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AN ERGODIC THEOREM FOR ASYMPTOTICALLY PERIODIC TIME-INHOMOGENEOUS MARKOV PROCESSES, WITH APPLICATION TO QUASI-STATIONARITY WITH MOVING BOUNDARIES

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Abstract

This paper deals with ergodic theorems for particular time-inhomogeneous Markov processes, whose time-inhomogeneity is asymptotically periodic. Under a Lyapunov/minorization condition, it is shown that, for any measurable bounded function f, the time average $\frac{1}{t}\int_0^t f(X_s)ds$ converges in $\mathbb{L}^2$ towards a limiting distribution, starting from any initial distribution for the process $(X_t)_{t\geq 0}$. This convergence can be improved to an almost sure convergence under an additional assumption on the initial measure. This result is then applied to show the existence of a quasi-ergodic distribution for processes absorbed by an asymptotically periodic moving boundary, satisfying a conditional Doeblin condition.

Keywords: Ergodic theorem; law of large numbers; time-inhomogeneous Markov processes; quasi-stationarity; quasi-ergodic distribution; moving boundaries

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1. Notation

Throughout, we shall use the following notation:

- $\mathbb{N} = \{1, 2, \ldots, \}$ and $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$.
- $\mathcal{M}_1(E)$ denotes the space of the probability measures whose support is included in E.
- $\mathcal{B}(E)$ denotes the set of the measurable bounded functions defined on E.
- $\mathcal{B}_1(E)$ denotes the set of the measurable functions f defined on E such that $\|f\|_\infty \leq 1$.
- For all $\mu \in \mathcal{M}_1(E)$ and $p \in \mathbb{N}$, $\mathbb{L}^p(\mu)$ denotes the set of the measurable functions $f : E \mapsto \mathbb{R}$ such that $\int_E |f(x)|^p \mu(dx) < +\infty$.
- For any $\mu \in \mathcal{M}_1(E)$ and $f \in \mathbb{L}^1(\mu)$, we define

$$\mu(f) := \int_E f(x)\mu(dx).$$

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- For any positive function ψ,

$$\mathcal{M}_1(\psi) := \{\mu \in \mathcal{M}_1(E) : \mu(\psi) < +\infty\}.$$

- Id denotes the identity operator.

2. Introduction

In general, an ergodic theorem for a Markov process $(X_t)_{t\geq 0}$ and probability measure π refers to the almost sure convergence

$$\frac{1}{t}\int_0^t f(X_s)ds \underset{t\to\infty}{\longrightarrow} \pi(f), \quad \forall f \in \mathbb{L}^1(\pi). \tag{1}$$

In the time-homogeneous setting, such an ergodic theorem holds for positive Harris-recurrent Markov processes with the limiting distribution π corresponding to an invariant measure for the underlying Markov process. For time-inhomogeneous Markov processes, such a result does not hold in general (in particular the notion of invariant measure is in general not well-defined), except for specific types of time-inhomogeneity such as *periodic time-inhomogeneous Markov processes*, defined as time-inhomogeneous Markov processes for which there exists $\gamma > 0$ such that, for any $s \leq t$, $k \in \mathbb{Z}_+$, and x,

$$\mathbb{P}[X_t \in \cdot | X_s = x] = \mathbb{P}[X_{t+k\gamma} \in \cdot | X_{s+k\gamma} = x]. \tag{2}$$

In other words, a time-inhomogeneous Markov process is periodic when the transition law between any times s and t remains unchanged when the time interval $[s, t]$ is shifted by a multiple of the period γ. In particular, this implies that, for any $s \in [0, \gamma)$, the Markov chain $(X_{s+n\gamma})_{n\in\mathbb{Z}_+}$ is time-homogeneous. This fact allowed Höpfner *et al.* (in [20, 21, 22]) to show that, if the skeleton Markov chain $(X_{n\gamma})_{n\in\mathbb{Z}_+}$ is Harris-recurrent, then the chains $(X_{s+n\gamma})_{n\in\mathbb{Z}_+}$, for all $s \in [0, \gamma)$, are also Harris-recurrent and

$$\frac{1}{t}\int_0^t f(X_s)ds \underset{t\to\infty}{\longrightarrow} \frac{1}{\gamma}\int_0^\gamma \pi_s(f)ds, \quad \text{almost surely, from any initial measure,}$$

where π_s is the invariant measure for $(X_{s+n\gamma})_{n\in\mathbb{Z}_+}$.

