

IRREDUCIBLE SUBGROUPS OF SYMPLECTIC GROUPS IN CHARACTERISTIC 2

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Abstract

Suppose that V is a finite dimensional vector space over a finite field of characteristic 2, G is the symplectic group on V and a is a non-zero vector of V . Here we classify irreducible subgroups of G containing a certain subgroup of $O_2(\text{Stab}_G(a))$ all of whose non-trivial elements are 2-transvections.

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1. Introduction

Let k denote the Galois field $\text{GF}(q)$, where $q = 2^m$, and suppose V is a finite dimensional vector space over k . An involution g of $\text{GL}(V)$ is a transvection (respectively a 2-transvection) of V if $C_V(g)$ has codimension 1 (respectively 2) in V . A subgroup K of $\text{GL}(V)$ is called a transvection subgroup if $C_V(K)$ has codimension 1, $[V, K]$ has dimension 1, and K is isomorphic to the additive group of k . Assume, additionally, that $\dim V = 2n$ where $n \geq 2$ and f is a non-degenerate alternating bilinear form on V . Letting G denote $\text{Sp}(V)$, the symplectic group on V defined by f , we may now state our main result.

THEOREM 1.1. *Let a be a non-zero vector of V and put $H = \text{Stab}_G(a)$. Suppose that X is a subgroup of $O_2(H)$ which satisfies*

- (i) $|X| = q^{2n-2}$; and
- (ii) *no element of X acts as a transvection on V .*

If L is a subgroup of G which contains X and acts irreducibly on V , then L acts naturally on V as one of $\mathrm{Sp}(V)$, $\Omega^\pm(V)$, $O^\pm(V)$ or $q = n = 2$ and $L \cong \mathrm{Alt}(6) \cong \mathrm{Sp}_4(2)'$ or $\Gamma \mathrm{L}(2, 4) \cong \mathrm{Sym}(5)$.

We remark that the listed groups all satisfy the hypothesis of Theorem 1.1. The exceptional cases when $q = n = 2$ are caused by the non-simplicity of $\mathrm{Sp}_4(2) \cong \mathrm{Sym}(6)$ in the $\mathrm{Alt}(6)$ case and by the fact that $\mathrm{Sym}(6)$ has two classes of subgroups isomorphic to $\mathrm{Sym}(5)$ in the second case. To clarify the second case further take V to be the doubly deleted $\mathrm{GF}(2)$ -permutation module for $\mathrm{Sym}(6)$, the point stabilizer $\mathrm{Sym}(5)$ acts on V as $O_4^-(2)$, while the transitive $\mathrm{Sym}(5)$ when restricted to $\mathrm{Alt}(5)$ acts on V as $\mathrm{SL}_2(4)$.

Observe that, as the non-trivial elements of the elementary abelian 2-group $O_2(H)$ are either transvections or 2-transvections, every non-trivial element of X must act as a 2-transvection on V . So Theorem 1.1 may be seen as a kindred spirit to the results of McLaughlin [3, 4], on irreducible linear groups which contain transvection subgroups and also to work of Dempwolff's [1, 2]. In [1, 2] Dempwolff shows that an irreducible subgroup Y of $\mathrm{SL}(V)$ which is generated by 2-transvections either contains transvections, has a normal abelian subgroup of odd order which has 1-dimensional homogeneous components on V which Y permutes transitively, contains a normal complex of Stellmacher elements, or a normal complex of roots involutions.

Theorem 1.1 generalizes a result due to Timmesfeld for the case $q = 2$, which plays an important part in the proof of his Theorem 4.5 [9]. This more general theorem plays an equally vital role in the classification of symplectic amalgams [5]. The proof given here is modelled on Timmesfeld's approach. We remark that Theorem 1.1 is also the principal content of [7, Satz 3.10]. However, the proof there calls upon the (lengthy) classification of groups generated by root involutions and then deals with the resulting configurations case by case. Our proof is direct and elementary—the only substantial results we use being the classification of groups generated by transvections due to McLaughlin [3, 4].

Our notation follows that of [8]. The following elementary result will be used in the proof of Theorem 1.1.

LEMMA 1.2. *Let V be a vector space of dimension $2n$, $n \geq 1$ which is equipped with a non-degenerate alternating bilinear form f . Put $G = \mathrm{Sp}(V)$. If a and b are non-zero vectors of V with $f(a, b) \neq 0$, then*

$$G = \langle O_2(\mathrm{Stab}_G(a)), O_2(\mathrm{Stab}_G(b)) \rangle.$$

PROOF. See [6].

