Q20 (primary).
Keywords: K-stability, birational superrigidity, Kähler–Einstein metrics, Fano varieties.

Corollary 1.3 [Fuj19b]. *Let $X \subseteq \mathbb{P}^n$ be a smooth hypersurface of degree $n \geqslant 4$; then X is K-stable.*

Another application is to the K-stability of general index-two hypersurfaces. By [Che01] and [Puk16], such hypersurfaces have alpha invariant $\frac{1}{2}$ and (X, M) is lc for every movable boundary $M \sim_{\mathbb{Q}} -K_X$. Hence, by analyzing the equality case in Theorem 1.2, we prove the following result.

Corollary 1.4. *Let $n \geqslant 16$ and let $U \subseteq \mathbb{P}H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(n))$ be the parameter space of smooth index-two hypersurfaces. Let $T \subseteq U$ be the set of hypersurfaces that are not K-stable. Then* $\mathrm{codim}_U T \geqslant \frac{1}{2}(n-11)(n-10) - 10$.

Although alpha invariants are in general hard to estimate, the known examples seem to suggest that birationally (super)rigid varieties have large alpha invariants. In view of Theorem 1.2, it is therefore natural to ask the following question.

Question 1.5. Let X be a birationally superrigid Fano variety. Is it true that $\alpha(X) > \frac{1}{2}$?

Obviously, a positive answer to this question will confirm the K-stability of all birationally superrigid Fano varieties. At this point, we only have a weaker estimate.

Theorem 1.6. *Let X be a $\mathbb{Q}$-Fano variety of Picard number one and dimension $n \geqslant 3$. Assume that X is birationally superrigid (or, more generally, (X, M) is log canonical for every movable boundary $M \sim_{\mathbb{Q}} -K_X$),$-K_X$ generates the class group $\mathrm{Cl}(X)$ of X and $|-K_X|$ is base-point-free, then $\alpha(X) \geqslant 1/(n+1)$.*

Note that this is in line with the conjectural K-stability of such varieties, since, by [FO18, Theorem 3.5], the alpha invariant of a K-semistable Fano variety is always $\geqslant 1/(n+1)$. We also remark that the assumptions about the index and base-point freeness in the above theorem seem to be mild and they are satisfied by most known examples of birationally superrigid varieties.

As other evidence towards a positive answer of Question 1.5 and K-stability of birationally superrigid Fano varieties, we prove the following result.

Theorem 1.7. *Let $f : X \to C$, $g : Y \to C$ be two flat families of $\mathbb{Q}$-Fano varieties (i.e. all geometric fibers are integral, normal and $\mathbb{Q}$-Fano) over a smooth pointed curve $0 \in C$. Assume that the central fibers $X_0 = f^{-1}(0)$ and $Y_0 = g^{-1}(0)$ are birationally superrigid and there exists an isomorphism $\rho : X \backslash X_0 \cong Y \backslash Y_0$ over the punctured curve $C \backslash 0$. Then ρ induces an isomorphism $X \cong Y$ over C.*

In other words, the moduli space (if it exists) of birationally superrigid Fano varieties is separated. A similar statement is also conjectured for families of K-stable Fano varieties (postscript note: this has now been proved by Blum and Xu) and our proof of Theorem 1.7 is indeed inspired by the recent work [BX18] in the uniformly K-stable case. One should also note that if the answer to Question 1.5 is positive, then Theorem 1.7 follows immediately from [Che09, Theorem 1.5].

https://doi.org/10.1112/S0010437X19007498 Published online by Cambridge University Press

