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THE STABILITY OF SOLUTIONS IN AN INITIAL-BOUNDARY REACTION-DIFFUSION SYSTEM

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We study the asymptotic behaviour as $t \to \infty$ of solutions of the initial-boundary value problem $v_t = G(u, v)$, $u_t = u_{xx} + F(u, v)$, and $t > 0$, $x \in \mathbb{R}$ or $x \in \mathbb{R}^+$ for a wide class of initial and boundary values, where F and G are smooth functions so that the system has three rest points.

1. INTRODUCTION

In this paper we study the system

$$(1.1) \qquad u_t = u_{xx} + F(u, v), \quad v_t = G(u, v), \quad (x, t) \in D$$

where $D = \mathbb{R} \times \mathbb{R}^+$ or $D = \mathbb{R}^+ \times \mathbb{R}^+$, with $G(u, v) = \gamma(u)(k(u) - v)$, $u \geqslant 0$, $v \geqslant 0$ (see [**2**, **6**]) and we assume:

1. There is an interval $[a, b] \subseteq [0, \infty)$ such that $F(u, v)$ is analytic on $[a, b] \times [0, \infty)$, and $\gamma(u)$, $k(u)$ are analytic and positive on $[a, b]$.
2. $F_u < 0$, $F_v < 0$, $G_u < 0$, $G_v < 0 \; \forall (u, v) \in [a, b] \times [0, \infty)$.
3. There exists a function $h(u)$, analytic and positive on $[a, b]$ such that $F(u, v) = 0 \Leftrightarrow v = h(u)$.
4. The equation $h(u) = k(u)$ has exactly three roots, $u_0 < u_1 < u_2$ in (a, b) such that: $h'(u_0) < k'(u_0)$, $h'(u_1) > k'(u_1)$, $h'(u_2) < k(u_2)$.
5. $I(u_2) = \int_{u_0}^{u_2} F(u, k(u))du > 0$.

If $v_i = h(u_i)$, $i = 1, 2, 3$, then, from the above assumptions we have:

(a) $h'(u) < 0$, $k'(u) < 0$ on $[a, b]$;
(b) $(F_u G_v - F_v G_u) |_{(u_i, v_i)} > 0$, $i = 0, 2$;
(c) $F(u, v) > 0$ for $v < h(u)$,
$G(u, v) > 0$ for $v < k(u)$, $\forall u \in (a, b)$.

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This system appears in equations of nerve conduction models, chemical reaction *et cetera* [**2, 3**].

We investigate the asymptotic behaviour, as $t \to \infty$, of solutions of (1.1) under the initial-boundary conditions:

$$u(x, 0) = f(x) \quad v(x, 0) = g(x), \quad x \geqslant 0, \tag{1.3}$$

$$u(0, t) = h(t), \quad t \geqslant 0. \tag{1.4}$$

Since u and v represent chemical concentration, it is natural to impose the conditions

$$u_0 \leqslant f(x), \quad h(t) \leqslant u_2, \quad v_2 \leqslant g(x) \leqslant v_0, \quad x \geqslant 0, \quad t \geqslant 0. \tag{1.5}$$

We are interested in the stability of the equilibrium states (u_i, v_i), $i = 0, 1, 2$, and to show that (u_0, v_0), (u_2, v_2) are stable states while (u_1, v_1) is unstable. So we may expect to have a threshold phenomenon in this case.

The main tool to be used is the following comparison principle: let $N(u, v) =: u_t - u_{xx} - F(u, v)$, $M(u, v) =: v_t - G(u, v)$.

Comparison Principle. (see [**5**]) Let $U(x, t) = (u(x, t), v(x, t))$, $\overline{U}(x, t) = (\overline{u}(x, t), \overline{v}(x, t))$ be bounded and of class C^2 with $N(U) \leqslant 0$, $M(U) \geqslant 0$, $N(\overline{U}) \geqslant 0$ and $M(\overline{U}) \leqslant 0$ on $\mathbb{R}^+ \times \mathbb{R}^+$. If $u(x, 0) \leqslant \overline{u}(x, 0)$, $v(x, 0) \geqslant \overline{v}(x, 0)$ and $u(0, t) \leqslant \overline{u}(0, t)$ $\forall x \in \mathbb{R}^+$, $\forall t \in \mathbb{R}^+$, then

$$u(x, t) \leqslant \overline{u}(x, t) \quad \text{and} \quad \overline{v}(x, t) \leqslant v(x, t), \quad \forall (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

Remark. A similar comparison principle holds for the pure initial value problem (1.1)–(1.2) (see [**5**]).