□

2. Orbits on points and vectors and a quadratic form

From now on we assume the situation depicted in Theorem 1.1. If L contains a transvection subgroup, then we may avail ourselves of McLaughlin's results ([3, 4]) to conclude, since $L \leq \text{Sp}(V)$, that L must be isomorphic to one of $\text{Sp}(V)$, $O^\pm(V)$, $\text{Sym}(2n + 1)$, or $\text{Sym}(2n + 2)$. Of these groups only $\text{Sp}(V)$ and $O^\pm(V)$ can contain an elementary abelian 2-group as large as X and so Theorem 1.1 holds in this case. Henceforth, therefore, we shall suppose that L contains no transvection subgroups.

LEMMA 2.1. (i) $[a^\perp, X] = \langle a \rangle$;

(ii) $[V, X] = a^\perp$;

(iii) $C_V(X) = \langle a \rangle$; and

(iv) Let Z be a subgroup of index q in X and let $U = \langle a, c \rangle$ be a 2-dimensional subspace of a^\perp . If Z centralizes U , then $C_V(Z) = U$.

PROOF. Since $O_2(H) \geq X$, $\langle a \rangle = [a^\perp, O_2(H)] \geq [a^\perp, X]$ and part (i) follows as X contains no transvections. Now suppose that $[V, X] < a^\perp$. Then $|[V, X]| \leq q^{2n-2}$. So, letting $v \in V \setminus a^\perp$, $\mathcal{V} = \{[v, x] \mid x \in X\}$ contains at most q^{2n-2} vectors. If $|\mathcal{V}| = q^{2n-2}$, then $\mathcal{V} = [V, X]$ and hence, by part (i), $[v, x] = a$, for some $x \in X$. But then $[V, x] \leq \langle [a^\perp, x], [v, x] \rangle \leq \langle a \rangle$, which implies that x is a transvection on V , a contradiction. Thus $|\mathcal{V}| < q^{2n-2}$. Since $|X| = q^{2n-2}$, there must exist distinct $x_1, x_2 \in X$ with $[v, x_1] = [v, x_2]$, whence $[v, x_1x_2] = 0$. This then implies that x_1x_2 is a transvection on V . Hence we infer that $[V, X] = a^\perp$, and we have (ii). Using part (ii) gives $C_V(X) = [V, X]^\perp = (a^\perp)^\perp = \langle a \rangle$, so proving (iii).

For part (iv), since Z centralizes a and c , we have $[V, Z] \leq a^\perp \cap c^\perp = U^\perp$ with $|U^\perp| = q^{2n-2}$. Now, as $|Z| = q^{2n-3}$, we may argue as in part (ii) to obtain $[V, Z] = U^\perp$. Hence $C_V(Z) = [V, Z]^\perp = U$. □

By hypothesis L acts irreducibly upon V and so $\langle a^\perp \rangle = V$. Thus we can find an image $\langle b \rangle$ of $\langle a \rangle$ (under L) with $b \in V \setminus a^\perp$. Select the vector representative b of $\langle b \rangle$ so as $f(a, b) = 1$. Then, for $\lambda \in k$ we put $\mathcal{O}_\lambda = \{(b^x + \lambda a) \mid x \in X\}$, and note that each \mathcal{O}_λ is an X -orbit which consists of points (1-spaces of V) in $V \setminus a^\perp$.

LEMMA 2.2. X has q regular orbits on the points in $V \setminus a^\perp$. These X -orbits are \mathcal{O}_λ , $\lambda \in k$.

PROOF. Suppose $\langle v \rangle$ is a point of $V \setminus a^\perp$ which is fixed by $x \in X^\#$. Then, by Lemma 2.1 (ii), $[\langle v \rangle, x] \leq [V, X] \cap \langle v \rangle = a^\perp \cap \langle v \rangle = 0$. Hence

$$[V, x] = [a^\perp + \langle v \rangle, x] = [a^\perp, x] \leq \langle a \rangle,$$

a contradiction as X contains no transvections. Therefore the X -orbits of points of V not contained in a^\perp are regular. So, by counting, we see that X has exactly q orbits on the points in $V \setminus a^\perp$. To complete the proof of the lemma we must show that for $\lambda, \mu \in k$, $\mathcal{O}_\lambda = \mathcal{O}_\mu$ implies that $\lambda = \mu$. Let $\lambda, \mu \in k$ be such that $\mathcal{O}_\lambda = \mathcal{O}_\mu$. Then $b + \lambda a = b^x + \mu a$ for some $x \in X$. So $[b, x] = b + b^x = (\mu + \lambda)a \in \langle a \rangle$ and consequently $[V, x] \leq \langle a \rangle$. Since no element of $X^\#$ is a transvection on V , we must have $x = 1$ and then $\lambda a = \mu a$. Hence $\lambda = \mu$. \square

For each $x \in X^\#$ we define a 2-dimensional subspace T_x of a^\perp by

$$T_x = [V, x] = \langle a, [b, x] \rangle = \langle a \rangle + \langle [b, x] \rangle.$$

LEMMA 2.3. For $x \in X^\#$ we have $T_x \cap b^\perp = \langle [b, x] + f(b, b^x)a \rangle$.