This paper aims to prove a similar result for time-inhomogeneous Markov processes said to be *asymptotically periodic*. Roughly speaking (a precise definition will be explicitly given later), an asymptotically periodic Markov process is such that, given a time interval $T \geq 0$, its transition law on the interval $[s, s+T]$ is asymptotically 'close to' the transition law, on the same interval, of a periodic time-inhomogeneous Markov process called an *auxiliary Markov process*, when $s \to \infty$. This definition is very similar to the notion of *asymptotic homogenization*, defined as follows in [1, Subsection 3.3]. A time-inhomogeneous Markov process $(X_t)_{t\geq 0}$ is said to be *asymptotically homogeneous* if there exists a time-homogeneous Markovian semigroup $(Q_t)_{t\geq 0}$ such that, for all $s \geq 0$,

$$\lim_{t\to\infty} \sup_x \|\mathbb{P}[X_{t+s} \in \cdot | X_t = x] - \delta_x Q_s\|_{TV} = 0, \tag{3}$$

where, for two positive measures with finite mass μ_1 and μ_2, $\|\mu_1 - \mu_2\|_{TV}$ is the *total variation distance* between μ_1 and μ_2:

$$\|\mu_1 - \mu_2\|_{TV} := \sup_{f\in\mathcal{B}_1(E)} |\mu_1(f) - \mu_2(f)|. \tag{4}$$

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In particular, it is well known (see [1, Theorem 3.11]) that, under this and suitable additional conditions, an asymptotically homogeneous Markov process converges towards a probability measure which is invariant for $(Q_t)_{t\geq 0}$. It is similarly expected that an asymptotically periodic process has the same asymptotic properties as a periodic Markov process; in particular an ergodic theorem holds for the asymptotically periodic process.

The main result of this paper provides for an asymptotically periodic Markov process to satisfy

$$\frac{1}{t}\int_0^t f(X_s)ds \xrightarrow[t\to\infty]{\mathbb{L}^2(\mathbb{P}_{0,\mu})} \frac{1}{\gamma}\int_0^\gamma \beta_s(f)ds, \quad \forall f\in\mathcal{B}(E), \forall\mu\in\mathcal{M}_1(E), \tag{5}$$

where $\mathbb{P}_{0,\mu}$ is a probability measure under which $X_0 \sim \mu$, and where β_s is the limiting distribution of the skeleton Markov chain $(X_{s+n\gamma})_{n\in\mathbb{Z}_+}$, if it satisfies a Lyapunov-type condition and a local Doeblin condition (defined further in Section 3), and is such that its auxiliary process satisfies a Lyapunov/minorization condition.

Furthermore, this convergence result holds almost surely if a Lyapunov function of the process $(X_t)_{t\geq 0}$, denoted by ψ, is integrable with respect to the initial measure:

$$\frac{1}{t}\int_0^t f(X_s)ds \xrightarrow[t\to\infty]{\mathbb{P}_{0,\mu}\text{-almost surely}} \frac{1}{\gamma}\int_0^\gamma \beta_s(f)ds, \quad \forall\mu\in\mathcal{M}_1(\psi).$$

This will be more precisely stated and proved in Section 3.

The main motivation of this paper is then to deal with *quasi-stationarity with moving boundaries*, that is, the study of asymptotic properties for the process X, conditioned not to reach some moving subset of the state space. In particular, such a study is motivated by models such as those presented in [3], which studies Brownian particles absorbed by cells whose volume may vary over time.

Quasi-stationarity with moving boundaries has been studied in particular in [24, 25], where a 'conditional ergodic theorem' (see further the definition of a *quasi-ergodic distribution*) has been shown when the absorbing boundaries move periodically. In this paper, we show that a similar result holds when the boundary is asymptotically periodic, assuming that the process satisfies a conditional Doeblin condition (see Assumption (A′)). This will be dealt with in Section 4.

The paper will be concluded by using these results in two examples: an ergodic theorem for an asymptotically periodic Ornstein–Uhlenbeck process, and the existence of a unique quasi-ergodic distribution for a Brownian motion confined between two symmetric asymptotically periodic functions.