In Section 2 of this paper we analyse the stability of the rest points for the pure initial value problem, while in Section 3 we study its stability for the initial boundary value problem.

2. Initial value problem

A steady state solution of (1.1) in (a, b) is a solution $(\tau(x), s(x))$ of the equation

$$\tau''(x) + F(\tau(x), k(\tau(x))) = 0 \tag{2.1}$$

where $s(x) = k(\tau(x))$.

We require the following lemma.

LEMMA 1. *Let $(\tau(x), k(\tau(x))) \in [u_0, u_2] \times [v_2, v_0]$ be a steady state solution of (1.1) on (a, b) with $-\infty \leqslant a < b \leqslant +\infty$. If $a > -\infty$ we suppose that $\tau(a) = u_0$, $k(\tau(a)) = v_0$, and if $b < \infty$ suppose that $\tau(b) = u_0$, $k(\tau(b)) = v_0$. If $(w_1(x, t), w_2(x, t))$ is a solution of (1.1) with initial conditions:*

$$w_1(x, 0) = \begin{cases} \tau(x) & x \in (a, b), \\ u_0 & \text{otherwise}, \end{cases}$$
$$w_2(x, 0) = \begin{cases} k(\tau(x)) & x \in (a, b), \\ v_0 & \text{otherwise}, \end{cases}$$

then $w_1(x, t)$ (respectively $w_2(x, t)$) is nondecreasing (nonincreasing) in t, for each x, fixed. Furthermore

$$\lim_{t \to \infty} (w_1(x, t), w_2(x, t)) = (q(x), r(x))$$

uniformly in each x-bounded interval, where $q(x)$ (respectively $r(x)$) is the smallest (biggest) steady state solution of (1.1) in $[u_0, u_2]$ (respectively $[v_2, v_0]$) in the sense that:

$$q(x) \geqslant \tau(x), \quad r(x) \leqslant k(\tau(x)) \quad \text{in } (a, b).$$

PROOF: The proof of this lemma mimics that in [1], for a single equation, and we have omitted it for sake of brevity. ∎

REMARK. Since $I(u_2) > 0$, there exists $K \in [u_1, u_2]$ such that $I(K) = 0$. Moreover $I(q(x)) > 0$ and $I'(q) = F(q, k(q)) > 0$ for $q \in (K, u_2)$. Then for any $\beta \in (K, u_2)$ the solution $q_\beta(x)$ of (2.1) with first integral $q'^2 + 2I(q) = 2I(\beta)$ such that $q(0) = u_0$, $q'(0) = \sqrt{2I(\beta)}$ satisfy: $q_\beta > u_0$ on $(0, b_\beta)$ $q_\beta(0) = q_\beta(b_\beta) = u_0$, and $q_\beta(x) \leqslant q_\beta(b_\beta/2) = \beta$ on $[0, b_\beta]$ where

$$b_\beta = 2\int_{u_0}^{\beta} \{2(I(\beta) - I(u))\}^{-1/2} du.$$

Then with this remark, we can state:

THEOREM 2.1. *Let $(u(x, t), v(x, t))$ be a solution of (1.1) on $\mathbb{R} \times \mathbb{R}^+$ such that $I(u_2) > 0$. If for some $\beta \in (K, u_2)$ and some x_0 so that $u(x, 0) \geqslant q_\beta(x - x_0)$, $v(x, 0) \leqslant k(q_\beta(x - x_0))$ on $(x_0, x_0 + b_\beta)$, then we have*

$$\lim_{t \to \infty} (u(x, t), v(x, t)) = (u_2, v_2).$$

PROOF: Since (u_0, v_0) is a solution of (1.1) and $u(x, 0) \geqslant u_0$ $v(x, 0) \leqslant v_0$, $\forall x \in \mathbb{R}$. Then by the comparison theorem we have that $u(x, t) \geqslant u_0$, $v(x, t) \leqslant v_0$,

$\forall (x, t) \in \mathbb{R} \times \mathbb{R}^+$. Let $u_1(x, t)$, $v_1(x, t)$ be a solution of (1.1) such that:

$$u_1(x, 0) = \begin{cases} q_\beta(x - x_0) & \text{on } (x_0, x_0 + b_\beta) \\ u_0 & \text{otherwise,} \end{cases}$$

$$v_1(x, 0) = \begin{cases} k(q_\beta(x - x_0)) & \text{on } (x_0, x_0 + b_\beta) \\ v_0 & \text{otherwise.} \end{cases}$$