PROOF. Since $\dim b^\perp = 2n - 1$ and $\langle a \rangle \not\leq b^\perp$, $\dim(T_x \cap b^\perp) = 1$. Now

$$\begin{aligned} f(b, [b, x] + f(b, b^x)a) &= f(b, b + b^x) + f(b, b^x)f(b, a) \\ &= f(b, b) + f(b, b^x) + f(b, b^x) = 0, \end{aligned}$$

and so $[b, x] + f(b, b^x)a \in T_x \cap b^\perp$. Thus, as $[b, x] + f(b, b^x)a \neq 0$, Lemma 2.3 holds. \square

LEMMA 2.4. Suppose T is a 2-dimensional subspace of a^\perp which contains $\langle a \rangle$. Then

- (i) there exists $x \in X^\#$ such that $T_x = T$; and
- (ii) $X_T := \{x \in X^\# \mid T_x = T\} \cup \{1\}$ is a subgroup of X of order q .

PROOF. Let T be a 2-dimensional subspace of a^\perp containing $\langle a \rangle$, and suppose that $\{x \in X^\# \mid T_x = T\}$ has at least q distinct elements. Then, since $\dim(T \cap b^\perp) = 1$, Lemma 2.3 implies there exists $x_1, x_2 \in X^\#$ with $x_1 \neq x_2$ such that

$$[b, x_1] + f(b, b^{x_1})a = [b, x_2] + f(b, b^{x_2})a.$$

So $b + b^{x_1} + b + b^{x_2} = (f(b, b^{x_1}) + f(b, b^{x_2}))a$, whence

$$b + b^{x_1 x_2^{-1}} = (f(b, b^{x_1}) + f(b, b^{x_2}))a^{x_2^{-1}} = (f(b, b^{x_1}) + f(b, b^{x_2}))a \in \langle a \rangle.$$

This forces $x_1 x_2^{-1}$ to be a transvection of V and so we conclude that

$$|\{x \in X^\# \mid T_x = T\}| \leq q - 1.$$

So there must be at least $(|X| - 1)/(q - 1) = (q^{2n-2} - 1)/(q - 1)$ 2-dimensional subspaces of V of the form T_x , for some $x \in X^\#$. However, this is also the number of 2-dimensional subspaces of a^\perp which contain $\langle a \rangle$ and so (i) holds. Moreover, $|\{x \in X^\# \mid T_x = T\}| = q - 1$. Since, for $x_1, x_2 \in X_T$, $[b, x_1 x_2] = [b, x_1]^{x_2} + [b, x_2] \in T$ we infer that X_T is a subgroup of X of order q . \square

LEMMA 2.5. *If T is a 2-dimensional subspace of a^\perp containing $\langle a \rangle$, then*

- (i) $|X/C_X(T)| = q$; and
- (ii) X has two orbits on the points of T , with one of length q and one consisting of $\langle a \rangle$.

PROOF. By definition $T = T_x$ for all $x \in X^\#$. Thus $C_V(x) = [V, x]^\perp = T_x^\perp = T^\perp$. Hence $C_V(X_T) = T^\perp$. Select a 2-dimensional subspace U of a^\perp containing $\langle a \rangle$ so as $U + T^\perp = a^\perp$. Since $T^\perp \neq a^\perp$, $U^\perp \cap T = \langle a \rangle$. Because $C_V(X_U) = U^\perp$ we then deduce that $C_T(X_U) = \langle a \rangle$ and therefore X_U has orbits of size 1 and q on the points of T . Since $X/C_X(T)$ embeds into $GL_2(q)$, this gives the lemma. □

LEMMA 2.6. *Suppose that $\langle c \rangle \leq a^\perp$ is in the same L -orbit as $\langle a \rangle$ and $\langle c \rangle \neq \langle a \rangle$. Then*

- (i) *the non-zero vectors of $\langle a, c \rangle$ are all in the same L -orbit; and*
- (ii) $N_L(\langle a \rangle)/C_L(\langle a \rangle)$ has order $q - 1$.

PROOF. Put $T = \langle a, c \rangle$. By assumption $\langle c \rangle = \langle a \rangle^g$ for some $g \in L$. Applying Lemma 2.5 (i) to the pair $T, \langle c \rangle$ gives $|X^g/C_{X^g}(T)| = q$. Since X and X^g act differently on T , we deduce that $\langle X, X^g \rangle/C_{\langle X, X^g \rangle}(T) \cong SL_2(q)$, and this gives the result. □

An immediate consequence of Lemma 2.6 is

LEMMA 2.7. *If L is transitive on the points of V , then L is transitive on $V^\#$.*

LEMMA 2.8. *Suppose that L is not transitive on the points of V . Then there exists a 2-dimensional subspace T of a^\perp containing $\langle a \rangle$ such that $\langle a \rangle^\perp \cap T = \{\langle a \rangle\}$. Moreover, for $U = \langle b, T \cap b^\perp \rangle$, $N_X(U) = 1$.*

PROOF. Suppose no such 2-dimensional subspace of a^\perp exists. Then all points in a^\perp are in $\langle a \rangle^\perp$ by Lemma 2.6. Let $\langle d \rangle$ be a point of V not in a^\perp . Since $\dim V \geq 4$, there exists $0 \neq c \in a^\perp \cap d^\perp$ whence $\langle c \rangle \in \langle a \rangle^\perp$. And then $\langle d \rangle \in \langle a \rangle^\perp$. So L is transitive on the points of V , a contradiction.

