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On the Width of Lattice-Free Simplices

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Abstract. We consider lattice-free simplices, simplices with vertices on the lattice $\mathbb{Z}^d$ in $\mathbb{R}^d$ and *no* other integral points; we show, by elementary methods, that there exist such simplices in dimension d with width (see Definition 2) going to infinity with d.

Key words: Lattice, lattice-free (empty) polytopes, polytopes, simplices, width.

1. Introduction

Integral polytopes (see [Br, K] for the basic definitions) are of interest in combinatorics, linear programming, algebraic geometry-toric varieties [D,O], number theory [K-L.].

We study here lattice-free simplices, i.e., simplices intersecting the lattice only at their vertices.

A natural question is to measure the 'flatness' of these polytopes, with respect to integral dual vectors. This (arithmetical) notion plays a crucial role in:

- the classification (up to affine unimodular maps) of lattice-free simplices in dimension 3 (see [O,MMM]) and
- the construction of a polynomial-time algorithm for integral linear programming (flatness permits induction on the dimension [K-L]).

Unfortunately, there were no known examples (in any dimension) of lattice-free polytopes with width greater than 2. We prove here the following theorem:

THEOREM. *Given any positive number α strictly inferior to $1/e$, for d large enough, there exists a lattice-free simplex of dimension d and width superior to αd.*

The proof is nonconstructive and involves replacing the search for lattice-free simplices in $\mathbb{Z}^d$ by the search for 'lattice-free lattices' containing $\mathbb{Z}^d$ ('turning the problem inside out', see Section 1.3), specializing in the next step to lattices of a simple kind, depending on a prime number p. The existence of lattice-free simplices of large width is then deduced by elementary computations, through a

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sufficient inequality involving the dimension d, the width k and the prime p (see (12)).

2. Notations

$\mathcal{P}_d$: The set of integral polytopes in $\mathbb{R}^d$; if P is such a polytope, P is a convex compact set, the set $\mathrm{Vert}(P)$ of vertices of P is a subset of $\mathbb{Z}^d$.

$\mathcal{S}_d$: The set of integral simplices in $\mathbb{R}^d$. In particular, σ_d will denote the canonical simplex with vertices at the origin and $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) - 1$ at the ith coordinate

G_d: The group of affine unimodular maps $G_d = \mathbb{Z}^d \rtimes \mathrm{GL}(d, \mathbb{Z})$ acts on $\mathbb{R}^d$ (preserving $\mathbb{Z}^d$), $\mathcal{P}_d$, and $\mathcal{S}_d$. A d-lattice M is a lattice with $\mathbb{Z}^d \subset M \subset (1/m)\mathbb{Z}^d$ for some $m \in \mathbb{N}^\star$.

2.1. LATTICE-FREE SIMPLICES AND THEIR WIDTH

Recall the following definition [K].

DEFINITION 1. An integral polytope P in $\mathbb{R}^d$ is *lattice-free* if $P \cap \mathbb{Z}^d = \mathrm{Vert}(P)$.

DEFINITION 2. Given an integral nonzero vector u in $(\mathbb{Z}^d)^\star$, the u-width of the polytope P of $\mathcal{P}_d$ is defined by

$$w_u(P) = \max_{x,y \in P} \langle u, x - y \rangle. \tag{1}$$

The *width of P* is

$$w(P) = \min_{\substack{u \in (\mathbb{Z}^d)^\star \\ u \neq 0}} w_u(P). \tag{2}$$

Remark. The width is the minimal length of all integral projections $u(P)$ for nonzero u.

2.2. KNOWN RESULTS ON THE WIDTH OF LATTICE-FREE POLYTOPES IN DIMENSION d

$d = 2$: Lattice-free simplices are all integral triangles of area $\frac{1}{2}$; they are equivalent to σ_2. This is elementary.

$d = 3$: Lattice-free polytopes have width one; in the case of simplices, this result has various proofs and applications (it is sometimes known as the 'terminal lemma', see [F, MS, O, Wh]).

$d = 4$: All lattice-free simplices have at least one basic facet (face with codimension one) [W] – this fact is not true in higher dimensions.

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EXAMPLES. There exist some interesting examples:

- L. Schlafli's polytope, studied by Coxeter [C];
- A recent example given by H. Scarf [private communication]: the simplex in dimension 5 with vertices, the first five unit vectors e_i and for last vertex $(23, 29, 31, 43, 57)$, has width 3.
- We have found with the help of a computer, some examples of widths 2, 3 and 4 in dimensions 4 and 5.

No other results seem to be known, apart from the following asymptotic result.