Then by Lemma 1 there exists a stationary solution $(\tau_1(x), s_1(x))$ so that

$$\lim_{t \to \infty} (u_1(x, t), v_1(x, t)) = (\tau_1(x), s_1(x))$$

uniformly in each x-bounded interval, where for some x_0:

$$\tau_1(x) \geqslant q(x - x_0), \quad s_1(x) \leqslant k(q_\beta(x - x_0)) \quad \text{in } (x_0, x_0 + b_\beta).$$

Then by the hypothesis on the initial conditions and the comparison principle we have: $u(x, t) \geqslant u_1(x, t)$ and $v(x, t) \leqslant v_1(x, t)$. On the other hand $u_0 \leqslant u(x, t) \leqslant u_2$, $v_2 \leqslant v(x, t) \leqslant v_0$ so it is sufficient to prove that $\tau_1(x) = u_2$ and $s_1(x) = k(u_2) = v_2$. Let us suppose that $\tau_1(x) < u_2$. Since $\tau_1(x)$ satisfies: $\tau_1'^2(x)/2 + I(\tau_1(x)) = P$ for some constant $P \geqslant I(\beta) > 0$, we may assume that there exists x_1 such that $\tau_1(x_1) = \gamma \in [u_0, u_2)$ so we have:

$$x - x_1 = \mp \int_{\tau_1}^{\gamma} \{2(P - I(u))\}^{-1/2} du$$

where the sign depends on the sign of $\tau_1'(x)$. From this it follows that for finite x^*, $\tau(x^*) = u_0$ with $\tau'(x^*) \neq 0$, hence $\tau_1(x)$ takes values smaller than u_0, which is not possible. Therefore $\tau_1(x) = u_2$ and so

$$\lim_{t \to \infty} (u(x, t), v(x, t)) = (u_2, k(u_2)) = (u_2, v_2).$$

∎

REMARK. In order to study the stability of the equilibrium point (u_0, v_0) and to estimate "how big" the initial condition must be to obtain the stability of this point, we use contracting rectangles for the vector field

$$H(p, r) = (F(p + u_0, v_0 - r), G(p + u_0, v_0 - r))^t,$$

of equation (1.1), in the following sense.

DEFINITION: A bounded convex set $R \subseteq \mathbb{R}^2$ is contracting for the vector field $H(p, r)$ if for any point $(p, r) \in \partial R$ and every outward unit normal $\vec{n}$ at (p, r): $H(p, r) \cdot \vec{n} < 0$.

THEOREM 2.2. *Let $u(x, t)$, $v(x, t)$ be a solution of (1.1) and let R be the rectangle*

$$R = \{(u, v) \mid u_0 - \varepsilon \leqslant u \leqslant u_1 - \varepsilon,\ v^* \leqslant v < v^{**},\ 0 < \varepsilon < u_1 - u_0,\\ v^* = \frac{1}{2}(h + k)(u_1 - \varepsilon),\ v^{**} = \frac{1}{2}(h + k)(u_0 - \varepsilon)\}.$$

If $(u(x, 0), v(x, 0)) = U(x, 0) \in R$, $\forall x \in \mathbb{R}$ and $U(x, 0)$ tend to (u_0, v_0) as $x \to \infty$, then there exist positive constants c, K such that:

$$\|(u(x, t) - u_0, v_0 - v(x, t))\|_\infty \leqslant K e^{-ct}\ \forall t > 0.$$

PROOF: It is easy to check that R is a contracting set for the given vector field and that $\tau R = \{(\tau p, \tau r) \mid (p, r) \in R\}$ is a contraction of R about (u_0, v_0), for any $\tau \in (0, 1]$. Since $U(x, 0) \in R\ \forall x \in \mathbb{R}$, there exists $\tau \in (0, 1]$ such that $U(x, 0) \in \tau R$. If L is the largest side of the rectangle τR then by the basic lemma of Rauch and Smoller [4, Lemma 3.8] there exists $s \in \mathbb{R}^+$ such that the upper Dini derivative satisfies:

$$\overline{D} q_{\tau R}(U(\ ,t)) \leqslant -(s/L) q_{\tau_R}(U(\ ,t)); \quad q_{\tau R}(U(x, 0)) = \tau \leqslant 1.$$

where

$$q_R(p(\ ,t), r(\ ,t)) = \operatorname*{Sup}_{x \in \mathbb{R}} \operatorname{Inf}\{\tau \geqslant 0 \mid (p(x, t), r(x, t)) \in \tau R\}.$$