We now let T be such a 2-dimensional subspace and prove that, for $U = \langle b, T \cap b^\perp \rangle$, $N_X(U) = 1$. Let $1 \neq x \in X$. We claim that $[b, x] \notin T \cap b^\perp$. For if $[b, x] \in T \cap b^\perp$, then $\langle b \rangle \leq \langle [b, x], b \rangle = \langle b^x, b \rangle \leq b^\perp$. Since $\langle b^x, b \rangle$ is a 2-dimensional subspace by Lemma 2.1 (ii), we may apply Lemma 2.6 (with b in place of a) to conclude that $\langle [b, x] \rangle \in \langle b \rangle^\perp$. Hence, as $\langle a \rangle$ and $\langle b \rangle$ are in the same L -orbit, $\langle [b, x] \rangle \in \langle a \rangle^\perp \cap T = \{\langle a \rangle\}$, a contradiction which establishes the claim. Since $b \notin T$, U is a 2-dimensional subspace and $U \cap a^\perp = T \cap b^\perp$. If there exists $1 \neq x \in N_X(U)$, then, since $\langle b \rangle^X$ is a regular X -orbit, $\langle b, b^x \rangle = U$ and consequently $[b, x] \in U \cap a^\perp = T \cap b^\perp$, contrary to $[b, x] \notin T \cap b^\perp$. Therefore, $N_X(U) = 1$. □

LEMMA 2.9. *Either L is transitive on the points of V or L has two orbits on the points of V . In the latter case we have*

$$\langle a \rangle^L \setminus a^\perp = \langle b \rangle^X = \mathcal{O}_0 \quad \text{and} \quad \langle a + b \rangle^L \setminus a^\perp = \bigcup_{\lambda \in k^*} \mathcal{O}_\lambda.$$

PROOF. Suppose that L is not transitive on the points of V . Then, by Lemma 2.8, there exists a 2-dimensional subspace T with $\langle a \rangle \leq T \leq a^\perp$ and $\langle a \rangle^L \cap T = \{\langle a \rangle\}$. Setting $U = \langle b, T \cap b^\perp \rangle$ we also have $N_X(U) = 1$ by Lemma 2.8. Since the elements of $X^{\#}$ all have order 2, it follows that no two points of $U \setminus a^\perp$ are in the same X -orbit. Hence the q points of $U \setminus a^\perp$ may be chosen as representative of the X -orbits on $V \setminus a^\perp$. Let $g \in L$ be such that $\langle a \rangle^g = \langle b \rangle$, and set $Y = X^g$. Then, by Lemma 2.5 (ii), Y has orbits of length 1 and q on the points of U and so, as $\langle b \rangle^Y = \{\langle b \rangle\}$, $\bigcup_{\lambda \in k^*} \mathcal{O}_\lambda$ is contained in an L -orbit.

For $\langle c \rangle \leq a^\perp$, if $\langle c \rangle \notin \langle a \rangle^L$, then we may use the above argument with $\langle a, c \rangle$ in place of T to show that $\langle c \rangle$ is in the same L -orbit as $\bigcup_{\lambda \in k^*} \mathcal{O}_\lambda$. Thus L has two orbits on the points of V with $\langle a \rangle^L \setminus a^\perp = \mathcal{O}_0$ and $\langle a + b \rangle^L \setminus a^\perp = \bigcup_{\lambda \in k^*} \mathcal{O}_\lambda$, and the lemma is proven. □

So far we have been dealing with points of V . The next lemma tells us about the orbits of L on the vectors of V . As will shortly be seen this plays an important role in the remainder of the proof of Theorem 1.1.

