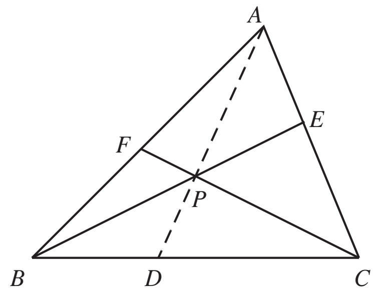

FIGURE 5

Since a triangle has many centres, the paragraph above can be a source for composing problems of the olympiad type. This line of research is pursued in [4] and also in [5].

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107.12 The Steiner-Lehmus theorem à la Euclid

The Steiner-Lehmus theorem states that if ABC is a triangle in which BE and CF are the (internal) angle bisectors of angles B and C respectively, and if $BE = CF$, then $AB = AC$; see Figure 1. Proofs of this abound in the literature, and many of these prove the stronger statement

$$AB > AC \Rightarrow BE > CF. \tag{1}$$

This Note provides yet another proof of (1) which is short and has the advantage of being closest to Euclidean, in the sense that Euclid himself would have found a place for it somewhere in his *Elements*, namely in Book VI. The proof also meets the standards of being purely geometric that were imposed by Professor C. L. Lehmus when he first posed the problem in 1840.

There are two statements from the *Elements* which will play a key role in the proof to be described. Proposition 18 of Book I states that if $AB > AC$ then $\angle C > \angle B$; this is to be found in [1]. Proposition 24, also from Book I, is called the *open mouth theorem* in [2, Theorem 6.3.9, p. 14], the *scissors lemma* in [3, p163] and sometimes the *hinge lemma*. It says that

If UVW and $U'V'W'$ are triangles in which $UV = U'V'$, $UW = U'W'$ and $\angle U > \angle U'$, then $VW > V'W'$.

This result will play the decisive role in the proof of the Steiner-Lehmus theorem.

Starting with a triangle ABC with $AB > AC$, and with angle bisectors BE and CF, as shown in Figure 1, we draw from E and F lines parallel to BC that meet AB and AC at E' and F' respectively. The Figure shows that F' lies between E and A (and hence $FF' < EE'$), but this is something that we will prove below, and we shall not use it until then.

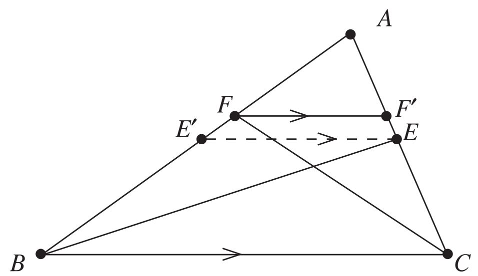

FIGURE 1

Observe, by simple angle-chasing, that the triangles BEE' and CFF' are both isosceles. Using similarity, the angle bisector theorem and the assumption $AC < AB$, we obtain

$$\frac{AF'}{F'C} = \frac{AF}{FB} = \frac{AC}{CB} < \frac{AB}{CB} = \frac{AE}{EC},$$

and so, adding 1 to the first and last expressions, $\dfrac{AC}{F'C} < \dfrac{AC}{EC}$ and hence $F'C > EC$. Thus F' lies between E and A, and hence $FF' < EE'$ as claimed earlier.

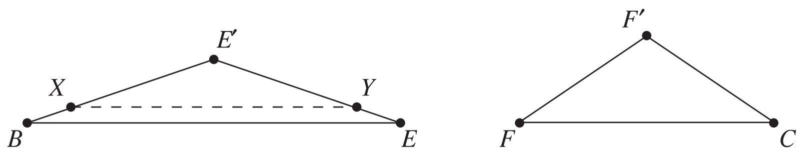

FIGURE 2

Figure 2 shows the isosceles triangles $EE'B$ and $FF'C$. Since $F'F < E'E$ it follows that there are points X and Y on BE' and EE' respectively such that $E'E = E'B > E'X = EY = F'F = F'C$. Thus we

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have $F'F = E'X$, $F'C = E'Y$ and $\angle F' < \angle E'$ and so, by the open mouth theorem, it follows that $BE > XY > CF$ as required.

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107.13 An interesting generator of Archimedean circles

A very simple but, we think, very hard to prove, Proposition 1 for Archimedean circles (see [1, 2, 3]) led us to an interesting generalisation, and unexpectedly not so hard to prove, Proposition 2.

Proposition 1: From a point P on a circle with centre O and diameter AB we drop the perpendicular PC to AB such that $AC = 2a$, $CB = 2b$. From P we draw the tangents PQ, PR, to these circles and the perpendicular from O meets the line CP at the point X. The symmetric circles relative to XO, namely $I(r)$ and $J(r)$ that are tangent to the line QR and internally tangent to the circles with diameters AB, XO are Archimedean circles i.e.

$$r = \frac{ab}{a + b}.$$

If S is the external centre of similitude (Figure 1) of the circles with diameters AC, CB then the inversion with pole S and power $SA \cdot SB = SC^2$ transforms the circles with diameters AC, CB and maps to itself the circle $P(PC)$ that passes through Q, R. Hence the inverse of Q lies on the circle $P(PC)$ and the circle with diameter CB and hence this point is R, which means that the line QR passes through S (Figure 1).