3. Ergodic theorem for asymptotically periodic time-inhomogeneous semigroup.

Asymptotic periodicity: the definition. Let $(E, \mathcal{E})$ be a measurable space. Consider $\{(E_t, \mathcal{E}_t)_{t\geq 0}, (P_{s,t})_{s\leq t}\}$ a Markovian time-inhomogeneous semigroup, giving a family of measurable subspaces of $(E, \mathcal{E})$, denoted by $(E_t, \mathcal{E}_t)_{t\geq 0}$, and a family of linear operator $(P_{s,t})_{s\leq t}$, with $P_{s,t} : \mathcal{B}(E_t) \to \mathcal{B}(E_s)$, satisfying for any $r \leq s \leq t$,

$$P_{s,s} = \mathrm{Id}, \qquad P_{s,t}\mathbb{1}_{E_t} = \mathbb{1}_{E_s}, \qquad P_{r,s}P_{s,t} = P_{r,t}.$$

In particular, associated to $\{(E_t, \mathcal{E}_t)_{t\geq 0}, (P_{s,t})_{s\leq t}\}$ is a Markov process $(X_t)_{t\geq 0}$ and a family of probability measures $(\mathbb{P}_{s,x})_{s\geq 0, x\in E_s}$ such that, for any $s \leq t$, $x \in E_s$, and $A \in \mathcal{E}_t$,

$$\mathbb{P}_{s,x}[X_t \in A] = P_{s,t}\mathbb{1}_A(x).$$

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We denote by $\mathbb{P}_{s,\mu} := \int_{E_s} \mathbb{P}_{s,x}\mu(dx)$ any probability measure μ supported on E_s. We also denote by $\mathbb{E}_{s,x}$ and $\mathbb{E}_{s,\mu}$ the expectations associated to $\mathbb{P}_{s,x}$ and $\mathbb{P}_{s,\mu}$ respectively. Finally, the following notation will be used for $\mu \in \mathcal{M}_1(E_s)$, $s \leq t$, and $f \in \mathcal{B}(E_t)$:

$$\mu P_{s,t}f := \mathbb{E}_{s,\mu}[f(X_t)], \qquad \mu P_{s,t} := \mathbb{P}_{s,\mu}[X_t \in \cdot].$$

The periodicity of a time-inhomogeneous semigroup is defined as follows. We say a semigroup $\{(F_t, \mathcal{F}_t)_{t\geq 0}, (Q_{s,t})_{s\leq t}\}$ is γ*-periodic* (for $\gamma > 0$) if, for any $s \leq t$,

$$(F_t, \mathcal{F}_t) = (F_{t+k\gamma}, \mathcal{F}_{t+k\gamma}), \quad Q_{s,t} = Q_{s+k\gamma,t+k\gamma}, \quad \forall k \in \mathbb{Z}_+.$$

It is now possible to define an *asymptotically periodic semigroup*.

Definition 1. (*Asymptotically periodic semigroups.*) A time-inhomogeneous semigroup $\{(E_t, \mathcal{E}_t)_{t\geq 0}, (P_{s,t})_{s\leq t}\}$ is said to be *asymptotically periodic* if (for some $\gamma > 0$) there exist a γ-periodic semigroup $\{(F_t, \mathcal{F}_t)_{t\geq 0}, (Q_{s,t})_{s\leq t}\}$ and two families of functions $(\psi_s)_{s\geq 0}$ and $(\tilde{\psi}_s)_{s\geq 0}$ such that $\tilde{\psi}_{s+\gamma} = \tilde{\psi}_s$ for all $s \geq 0$, and for any $s \in [0, \gamma)$, the following hold:

1. $\bigcup_{k=0}^{\infty} \bigcap_{l\geq k} E_{s+l\gamma} \cap F_s \neq \emptyset$.
2. There exists $x_s \in \bigcup_{k=0}^{\infty} \bigcap_{l\geq k} E_{s+l\gamma} \cap F_s$ such that, for any $n \in \mathbb{Z}_+$,

$$\|\delta_{x_s} P_{s+k\gamma,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma} \times \cdot] - \delta_{x_s} Q_{s,s+n\gamma}[\tilde{\psi}_s \times \cdot]\|_{TV} \underset{k\to\infty}{\longrightarrow} 0. \tag{6}$$

The semigroup $\{(F_t, \mathcal{F}_t)_{t\geq 0}, (Q_{s,t})_{s\leq t}\}$ is then called the *auxiliary semigroup of* $(P_{s,t})_{s\leq t}$.