2. Preliminary

2.1 Notation and conventions

We work over the field $\mathbb{C}$ of complex numbers. Unless otherwise specified, all varieties are assumed to be projective and normal and divisors are understood as $\mathbb{Q}$-divisors. The notions of canonical, klt and lc singularities are defined in the sense of [Kol97, Definition 3.5]. A movable boundary is defined as an expression of the form $a\mathcal{M}$, where $a \in \mathbb{Q}$ and $\mathcal{M}$ is a movable linear system. Its $\mathbb{Q}$-linear equivalence class is defined in an evident way. If $M = a\mathcal{M}$ is a movable boundary on X, we say that (X, M) is klt (respectively canonical, lc) if for $k \gg 0$ and for general members $D_1, \dots, D_k$ of the linear system $\mathcal{M}$, the pair (X, M_k) (where $M_k = (a/k)\sum_{i=1}^{k} D_i$) is klt (respectively canonical, lc) in the usual sense. The lc threshold [Kol97, Definition 8.1] of a divisor D on X is denoted by $\mathrm{lct}(X; D)$.

2.2 K-stability

We refer to [Tia97, Don02] for the original definition of K-stability using test configurations. In this paper we use the following equivalent valuative criterion.

DEFINITION 2.1 [Fuj19a, Definition 1.1]. Let X be a $\mathbb{Q}$-Fano variety of dimension n. Let F be a prime divisor over X, i.e. there exists a projective birational morphism $\pi : Y \to X$ with Y normal such that F is a prime divisor on Y.

(i) For any $x \geqslant 0$, we define $\mathrm{vol}_X(-K_X - xF) := \mathrm{vol}_Y(-\pi^* K_X - xF)$.

(ii) The *pseudo-effective threshold* $\tau(F)$ of F with respect to $-K_X$ is defined as

$$\tau(F) := \sup\{\tau > 0 \mid \mathrm{vol}_X(-K_X - \tau F) > 0\}.$$

(iii) Let $A_X(F)$ be the log discrepancy of F with respect to X. We set

$$\beta(F) := A_X(F) \cdot ((-K_X)^n) - \int_0^\infty \mathrm{vol}_X(-K_X - xF)\, dx.$$

(iv) F is said to be *dreamy* if the graded algebra

$$\bigoplus_{k,j \in \mathbb{Z}_{\geqslant 0}} H^0(Y, -kr\pi^* K_X - jF)$$

is finitely generated for some (hence, for any) $r \in \mathbb{Z}> 0$ with rK_X Cartier.

THEOREM 2.2 [Fuj19a, Theorems 1.3 and 1.4] and [Li17, Theorem 3.7]. *Let X be a $\mathbb{Q}$-Fano variety. Then X is K-stable (respectively K-semistable) if and only if $\beta(F) > 0$ (respectively $\beta(F) \geqslant 0$) holds for any dreamy prime divisor F over X.*

2.3 Birational superrigidity

A Fano variety X is said to be birationally superrigid if it has terminal singularities, is $\mathbb{Q}$-factorial of Picard number one and every birational map $f : X \dashrightarrow Y$ from X to a Mori fiber space is an isomorphism (see e.g. [CS08, Definition 1.25]). We have the following equivalent characterization of birational superrigidity.

THEOREM 2.3 [CS08, Theorem 1.26]. *Let X be a Fano variety. Then it is birationally superrigid if and only if it has $\mathbb{Q}$-factorial terminal singularities, Picard number one and, for every movable boundary $M \sim_{\mathbb{Q}} -K_X$ on X, the pair (X, M) has canonical singularities.*

3. Proofs

In this section we prove the results stated in the introduction.

Proof of Theorem 1.2. The proof strategy is similar to those of [Fuj19c, Proposition 2.1]. By assumption, we have $\mathrm{lct}(X;D) \geqslant \frac{1}{2}$ (respectively $> \frac{1}{2}$) for every effective divisor $D \sim_{\mathbb{Q}} -K_X$. For simplicity we only prove the K-semistability, since the K-stability part is almost identical. As such, we assume that $\mathrm{lct}(X;D) \geqslant \frac{1}{2}$ in the rest of the proof. Let F be a dreamy divisor over X and let $\tau = \tau(F)$. Let $\pi : Y \to X$ be a projective birational morphism such that F is a prime divisor on Y and let

$$b = \frac{1}{(-K_X)^n} \int_0^\tau \mathrm{vol}_X(-K_X - xF)\, dx.$$

By Theorem 2.2, it suffices to show that $b \leqslant A = A_X(F)$.