LEMMA 2.10. *Suppose that L is not transitive on the points of V . Then*

- (i) *the vectors of $\langle a + b \rangle$ are in distinct L -orbits; and*
- (ii) *the vectors of $\langle a \rangle^{\#}$ are in a single L -orbit.*

PROOF. Clearly, if $q = 2$ there is nothing to prove, so we may assume $q > 2$. Again we choose a 2-dimensional subspace T with $\langle a \rangle \leq T \leq a^\perp$ and $\langle a \rangle^L \cap T = \{\langle a \rangle\}$. Let $x \in X^{\#}$ be such that $T = T_x$. Then, by Lemma 2.3, $c = [b, x] + f(b, b^x)a \in T \cap b^\perp$. Put $U = \langle b, c \rangle$. Then T and U are both subspaces of c^\perp which contain $\langle c \rangle$. Let $g \in L$ be such that $\langle a \rangle^g = \langle b \rangle$, and set $Y = X^g$. Put $K = \langle C_X(c), C_Y(c) \rangle$ and $W = c^\perp / \langle c \rangle$. By Lemma 2.5 (i), $|C_X(c)| = |C_Y(c)| = q^{2n-3}$ and clearly K acts upon W . From Lemma 2.8 $N_X(U) = 1$ and so, as $\langle c \rangle \leq U \leq c^\perp$, $C_{C_X(c)}(W) = 1$. Likewise $C_{C_Y(c)}(W) = 1$. Put $\bar{K} = K / C_K(W)$. By orders $\overline{C_X(c)}$ and $\overline{C_Y(c)}$ are, respectively, the largest normal 2-subgroups of the stabilizer, in \bar{K} , of $a + \langle c \rangle$ and $b + \langle c \rangle$. Since $f(a + \langle c \rangle, b + \langle c \rangle) = 1$, using Lemma 1.2 we deduce that $\bar{K} \cong \text{Sp}(W)$. Because L is assumed to contain no transvection subgroups, the structure of $C_G(c)$ and $q > 2$ imply that $O_2(C_L(c)) = 1$. So $C_K(W) = 1$ and hence $K \cong \text{Sp}(W)$. Since $\text{Sp}(W)$ acts transitively upon the non-zero vectors of W and $\langle a \rangle^L \cap T = \{\langle a \rangle\}$, it follows that the vectors of $\langle a \rangle^{\#}$ are in the same L -orbit, so (ii) holds. If (i) were

false, then there would exist an element in L acting non-trivially upon $\langle c \rangle$. Hence, as $K \cong \text{Sp}(W)$ and $O_2(C_L(c)) = 1$, $\text{Stab}_L(\langle c \rangle) = Z \times K$ with $Z \neq 1$ and Z centralizing W . Noting that $|Z|$ divides $(q - 1)$ and Z must act non-trivially on V/c^\perp , we see that $V = [V, Z] \times C_V(Z)$ with $\dim[V, Z] = 2$ and K inducing the full symplectic group on $C_V(Z)$. Since K is a simple group (recall $q > 2$) and K centralizes $c \in [V, Z]$, it follows that K centralizes $[V, Z]$. But then K , and hence L , contains a transvection subgroup. Thus we conclude that (i) also holds. \square

LEMMA 2.11. *Suppose that L has two orbits on the points of V . Then for $\lambda, \mu \in k, \mu \neq 0$, the vectors $\lambda a + \mu b$ and $\lambda \mu a + b$ are in the same L -orbit.*

PROOF. Since all the non-zero vectors of $\langle a \rangle$ are in the same L -orbit by Lemma 2.10 (ii), $a^g = \mu a$ for some $g \in L$. So g stabilizes $\langle a \rangle$ and hence leaves a^\perp invariant. Consequently g leaves invariant $\langle a \rangle^L \setminus a^\perp = \langle b \rangle^X$, using Lemma 2.9. So for a suitable $x \in X$, gx stabilizes $\langle b \rangle$. Set $g_1 = gx$, and notice that $a^{g_1} = a^{g^x} = \mu a^x = \mu a$. Now $1 = f(a, b) = f(a^{g_1}, b^{g_1}) = f(\mu a, b^{g_1}) = f(a, \mu b^{g_1})$. Since $\mu b^{g_1} \in \langle b \rangle$, we must have $\mu b^{g_1} = b$ by the original choice of b . So $b^{g_1} = \mu^{-1}b$ and then

$$(\lambda a + \mu b)^{g_1} = \lambda a^{g_1} + \mu b^{g_1} = \lambda \mu a + \mu \mu^{-1}b = \lambda \mu a + b,$$

which gives the result. \square

For the moment we suppose that L has two orbits on the points of V . For $v \in V$ define $Q : V \rightarrow k$ by

$$Q(v) = \begin{cases} 0 & \text{if } v \in a^\perp \cup \{0\} \\ \lambda^2 & \text{if } v \in (\lambda a + \lambda b)^\perp. \end{cases}$$

In view of Lemma 2.10, Q is well defined. Observe that for $v = \mu a + b, \mu \neq 0$, v is in $(\sqrt{\mu}a + \sqrt{\mu}b)^\perp$ by Lemma 2.11 and therefore $Q(v) = \mu$. Also for all $\mu \in k, v \in V$, we have $Q(\mu v) = \mu^2 Q(v)$. It is immediate from its definition that Q is L -invariant.