PROPOSITION 1. *There exists a universal constant C such that for any lattice-free polytope of dimension d*

$$w(P) \leqslant Cd^2. \tag{3}$$

Proof. The 'Flatness Theorem' of [K-L] asserts that there exists C such that any convex compact set K in $\mathbb{R}^d$ with $K \cap \mathbb{Z}^d = \phi$ satisfies

$$w(K) \leqslant C\, d^2, \tag{4}$$

where w is defined as in 2.2.

If P is any lattice-free polytope, take a point a in the relative interior of P and apply the previous Flatness Theorem to the homothetic $\widetilde{P}$ of P with respect to a and fixed ratio α strictly less than one. Then formula (4) shows that the width of P, which is proportional to the width of $\widetilde{P}$, is also bounded by a function of type (6).

Remark. Recent results of [Ba] show that (3) is true with a right-hand side proportional to $d \log d$.

2.3. TURNING THE WIDTH INSIDE OUT

Let us define a new norm on $\mathbb{R}^d$: If $\xi = (\xi_i)$ is a vector in $\mathbb{R}^d$, take $\|\xi\| = \max_i(0, \xi_i) - \min_i(0, \xi_i)$.

DEFINITION 3. Let

$$w(M) = \min_{\substack{\xi \in M^\star \\ \xi \neq 0}} \|\xi\|. \tag{5}$$

It is easy to show that the existence of an integral lattice-free simplex of dimension d, volume $v/d!$ and width at least k is equivalent with the existence of a d-lattice M, containing $\mathbb{Z}^d$, with

$$M \cap \sigma_d = \operatorname{Vert} \sigma_d, \qquad w(M) \geqslant k, \qquad \det(M) = \frac{1}{v}. \tag{6}$$

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3. In Search of Lattice-Free Simplices (Asymptotically)

3.1.

We restrict our study to d-lattices given by

$$y \in \mathbb{Z}^d, \qquad M(y) = \mathbb{Z}^d + \mathbb{Z}\frac{1}{p}y, \qquad M(y) \neq \mathbb{Z}^d, \tag{7}$$

where p is a prime number; this lattice clearly depends only on the class of y in $(\mathbb{Z}/p\mathbb{Z})^d$.

LEMMA 1. *The set of lattices M (for a fixed p) can be identified with the space of lines in $(\mathbb{Z}/p\mathbb{Z})^d$.*

In particular, the number of such lattices is

$$m(d, p) = \frac{p^d - 1}{p - 1}. \tag{8}$$

Let $f(d, p)$ be the number of lattices M such as (7) satisfying

$$M \cap \check{\sigma}_d \neq \phi, \tag{9}$$

where $\check{\sigma}_d = \sigma_d \backslash \mathrm{Vert}(\sigma_d)$.
(The lattice M intersects σ_d in other points than the vertices.)

LEMMA 2. *The number $f(d, p)$ satisfies*

$$f(d, p) \leqslant \frac{(p+1)\cdots(p+d)}{d!} - (d+1).$$

Proof. Suppose x is a point in $\mathbb{M}(y)$ belonging to $\check{\sigma}_d$. Then it can be written as $x = z + my/p$ with m nondivisible by p.

Writing my/p as the sum of an integral vector and a remainder, we get

$$x = z + z' + \frac{\tilde{y}}{p}, \quad 0 \leqslant \tilde{y}_i < p, \quad \tilde{y}_i \in \mathbb{N}, \quad x \in \check{\sigma}_d.$$

This implies

$$z + z' = 0, \qquad x = \frac{\tilde{y}}{p}, \qquad \tilde{y} \in p\check{\sigma}_d \cap \mathbb{Z}^d.$$

The vectors $y, my, \tilde{y}$ define the same line in $(\mathbb{Z}/p\mathbb{Z})^d$. This shows that the number of lattices $M(y)$ satisfying (9) is less than the number of points in $p\check{\sigma}_d \cap \mathbb{Z}^d$, given by the right-hand side of Lemma 2 [E].

Now let $g(d, p, k)$ be the number of lattices $M(y)$, as in (7), with $w(M(y)) \leqslant k$.

LEMMA 3. *The number* $g(d, p, k)$ *satisfies*

$$g(p, d, k) \leqslant 2[(k+1)^{d+1} - k^{d+1}]p^{d-2}. \tag{10}$$

Proof. The assumption on the lattice means the existence of a nonzero vector ξ in $\mathbb{Z}^d$ with

$$y = (y_1, \dots y_d), \qquad \xi = (\xi_1, \dots \xi_d), \quad \sum \xi_i y_i \in p\mathbb{Z}$$

and we have $\|\xi\| \leqslant k \Rightarrow \|\xi\|_\infty \leqslant k$.