Suppose that this is not the case, i.e. $b > A$. As in the proof of [Fuj19c, Proposition 2.1], we have

$$\int_0^\tau (x-b) \cdot \mathrm{vol}_{Y|F}(-\pi^* K_X - xF)\, dx = 0, \tag{1}$$

where $\mathrm{vol}_{Y|F}$ denotes the restricted volume of a divisor to F (see [ELM09]). Since X has Picard number one, every divisor on X is linearly equivalent to a multiple of $-K_X$ and hence by our assumption that (X, M) is lc for every movable boundary M, we see that there exists at most one irreducible divisor $D \sim_{\mathbb{Q}} -K_X$ such that $\mathrm{ord}_F(D) > A$. It follows that $\mathrm{ord}_F(D) = \tau$ by the definition of $\tau(F)$. Moreover if $x \geqslant A$ and $D' \sim_{\mathbb{Q}} -K_X$ is an effective divisor for which $\mathrm{ord}_F(D') \geqslant x$ and we write $D' = aD + \Gamma$, where $D \not\subseteq \mathrm{Supp}\ (\Gamma)$, then $\mathrm{ord}_F(\Gamma) \leqslant A$. As

$$-\pi^* K_X - xF = \frac{\tau - x}{\tau - A}(-\pi^* K_X - AF) + \frac{x - A}{\tau - A}(-\pi^* K_X - \tau F),$$

we see that $a \geqslant (x-A)/(\tau - A)$ and, therefore, if in addition $x \in \mathbb{Q}$ and m is sufficiently divisible, then the natural inclusion

$$H^0\left(Y, \frac{\tau - x}{\tau - A}(-m\pi^* K_X - mAF)\right) \hookrightarrow H^0(Y, -m\pi^* K_X - mxF)$$

given by the multiplication of $m \cdot ((x-A)/(\tau - A))(\pi^* D - \tau F)$ is an isomorphism. By the definition and the continuity of restricted volume, this implies (note that F is not in the support of $\pi^* D - \tau F$) that

$$\mathrm{vol}_{Y|F}(-\pi^* K_X - xF) = \left(\frac{\tau - x}{\tau - A}\right)^{n-1} \mathrm{vol}_{Y|F}(-\pi^* K_X - AF) \tag{2}$$

when $A \leqslant x \leqslant \tau$. On the other hand, by the log-concavity of restricted volume [ELM09, Theorem A], we have

$$\mathrm{vol}_{Y|F}(-\pi^* K_X - xF) \geqslant \left(\frac{x}{A}\right)^{n-1} \mathrm{vol}_{Y|F}(-\pi^* K_X - AF) \tag{3}$$

whenever $x \in [0, A]$. Since $b > A$, combining (1), (2) and (3), we get the inequality

$$0 \leqslant \int_0^A (x-b)\left(\frac{x}{A}\right)^{n-1} dx + \int_A^\tau (x-b)\left(\frac{\tau - x}{\tau - A}\right)^{n-1} dx, \tag{4}$$

https://doi.org/10.1112/S0010437X19007498 Published online by Cambridge University Press

which is equivalent to

$$\frac{\tau - 2A}{n(n+1)} + \frac{A-b}{n} \geqslant 0. \tag{5}$$

As $b > A$, we have $\mathrm{ord}_F(D) = \tau > 2A$, which implies that $\mathrm{lct}(X; D) < \frac{1}{2}$, contradicting our assumption. □

Remark 3.1. Let us also give a somewhat conceptual summary of the proof (for K-semistability). In terms of the equation (1), it suffices to check that the center of mass of the interval $[0, \tau]$ with density function $f(x) = (1/(-K_X)^n)\mathrm{vol}_{Y|F}(-\pi^*K_X - xF)$ is at most A. The assumption that (X, M) is lc for every movable boundary $M \sim_{\mathbb{Q}} -K_X$ implies that $g(x) = f(x)^{1/(n-1)}$ is linear when $x \geqslant A$ and $g(\tau) = 0$ if $\tau > A$, while $\alpha(X) \geqslant \frac{1}{2}$ implies that the length of the interval is at most $2A$. Since $g(x)$ is also concave, it is clear (by looking at the graph of $g(x)$) that the center of mass is at most A.