LEMMA 2.12. *Assume that $v \in a^\perp$. Then for $w \in V$ and $\alpha, \beta \in k$*

$$Q(\alpha v + \beta w) = \alpha^2 Q(v) + \beta^2 Q(w) + \alpha \beta f(v, w).$$

PROOF. Clearly, if $\beta = 0$ then we are done—so we assume $\beta \neq 0$. Without loss of generality we may suppose that $v = a$. Suppose first that $w \in a^\perp$. In this case we need to show that $Q(\alpha a + \beta w) = \beta^2 Q(w) = Q(\beta w)$. This means we need to show that $\alpha a + \beta w$ is in the same L -orbit as βw . If $w \in \langle a \rangle$, then $Q(\alpha a + \beta w) = 0 = Q(\beta w)$. Therefore, we may also assume $w \notin \langle a \rangle$ and so $\langle w, a \rangle$ is a 2-dimensional subspace of a^\perp . By Lemma 2.5 (i) $X/C_X(\langle w, a \rangle)$ has order q and hence there exists $x \in X$ such

that $[\beta w, x] = \alpha a$. Thus $\beta w + \beta w^x = \alpha a$, which is to say that $\beta w = (\alpha a + \beta w)^x$ and we are done.

Next we suppose that $w \notin a^\perp$. Then we have two cases, the first being when $w \in a^L$. Here we have $w = \tau b^x$ for some $\tau \in k$ ($\tau \neq 0$), $x \in X$ by Lemma 2.9. Hence, as X fixes a , $\alpha a + \beta w = \alpha a + \beta \tau b^x = (\alpha a + \beta \tau b)^x$. Therefore, by Lemma 2.11, $\alpha a + \beta w$ is in the same L -orbit as $\alpha \beta \tau a + b$, whence $Q(\alpha a + \beta w) = \alpha \beta \tau$. Now $f(a, w) = f(a, \tau b^x) = f(a, \tau b) = \tau f(a, b) = \tau$. Therefore,

$$\alpha^2 Q(a) + \beta^2 Q(w) + \alpha \beta f(a, w) = \alpha^2 0 + \beta^2 0 + \alpha \beta \tau = \alpha \beta \tau = Q(\alpha a + \beta w),$$

as required. Turning to the second case, $w \notin a^L$, again using Lemma 2.9 we have $w = \tau b^x + \tau \lambda a$, for some $\lambda, \tau \in k$ ($\tau \neq 0 \neq \lambda$), $x \in X$. So, by Lemma 2.11, $\alpha a + \beta w$ and $\beta \tau (\alpha + \beta \tau \lambda) a + b$ are in the same L -orbit and hence $Q(\alpha a + \beta w) = \beta \tau (\alpha + \beta \tau \lambda)$. Because w and $\tau b + \tau \lambda a$ are in the same L -orbit, we obtain $Q(w) = \tau^2 \lambda$. Since $Q(a) = 0$ and $f(a, w) = \tau$,

$$\alpha^2 Q(a) + \beta^2 Q(w) + \alpha \beta f(a, w) = \beta^2 \tau^2 \lambda + \alpha \beta \tau = Q(\alpha a + \beta w).$$

This verifies the equation in the final case and so the lemma holds. □

LEMMA 2.13. *Q is a quadratic form on V.*

PROOF. Let $v, w \in V$ and $\alpha, \beta \in k$. So we must show that $Q(\alpha v + \beta w) = \alpha^2 Q(v) + \beta^2 Q(w) + \alpha \beta f(v, w)$. In view of Lemma 2.12 we only need examine the case when v and w are not in the same L -orbit as a . Also, we may assume that $w \notin a^\perp$. Then, by Lemma 2.2, $w \in \mathcal{O}_\lambda$ for some $\lambda \in k^\#$. Hence $w = b^x + \lambda a$ for some $x \in X$. Therefore, $w + \lambda a = b^x \in a^L$ and so, using Lemma 2.12,

$$\begin{aligned} Q(\alpha v + \beta w) &= Q(\alpha v + \beta \lambda a + \beta w + \beta \lambda a) \\ &= Q(\alpha v + \beta \lambda a) + Q(\beta w + \beta \lambda a) + f(\alpha v + \beta \lambda a, \beta w + \beta \lambda a) \\ &= \alpha^2 Q(v) + \beta^2 \lambda^2 Q(a) + \alpha \beta \lambda f(v, a) + \beta^2 Q(w) \\ &\quad + \beta^2 \lambda^2 Q(a) + \beta^2 \lambda f(w, a) + \alpha \beta f(v, w) \\ &\quad + \alpha \beta \lambda f(v, a) + \beta^2 \lambda f(a, w) + \beta^2 \lambda^2 f(a, a) \\ &= \alpha^2 Q(v) + \beta^2 Q(w) + \alpha \beta f(v, w). \end{aligned}$$