Then

$$q_{\tau R}((U(\ ,t))) \leqslant e^{-(s/L)t} q_{\tau R}(U(\ ,0)) < K e^{-(s/L)t}$$

and the theorem follows. ∎

REMARKS. (1) Since $\varepsilon > 0$ is arbitrary, we may choose it sufficiently small so that $U(x, 0) \in R$ for all $x \in \mathbb{R}$. Hence letting $\varepsilon \to 0^+$ we see that the initial conditions are bounded by

$$u_0 \leqslant u(x, t) \leqslant u_1, \quad v_1 \leqslant v(x, t) \leqslant v_0.$$

(2) In the same manner we may prove that the steady state (u_2, v_2) is asymptotically exponentially stable with domain of stability given by $u_1 \leqslant u(x, 0) \leqslant u_2$, $v_2 \leqslant v(x, 0) \leqslant v_1$.

(3) From the above, we see that the steady state (u_1, v_1) is unstable, that is, it is a threshold point.

3. Initial-boundary value problem

Let us consider the boundary value problem

$$(3.1) \qquad u_t = u_{xx} + F(u, v), \qquad v_t = G(u, v), \qquad x > 0, t > 0$$

$$(3.2) \qquad u(x, 0) = u_0, \qquad v(x, 0) = v_0, \qquad x > 0$$

$$(3.3) \qquad u(0, t) = h(t) \in [u_0, u_2] \qquad \forall t > 0.$$

An analogous lemma to Lemma 1, reads:

Lemma 2. *Let $(\tau(x), s(x))$ be a stationary solution of (3.1) in (a, b) with $a > 0$ and let $\tau(a) = \tau(b) = u_0$, $s(a) = s(b) = v_0$. Let $(w_1(x, t), w_2(x, t))$ be a solution of (3.1) with initial-boundary conditions*

$$w_1(x, 0) = \begin{cases} \tau(x) & \text{in } (a, b) \\ u_0 & \text{on } \mathbb{R}^+ \setminus (a, b), \end{cases}$$

$$w_2(x, 0) = \begin{cases} s(x) & \text{in } (a, b) \\ v_0 & \text{on } \mathbb{R}^+ \setminus (a, b) \end{cases}$$

$$w_1(x, 0) = \quad \psi(t) \quad \text{on } \mathbb{R}^+.$$

Suppose that $\psi(t)$ in nondecreasing and $\psi(0) = u_0$ with $\psi(t) \in [u_0, u_2]$. Then $w_1(x, t)$ (respectively $w_2(x, t)$) is non-decreasing (non-increasing) in t, for each x fixed. Furthermore

$$\lim_{t \to \infty} (w_1(x, t), w_2(x, t)) = (q(x), r(x))$$

uniformly in each x-bounded interval, where $(q(x), r(x))$ is a steady state solution of (3.1) and they satisfy:

$$q(0) \geqslant \lim_{t \to \infty} \psi(t) \quad \text{and} \quad q(x) \geqslant \tau(x),\ r(x) \leqslant s(x) \quad \text{in } (a, b).$$

Remark. Consider the problem

$$(3.4) \qquad \tau''(x) + F(\tau(x), k(\tau(x))) = 0 \text{ on } \mathbb{R}^+, \quad \tau(0) = \beta.$$

This equation has a unique solution on $[u_0, u_2]$ for each $\beta \in (K, u_2]$, which converges to u_2 as $x \to \infty$, and it has two solutions for $\beta \in [u_0, K)$, one of which converges to u_0 as $x \to \infty$.

Theorem 3.1. *Let $(u(x, t), v(x, t))$ be a solution of (3.1) – (3.3) and let $(p_\beta(x), k(p_\beta(x)))$ be a steady state solution of (3.1) such that $p_\beta(0) = \beta$, $p'_\beta(0) = 0$, for some $\beta \in [K, u_2]$. For any $\beta \in (K, u_2)$ there exist positive numbers a_β and*

t_β such that $p_\beta(\pm a_\beta) = u_0$ and if $h(t) \geqslant \beta$, $t \in (t_1, t_1 + t_\beta)$, some $t_1 > 0$. Then the solution $(u(x, t), v(x, t))$ of (3.1) satisfies $u(x, t_1 + t_\beta) \geqslant p_\beta(x - a_\beta - 1)$, $v(x, t_1 + t_\beta) \leqslant k(p_\beta(x - a_\beta - 1))$, for all $x \in (1, 1 + 2a_\beta)$, and

$$\lim_{x\to\infty} \lim_{t\to\infty} (\operatorname{Inf} u(x, t), \operatorname{Sup} v(x, t)) = (u_2, v_2). \tag{3.6}$$