It is not hard to characterize the equality case from the above proof.

Corollary 3.2. *Let X be a $\mathbb{Q}$-Fano variety of Picard number one. Assume that (X, M) is lc for every movable boundary $M \sim_{\mathbb{Q}} -K_X$, $\alpha(X) \geqslant \frac{1}{2}$ but X is not K-stable; then there exist a dreamy prime divisor F over X, a movable boundary $M \sim_{\mathbb{Q}} -K_X$ and an effective divisor $D \sim_{\mathbb{Q}} -K_X$ such that F is a log canonical place of (X, M) and $(X, \frac{1}{2}D)$. In particular, if X is birationally superrigid and $\alpha(X) \geqslant \frac{1}{2}$, then X is K-stable.*

Proof. We keep the notation from the above proof. Let $\eta = \eta(F)$ be the movable threshold of $-K_X$ with respect to F, i.e. the supremum of $c \geqslant 0$ such that there exists an effective divisor $D_0 \sim_{\mathbb{Q}} -K_X$ with $\mathrm{ord}_F(D_0) = c$ whose support does not contain D (see e.g. [Zhu18, Definition 4.1]). Since F is dreamy, this is indeed a maximum and there exists $D_1 \sim_{\mathbb{Q}} -K_X$ such that $\mathrm{ord}_F(D_1) = \eta$ (one can simply take D_1 to be the divisor corresponding to a generator of $\bigoplus_{k,j\in\mathbb{Z}_{\geqslant 0}} H^0(Y, -kr\pi^*K_X - jF)$ with largest slope j/kr among those that do not vanish on D).

Suppose that X is not K-stable and choose F to be a dreamy divisor over X such that $\beta(F) = 0$; then we have $b = A \geqslant \eta$. In this case by the same proof as above the (in)equalities (2), (3), (4) and (5) hold true with η in place of A and we have

$$\frac{\tau - 2\eta}{n(n+1)} + \frac{\eta - b}{n} \geqslant 0$$

or $(\tau - 2A) + (n-1)(\eta - A) \geqslant 0$ (note that $b = A$). But by assumption we have $\tau \leqslant 2A$ and $\eta \leqslant A$ and hence this is only possible when $\eta = A$ and $\tau = 2A$. Taking M to be the linear system generated by a sufficiently divisible multiple of D and D_1 (and then rescale so that $M \sim_{\mathbb{Q}} -K_X$) finishes the proof. □

Proof of Corollary 1.4. Let $S \subseteq U$ be the set of *regular* hypersurfaces as defined in [Puk17, § 0.2]. By [Puk17, Theorem 2], S is non-empty and the complement of S has codimension at least $\frac{1}{2}(n-11)(n-10) - 10$. Therefore, it suffices to show that every hypersurface in the set S is K-stable. Let X be such a hypersurface.

Let H be the hyperplane class and let $D \sim_{\mathbb{Q}} H \sim_{\mathbb{Q}} -\frac{1}{2}K_X$ be an effective divisor. By [Che01, Lemma 3.1], (X, D) is lc and indeed by [Puk02, Proposition 5] we have $\mathrm{mult}_x D \leqslant 1$ for all but finitely many $x \in X$; hence, by [Kol97, (3.14.1)], (X, D) has canonical singularities outside a

https://doi.org/10.1112/S0010437X19007498 Published online by Cambridge University Press

finite number of points. It follows that every lc center of (X, D) is either a divisor on X or an isolated point.

On the other hand, let $M \sim_{\mathbb{Q}} -K_X$ be a movable boundary; then, by the main result of [Puk17], the only possible center of maximal singularities of (X, M) is a linear section of X of codimension two. It follows that (X, M) is lc and every lc center of (X, M) is a linear section of codimension two.