This proves the lemma. □

LEMMA 2.14. *If L has two orbits on the points of V, then L acts naturally on V as one of $\Omega^\pm(V)$ and $O^\pm(V)$.*

PROOF. From Lemma 2.13 L preserves the quadratic form Q and so L is a subgroup of $O(V, Q) (\cong O^\pm(V))$. Now $X = O_2(\text{Stab}_{O(V, Q)}(a))$ and the orbits of L and $O(V, Q)$ upon the points of V are the same. Hence $L \geq \langle X^{O(V, Q)} \rangle \geq \Omega^\pm(V)$. Thus Lemma 2.14 holds. □

3. Proof of Theorem 1.1

In the light of Lemma 2.14 from now on we may assume that L is transitive on the points of V , whence, by Lemma 2.6, L is transitive on $V^\#$. We recall there is a one-to-one correspondence between vectors c of V and symplectic transvections τ_c of $\text{Sp}(V)$ where τ_c is defined by $\tau_c(v) = v + f(v, c)c$ ($v \in V$). Hence L acts transitively (by conjugation) upon the symplectic transvections. Therefore if L contains a transvection it must contain them all and, in particular, will then contain a transvection subgroup. So we conclude that L contains no transvections. For c a non-zero vector of V we define X_c to be X^g where $g \in L$ is such that $a^g = c$. If $a^{g_1} = c$ for $g_1 \in L$, then $X^g X^{g_1} \leq O_2(\text{Stab}_G(c))$ and if $X^g \neq X^{g_1}$, then $X^g X^{g_1}$ intersects the transvection subgroup of $O_2(\text{Stab}_G(c))$ non-trivially. So X_c is well defined; note that $X_a = X$. We now complete the proof of Theorem 1.1, arguing by induction on n and starting with $n = 2$. When $q = 2$ we have $G \cong \text{Sym}(6)$ and it is fairly straightforward to calculate in $\text{Sym}(6)$ to deduce that $L = \text{Alt}(6)$ or $\Gamma L(2, 4)$. Thus we may assume $q > 2$. We note that $|X_a| = |X| = q^2$. Let T be a 2-dimensional space of a^\perp containing $\langle a \rangle$, and let $t \in T \setminus \langle a \rangle$. From Lemmas 2.4 (ii) and 2.5 (i) we have that $X_T = C_{X_a}(T)$ has order q . Put $Y_{at} = \langle X_a, X_t \rangle$. Then $Y_{at} \leq \text{Stab}_G T \sim q^{1+2}(\text{SL}_2(q) \times (q - 1))$. Recall that $O_2(\text{Stab}_G(T))$ is an indecomposable $\text{GF}(q)$ $\text{Stab}_G(T)$ -module. Hence, since $Y_{at} \leq O^2(\text{Stab}_G(T))$ and L contains no transvections, $Y_{at} \cong q \times \text{SL}_2(q)$ with $Y_{at} \cap O_2(\text{Stab}_G(T)) = X_T$ having order q . Evidently $X_a \in \text{Syl}_2 Y_{at}$. Furthermore $N_{Y_{at}}(X_a)$ has order $q^2(q - 1)$, stabilizes $\langle a \rangle$ and induces $\{ \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \mid \lambda \in \text{GF}(q) \}$ on X_a . For 2-dimensional subspaces T_1, T_2 of a^\perp containing $\langle a \rangle$ with $T_1 \neq T_2$ we must have $X_{T_1} \neq X_{T_2}$ for otherwise $X_{T_1} = X_{T_2}$ centralizes $T_1 + T_2 = a^\perp$. Thus

$$N = \langle N_{Y_{at}}(X_a) \mid T = \langle t, a \rangle \text{ a subspace of } a^\perp \rangle (\leq \text{Stab}_G(a))$$

induces a 2-transitive action upon the points of X_a , and consequently $N/C_N(X_a) \cong \text{SL}_2(q) \times (q - 1)$. So, as L has no transvections, $N \sim q^2(\text{SL}_2(q) \times (q - 1))$. But this contradicts the structure of $\text{Stab}_G(a)$ and therefore we have verified the theorem for $n = 2$.

From now on we assume that $n \geq 3$. Again we consider a 2-dimensional subspace T of a^\perp which contains $\langle a \rangle$, and let $t \in T \setminus \langle a \rangle$. Put $X_{T,t} = \langle x \in X_t \mid [V, t] = T \rangle$, and let $K_{T,t}$ be a complement to $X_{T,t}$ in $C_{X_t}(T)$. By Lemmas 2.4 and 2.5 (i) $|K_{T,t}| = q^{2n-4}$. We next show that

- 3.1. (i) $|K_{T,t} O_2(H)/O_2(H)| = q^{2n-4}$;
- (ii) $K_{T,t} O_2(H)$ centralizes T^\perp/T ; and
- (iii) no element of $K_{T,t}$ induces a transvection on $a^\perp/\langle a \rangle$.