PROOF: Since (u_0, v_0) is a solution of (1.1)–(1.3) then by the comparison theorem: $u(x, t) \geqslant u_0$, $v(x, t) \leqslant v_0$ on $\mathbb{R}^+ \times \mathbb{R}^+$. Define $s(t)$ as a smooth and nondecreasing function so that $s(t) = u_0$, for $t \in (-\infty, 0)$, and $s(t) = \beta$, for $t \in (1, +\infty)$. Let $(w_1(x, t), w_2(x, t))$ be a solution of (3.1)–(3.3) on $\mathbb{R}^+ \times \mathbb{R}^+$ such that $w_1(0, t) = s(t)$, $t \in \mathbb{R}^+$. Then it is well-known that the solution $(w_1(x, t), w_2(x, t))$ converges, uniformly in x, as $t \to \infty$, to a steady state solution of (3.1)–(3.3), $(q(x), k(q(x)))$ with $q(0) \geqslant \lim_{t\to\infty} s(t) = \beta$. Since $\beta > K$ the problem:

$$q''(x) + F(q(x), k(q(x))) = 0, \quad x \in \mathbb{R}^+, q(0) = \beta. \tag{3.7}$$

has a unique solution $q(x)$ such that $q(x) \to u_2$, $k(q(x)) \to v_2$ as $x \to \infty$. Furthermore, from the phase portrait of (3.7) we learn that there exist a number a_β and a solution $p_\beta(x)$ defined on $(0, a_\beta)$ such that $p_\beta(0) = p_\beta(a) = u_0$ and $p_\beta(x) \leqslant p_\beta(a/2) = \tau$ on $(0, a_\beta)$. Thus $p_\beta(x - 1) < s(x)$ and $k(p_\beta(x - 1)) > k(s(x))$ on $(1, 1 + a_\beta)$. Since the convergence of (w_1, w_2) to $(s(x), k(s(x)))$ is uniform on $[1, 1 + a_\beta]$, there exist a time t_β for which, on $[1, 1 + a_\beta]$, we have:

$$w_1(x, t_\beta) \geqslant p_\beta(x - 1), \quad w_2(x, t_\beta) \leqslant k(p_\beta(x - 1)).$$

Then, by the comparison theorem, we have:

$$u(x, t + t_1) \geqslant w_1(x, t), \quad v(x, t + t_1) \leqslant w_2(x, t) \text{ on } \mathbb{R}^+ \times [0, t_\beta].$$

Therefore $\lim_{t\to\infty} \operatorname{Inf} u(x, t)$ (respectively, $\lim_{t\to\infty} \operatorname{Sup} v(x, t)$) is bounded below (respectively, above) by a stationary solution $s_1(x)$ (respectively, $k(s_1(x))$) of (3.7), such that

$$s_1(x) \geqslant p_\beta(x - 1), \quad k(s_1(x)) \leqslant k(p_\beta(x - 1)) \text{ on } [1, 1 + a_\beta].$$

In particular, $s_1(x + a_\beta/2) \geqslant \beta > K$. Hence $\lim_{x\to\infty} (s_1(x), k(s_1(x))) = (u_2, v_2)$ and the theorem follows. ∎

THEOREM 3.2. *Let $(u(x, t), v(x, t))$ be a solution of (3.1)–(3.3) and let $\beta = \operatorname{Sup} h(t) < K$. Then $u(x, t) \leqslant q_\beta(x)$, $v(x, t) \geqslant p_\beta(x)$. In particular,*

$$\lim_{x\to\infty} \lim_{t\to\infty} (\operatorname{Sup} u(x, t), \operatorname{Inf} v(x, t)) = (u_0, v_0)$$

where $(q_\beta(x), p_\beta(x))$ *is a steady state solution of (3.1).*

PROOF: From the remark we have $(q_\beta(x), p_\beta(x)) \to (u_0, v_0)$ as $x \to \infty$ and the result follows directly from the comparison principle, because $u(0, t) = h(t) < \beta$, $u(x, 0) = u_0 \leqslant q_\beta(x)$, $v(x, 0) = v_0 \geqslant p_\beta(x)$ $\forall x \in \mathbb{R}^+$. ($\beta < K$ implies $u_0 < q_\beta(x) < K$). ∎

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