Hence, X is K-semistable by Theorem 1.2. Suppose that it is not K-stable; then by Corollary 3.2 there exist a movable boundary $M \sim_{\mathbb{Q}} -K_X$ and an effective divisor $D \sim_{\mathbb{Q}} H$ such that (X, M) and (X, D) have a common lc center. But, by the previous analysis, this is impossible as the lc centers of (X, M) and (X, D) always have different dimension. Therefore, X is K-stable and the proof is complete. $\square$

Proof of Theorem 1.6. It suffices to show that $\mathrm{lct}(X; D) \geqslant 1/(n+1)$ for every $D \sim_{\mathbb{Q}} -K_X$. We may assume that D is irreducible. Since $\mathrm{Cl}(X)$ is generated by $-K_X$, we have $\mathrm{mult}_\eta D \leqslant 1$, where η is the generic point of D. It follows that (X, D) is lc in codimension one [Kol97, (3.14.1)] and hence the multiplier ideal $\mathcal{J}(X, (1-\epsilon)D)$ (where $0 < \epsilon \ll 1$) defines a subscheme of codimension at least two. By Nadel vanishing,

$$H^i(X, \mathcal{J}(X, (1-\epsilon)D) \otimes \mathcal{O}_X(-rK_X)) = 0$$

for every $i > 0$ and $r \geqslant 0$; therefore, by Castelnuovo–Mumford regularity (see e.g. [Laz04, § 1.8]), the sheaf $\mathcal{J}(X, (1-\epsilon)D) \otimes \mathcal{O}_X(-nK_X)$ is generated by its global sections and we get a movable linear system

$$\mathcal{M} = |\mathcal{J}(X, (1-\epsilon)D) \otimes \mathcal{O}_X(-nK_X)|.$$

Suppose that $\mathrm{lct}(X; D) < 1/(n+1)$, let E be an exceptional divisor over X that computes it, let $A = A_X(E)$ and let $\pi : Y \to X$ be a projective birational morphism such that the center of E on Y is a divisor; then $A \in \mathbb{Z}$ (since $-K_X$ is Cartier by assumption) and we have $d = \mathrm{ord}_E(D) > (n+1)A$ and $\mathcal{J}(X, (1-\epsilon)D) \subseteq \pi_*\mathcal{O}_Y((A-1-\lfloor(1-\epsilon)d\rfloor)E) \subseteq \pi_*\mathcal{O}_Y(-(nA+1)E)$. It follows that $\mathrm{ord}_E(\mathcal{M}) \geqslant nA+1$ and hence for the movable boundary $M = (1/n)\mathcal{M} \sim_{\mathbb{Q}} -K_X$ we have $\mathrm{ord}_E(M) > A$ and (X, M) is not lc, violating our assumption. Thus, $\mathrm{lct}(X; D) \geqslant 1/(n+1)$ and we are done. $\square$

Finally we prove the separatedness statement (Theorem 1.7). For this we recall the following criterion.

Lemma 3.3. *Let $f : X \to C$, $g : Y \to C$ be flat families of $\mathbb{Q}$-Fano varieties over a smooth pointed curve $0 \in C$ with central fibers X_0 and Y_0. Assume that K_X and K_Y are $\mathbb{Q}$-Cartier and let $D_X \sim_{\mathbb{Q}} -K_X$, $D_Y \sim_{\mathbb{Q}} -K_Y$ be effective divisors not containing X_0 or Y_0. Assume that there exists an isomorphism*

$$\rho : (X, D_X) \times_C C^\circ \cong (Y, D_Y) \times_C C^\circ$$

over $C^\circ = C \backslash 0$, that $(X_0, D_X|_{X_0})$ is klt and that $(Y_0, D_Y|_{Y_0})$ is lc. Then ρ extends to an isomorphism $(X, D_X) \cong (Y, D_Y)$.