The number of integral points ξ of norm less or equal to k is $n(k, d) = (k+1)^{d+1} - k^{d+1}$.

Proof. Let $m = \inf_i(0, \xi_i)$, $M = \sup_i(0, \xi_i)$.

The possible values of m are $m = -k, \dots -1, 0$.

(a) For all values except 0, one of the x_i has value m, and the others can take any value between m and $m+k$. For each m, the number of possibilities is equal to $S_1 = [k+1]^d - k^d$.

(b) When $m = 0$, all $x_i's$ are nonnegative, and the contribution is $S_2 = [k+1]^d$.

Adding up the contributions, we get

$$n(k, d) = k[(k+1)^d - k^d] + (k+1)^d = (k+1)^{d+1} - k^{d+1}].$$

Going back to the proof of Lemma 3, choose a vector ξ with norm smaller than k (strictly less than p): this implies that the linear form defined by ξ $\hat{\xi}$: $(\mathbb{Z}/p\mathbb{Z})^d \rightarrow \mathbb{Z}/p\mathbb{Z}$ is surjective, and its kernel has p^{d-1} elements; the number of corresponding lattices is

$$r(p, d) = \frac{p^{d-1} - 1}{p - 1} \leqslant 2p^{d-2}.$$

We can choose at most $n(k, d)$ vectors ξ. Hence

$$g(p, d, k) \leqslant 2[(k+1)^{d+1} - k^{d+1}]p^{d-2} \leqslant 2(k+1)^{d+1}p^{d-2}. \tag{11}$$

3.2.

From Lemmas 2 and 3 we conclude that for large d and k, the condition

$$2(d+1)(k+1)^d p^{d-2} + \frac{(p+d)^d}{d!} < p^{d-1} \tag{12}$$

ensures the existence of a lattice $M(y)$ of width greater than k, dimension d, and $M(y) \subset (1/p)\mathbb{Z}^d$.

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The following is well known.

LEMMA 4. *Given any sequence of numbers* (a_d) *going to infinity, there exists an equivalent sequence* (p_d) *of prime numbers*.

Proof. Given ε strictly positive, we know from the prime number theorem that for d large enough there exists a prime number p_d in the interval $[(1-\varepsilon)a_d, (1+\varepsilon)a_d]$. This implies $| p_d - a_d | < \varepsilon a_d$ for d large enough.

Choose now α arbitrary (we will soon fix it) and a sequence (p_d) of primes with $p_d \sim \alpha d!$ and let us find α and a sequence (k_d) such that

$$2(d+1)(k_d+1)^d p_d^{d-2} < \tfrac{1}{2} p_d^{d-1}, \tag{13}$$

$$\frac{(p_d+d)^d}{d!} < \tfrac{1}{2} p_d^{d-1}. \tag{14}$$

These two conditions imply (12).

The condition (14) is satisfied for large enough d if $\alpha < \frac{1}{2}$. Indeed $p_d{+}d \sim \alpha d!$; since $\alpha < \frac{1}{2}$.

Condition (14) follows if we can show that $(1 + d/p_d)^{d-1} \to 1 (d \to \infty)$. But

$$\log(1+d/p)^{d-1} \leqslant (d-1)d/p \sim d^2/\alpha d! \to 0.$$

Condition (13) becomes

$$k_d + 1 < \left[\frac{1}{4(d+1)} p_d\right]^{1/d+1}.$$

This last expression is equivalent, because of Stirling's formula, to d/e. Hence, if we choose any sequence of integral numbers (k_d) with $k_d < \alpha d$ and

$$0 < \alpha < \frac{1}{e} \tag{15}$$

then (13) and (14) are satisfied for large d.

THEOREM. *For any* α *strictly less than* $1/e$, *there exists for sufficiently large* d *a sequence of lattice-free simplices of dimension* d *and width* w_d, $w_d > \alpha d$.

Defining $w(d) = \sup_\sigma w(\sigma)$ supremum taken over all lattice-free simplices of dimension d, then the previous Theorem amounts to

$$\lim_{d\to\infty} \frac{w(d)}{d} \geqslant \frac{1}{e}.$$

Final Remark. The study above raises the hope of improving the bounds on the maximal width, by introducing more general lattices generated by a finite number

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of rational vectors, and replacing the prime p by powers in (7) (Note the study of general lattices of such type in [Sh].) Unfortunately (and rather mysteriously), our computations in these new cases give the *same* bounds.

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