Suppose that $1 \neq x \in K_{T,t} \cap O_2(H)$. Then $[a^\perp, x] \leq \langle a \rangle$ and $[t^\perp, x] \leq \langle t \rangle$. Since x is not a transvection on V this gives $[V, x] = T$ which implies $x \in K_{T,t} \cap X_{T,t} = 1$, a

contradiction. Therefore part (i) holds. Part (ii) follows from $[T^\perp, K_{T,t}] \leq [t^\perp, X_t] \leq \langle t \rangle \leq T$. Finally suppose that $1 \neq x \in K_{T,t}$ operates as a transvection on $a^\perp/\langle a \rangle$, and let W be the preimage of $C_{a^\perp/\langle a \rangle}(x)$. Then $[W, x] \leq \langle a \rangle$. Assume that $[W, x] = \langle a \rangle$ holds. Since x is a 2-transvection on V and $x \in X_t$, this gives $[V, x] = T$ and so $x \in K_{T,t} \cap X_{T,t} = 1$, a contradiction. Therefore $[W, x] = 0$ and so $W = C_V(x) \leq t^\perp$. Hence $C_V(x) = W = t^\perp \cap a^\perp = T^\perp$. But then $[V, x] = T^{\perp\perp} = T$ which again gives the impossible $x \in K_{T,t} \cap X_{T,t} = 1$. Thus (iii) holds. Put $\bar{a} = a^\perp/\langle a \rangle$ and

$$L_0 = \langle C_{X_t}(T) \mid t \in T \setminus \langle a \rangle \text{ and } T \text{ a 2-dimensional subspace of } a^\perp \text{ containing } \langle a \rangle \rangle.$$

3.2. L_0 acts irreducibly upon \bar{a} .

Suppose that W is an L_0 -invariant subspace with $\langle a \rangle < W < a^\perp$. Let $c \in a^\perp$ be a vector not in W , and put $U = \langle a, c \rangle$. Then, by Lemma 2.1 (i),

$$[c^\perp \cap W, C_{X_c}(U)] \leq W \cap \langle c \rangle = 0.$$

Therefore $\langle a \rangle \leq c^\perp \cap W \leq C_V(C_{X_c}(U))$. Hence as $c \notin W$, $c^\perp \cap W = \langle a \rangle$ by Lemma 2.1 (iv) and so $\dim \bar{W} = 1$. Thus every proper non-zero L_0 -invariant subspace of \bar{a} has dimension one. Since \bar{W}^\perp is L_0 -invariant, this forces $\bar{W} = \bar{W}^\perp$, whence $\dim V = 4$. However we have $\dim V > 4$, and so (3.2) holds.

Together (3.1) and (3.2) imply that L_0 acting on the $2(n-1)$ -dimensional symplectic space \bar{a} satisfies the hypotheses of the theorem. Therefore, by induction, $L_0/C_{L_0}(\bar{a})$ is isomorphic to one of $O^\pm(\bar{a})$, $\Omega^\pm(\bar{a})$, $Sp(\bar{a})$, $Alt(6)$, or $\Gamma L(2, 4)$ with $q = n - 1 = 2$ in the latter two cases. The orthogonal cases are impossible as for every 1-dimensional subspace \bar{T} of \bar{a} there is a corresponding subgroup of order q^{2n-4} centralizing T^\perp/T which is not the case in the orthogonal groups when we select a non-isotropic 1-dimensional subspace (stabilizer $Sp_{2n-4}(q)$). Thus $L_0/C_{L_0}(\bar{a}) \cong Sp(\bar{a})$ or $L_0/C_{L_0}(\bar{a}) \cong Alt(6)$ and $q = n - 1 = 2$. Now L containing no transvections forces L_0 to have index q (or 4 in the $L_0/C_{L_0}(\bar{a}) \cong Alt(6)$ case) in H both of which are impossible. Therefore, we have that $L_0/C_{L_0}(\bar{a}) \cong \Gamma L(2, 4)$ and $L_0 \sim 2^4 \Gamma L(2, 4)$ with $O_2(L_0)$ a ‘natural’ $GF(2)\Gamma L(2, 4)$ module. However, this means that L_0 is a subgroup of index 12 in $H \sim 2^{1+4} Sp(4, 2)$, and, since it has order coprime to 11, has an orbit of length at most 10 on its twelve right cosets in H . By considering an orbit on which the $O_2(L_0)$ acts non-trivially we find that L_0 must have a subgroup of index 10, which intersects $O_2(L_0)$ in a subgroup of order 8 and projects to $Sym(4)$ in $L_0/O_2(L_0)$. This contradicts the module structure of $O_2(L_0)$ and thus $L_0/C_{L_0}(\bar{a}) \not\cong \Gamma L(2, 4)$. This completes the proof of Theorem 1.1. □

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