Proof. This follows from the exact same proof of [LWX19, Theorem 5.2] (see also [BX18, Proposition 3.2]). $\square$

Proof of Theorem 1.7. Since birationally superrigid Fano varieties have terminal singularities, K_X and K_Y are $\mathbb{Q}$-Cartier by [dFH11, Proposition 3.5]. Hence, the result follows from Theorem 2.3 and the following more general statement. $\square$

https://doi.org/10.1112/S0010437X19007498 Published online by Cambridge University Press

Lemma 3.4. *Let $f : X \to C$, $g : Y \to C$ be flat families of $\mathbb{Q}$-Fano varieties over a smooth pointed curve $0 \in C$ that are isomorphic over $C^\circ = C\backslash 0$. Let X_0 and Y_0 be their central fibers. Assume that:*

(i) *K_X and K_Y are $\mathbb{Q}$-Cartier;*
(ii) *for every movable boundary $M_X \sim_{\mathbb{Q}} -K_{X_0}$, (X_0, M_X) is klt;*
(iii) *for every movable boundary $M_Y \sim_{\mathbb{Q}} -K_{Y_0}$, (Y_0, M_Y) is lc.*

Then $X \cong Y$ over C.

Proof. By assumption, X is birational to Y over C. Let m be a sufficiently large and divisible integer and let $D_1 \in |-mK_{X_0}|$, $D_2 \in |-mK_{Y_0}|$ be general divisors in the corresponding linear system. Choose effective divisors $D_{X,1} \sim -mK_X$, $D_{Y,2} \sim -mK_Y$ not containing X_0 or Y_0 such that $D_{X,1}|_{X_0} = D_1$ and $D_{Y,2}|_{Y_0} = D_2$. Let $D_{Y,1}$ and $D_{X,2}$ be their strict transforms to the other family. Since X and Y are isomorphic over C°, we have $D_{Y,1} \sim -mK_Y + W$, where W is supported on Y_0; but, as Y_0 is irreducible, we have $W = \ell Y_0 = \ell g^*(0)$ for some integer ℓ. Since the question is local around $0 \in C$, we may shrink C so that $Y_0 \sim 0$ and thus $D_{Y,1} \sim -mK_Y$. Similarly, we also have $D_{X,2} \sim -mK_X$. Let $D'_1 = D_{Y,1}|_{Y_0}$ and $D'_2 = D_{X,2}|_{X_0}$. Let $\mathcal{M}_X$ be the linear system spanned by $D_{X,1}$ and $D_{X,2}$ and let $M_X = (1/m)\mathcal{M}_X \sim_{\mathbb{Q}} -K_X$. Similarly, we have $\mathcal{M}_Y$ and $M_Y \sim_{\mathbb{Q}} -K_Y$. As D_1 and D_2 are general, D_1 and D'_2 have no common components; hence, the restriction of M_X to X_0 is still a movable boundary and, therefore, by our second assumption, $(X_0, M_X|_{X_0})$ is klt. Similarly, $(Y_0, M_Y|_{Y_0})$ is lc and we conclude by Lemma 3.3. □

Acknowledgements

The authors would like to thank their advisor János Kollár for constant support, encouragement and numerous inspiring conversations. The first author wishes to thank Harold Blum for a useful discussion as well. Both authors also wish to thank Yuchen Liu for helpful discussions.

References

BJ17 H. Blum and M. Jonsson, *Thresholds, valuations, and K-stability*, Preprint (2017), arXiv:1706.04548.

BX18 H. Blum and C. Xu, *Uniqueness of K-polystable degenerations of Fano varieties*, Ann. of Math., to appear. Preprint (2018), arXiv:1812.03538.

Che01 I. A. Cheltsov, *Log canonical thresholds on hypersurfaces*, Mat. Sb. **192** (2001), 155–172.

Che09 I. Cheltsov, *On singular cubic surfaces*, Asian J. Math. **13** (2009), 191–214.

CS08 I. A. Cheltsov and K. A. Shramov, *Log-canonical thresholds for nonsingular Fano threefolds*, Uspekhi Mat. Nauk **63** (2008), 73–180.

dF16 T. de Fernex, *Erratum to: Birationally rigid hypersurfaces*, Invent. Math. **203** (2016), 675–680.

dFEM03 T. de Fernex, L. Ein and M. Mustaţă, *Bounds for log canonical thresholds with applications to birational rigidity*, Math. Res. Lett. **10** (2003), 219–236.

dFH11 T. de Fernex and C. D. Hacon, *Deformations of canonical pairs and Fano varieties*, J. Reine Angew. Math. **651** (2011), 97–126.

Don02 S. K. Donaldson, *Scalar curvature and stability of toric varieties*, J. Differential Geom. **62** (2002), 289–349.

ELM09 L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye and M. Popa, *Restricted volumes and base loci of linear series*, Amer. J. Math. **131** (2009), 607–651.

Fuj19a K. Fujita, *A valuative criterion for uniform K-stability of $\mathbb{Q}$-Fano varieties*, J. Reine Angew. Math. **751** (2019), 309–338.

Fuj19b K. Fujita, *K-stability of Fano manifolds with not small alpha invariants*, J. Inst. Math. Jussieu **18** (2019), 519–530.

Fuj19c K. Fujita, *Uniform K-stability and plt blowups of log Fano pairs*, Kyoto J. Math. **59** (2019), 399–418.

FO18 K. Fujita and Y. Odaka, *On the K-stability of Fano varieties and anticanonical divisors*, Tohoku Math. J. (2) **70** (2018), 511–521.

IM71 V. A. Iskovskih and Ju. I. Manin, *Three-dimensional quartics and counterexamples to the Lüroth problem*, Mat. Sb. (N.S.) **86** (1971), 140–166.

KOW18 I.-K. Kim, T. Okada and J. Won, *Alpha invariants of birationally rigid Fano threefolds*, Int. Math. Res. Not. IMRN **2018** (2018), 2745–2800.

Kol97 J. Kollár, *Singularities of pairs*, in *Algebraic geometry—Santa Cruz 1995*, Proceedings of Symposia in Pure Mathematics, vol. 62 (American Mathematical Society, Providence, RI, 1997), 221–287.

Laz04 R. Lazarsfeld, *Positivity in algebraic geometry I. Classical setting: line bundles and linear series*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge [A Series of Modern Surveys in Mathematics], vol. 48 (Springer, Berlin, 2004).

Li17 C. Li, *K-semistability is equivariant volume minimization*, Duke Math. J. **166** (2017), 3147–3218.

LWX19 C. Li, X. Wang and C. Xu, *On the proper moduli spaces of smoothable Kähler–Einstein Fano varieties*, Duke Math. J. **168** (2019), 1387–1459.

OO13 Y. Odaka and T. Okada, *Birational superrigidity and slope stability of Fano manifolds*, Math. Z. **275** (2013), 1109–1119.

Puk02 A. V. Pukhlikov, *Birationally rigid Fano hypersurfaces*, Izv. Ross. Akad. Nauk Ser. Mat. **66** (2002), 159–186.

Puk16 A. V. Pukhlikov, *Birational geometry of Fano hypersurfaces of index two*, Math. Ann. **366** (2016), 721–782.

Puk17 A. V. Pukhlikov, *Birational geometry of singular Fano hypersurfaces of index two*, Preprint (2017), arXiv:1712.08796.

Tia97 G. Tian, *Kähler–Einstein metrics with positive scalar curvature*, Invent. Math. **130** (1997), 1–37.

Zhu18 Z. Zhuang, *Birational superrigidity and K-stability of Fano complete intersections of index one (with an appendix written jointly with Charlie Stibitz)*, Preprint (2018), arXiv:1802.08389.

Charlie Stibitz cstibitz@math.northwestern.edu
Department of Mathematics, Northwestern University,
Evanston, IL 60208, USA

Ziquan Zhuang zzhuang@math.princeton.edu
Department of Mathematics, Princeton University,
Princeton, NJ 08544-1000, USA

https://doi.org/10.1112/S0010437X19007498 Published online by Cambridge University Press