When $\psi_s = \tilde{\psi}_s = \mathbb{1}$ for all $s \geq 0$, we say that the semigroup $(P_{s,t})_{s\leq t}$ is *asymptotically periodic in total variation*. By extension, we will say that the process $(X_t)_{t\geq 0}$ is asymptotically periodic (in total variation) if the associated semigroup $\{(E_t, \mathcal{E}_t)_{t\geq 0}, (P_{s,t})_{s\leq t}\}$ is asymptotically periodic (in total variation).

In what follows, the functions $(\psi_s)_{s\geq 0}$ and $(\tilde{\psi}_s)_{s\in[0,\gamma)}$ will play the role of Lyapunov functions (that is to say, satisfying Assumption 1(ii) below) for the semigroups $(P_{s,t})_{s\leq t}$ and $(Q_{s,t})_{s\leq t}$, respectively. The introduction of these functions in the definition of asymptotically periodic semigroups will allow us to establish an ergodic theorem for processes satisfying the Lyapunov/minorization conditions stated below.

Lyapunov/minorization conditions. The main assumption of Theorem 1, which will be provided later, will be that the asymptotically periodic Markov process satisfies the following assumption.

Assumption 1. *There exist $t_1 \geq 0$, $n_0 \in \mathbb{N}$, $c > 0$, $\theta \in (0, 1)$, a family of measurable sets $(K_t)_{t\geq 0}$ such that $K_t \subset E_t$ for all $t \geq 0$, a family of probability measures $(\nu_s)_{s\geq 0}$ on $(K_s)_{s\geq 0}$, and a family of functions $(\psi_s)_{s\geq 0}$, all lower-bounded by 1, such that the following hold:*

(i) For any $s \geq 0$, $x \in K_s$, and $n \geq n_0$,

$$\delta_x P_{s,s+nt_1} \geq c\nu_{s+nt_1}.$$

(ii) For any $s \geq 0$,

$$P_{s,s+t_1}\psi_{s+t_1} \leq \theta\psi_s + C\mathbb{1}_{K_s}.$$

(iii) For any $s \geq 0$ and $t \in [0, t_1)$,

$$P_{s,s+t}\psi_{s+t} \leq C\psi_s.$$

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When a semigroup $(P_{s,t})_{s\leq t}$ satisfies Assumption 1 as stated above, we will say that the functions $(\psi_s)_{s\geq 0}$ are *Lyapunov functions* for the semigroup $(P_{s,t})_{s\leq t}$. In particular, under (ii) and (iii), it is easy to prove that for any $s \leq t$,

$$P_{s,t}\psi_t \leq C\left(1 + \frac{C}{1-\theta}\right)\psi_s. \tag{7}$$

We remark in particular that Assumption 1 implies an *exponential weak ergodicity in ψ_t-distance*; that is, we have the existence of two constants $C' > 0$ and $\kappa > 0$ such that, for all $s \leq t$ and for all probability measures $\mu_1, \mu_2 \in \mathcal{M}_1(E_s)$,

$$\|\mu_1 P_{s,t} - \mu_2 P_{s,t}\|_{\psi_t} \leq C'[\mu_1(\psi_s) + \mu_2(\psi_s)]e^{-\kappa(t-s)}, \tag{8}$$

where, for a given function ψ, $\|\mu - \nu\|_\psi$ is the ψ-distance, defined to be

$$\|\mu - \nu\|_\psi := \sup_{|f|\leq\psi} |\mu(f) - \nu(f)|, \quad \forall \mu, \nu \in \mathcal{M}_1(\psi).$$

In particular, when $\psi = \mathbb{1}$ for all $t \geq 0$, the ψ-distance is the total variation distance. If we have weak ergodicity (8) in the time-homogeneous setting (see in particular [15]), the proof of [15, Theorem 1.3] can be adapted to a general time-inhomogeneous framework (see for example [6, Subsection 9.5]).

The main theorem and proof. The main result of this paper is the following.

Theorem 1. *Let $\{(E_t, \mathcal{E}_t)_{t\geq 0}, (P_{s,t})_{s\leq t}, (X_t)_{t\geq 0}, (\mathbb{P}_{s,x})_{s\geq 0, x\in E_s}\}$ be an asymptotically γ-periodic time-inhomogeneous Markov process, with $\gamma > 0$, and denote by $\{(F_t, \mathcal{F}_t)_{t\geq 0}, (Q_{s,t})_{s\leq t}\}$ its periodic auxiliary semigroup. Also, denote by $(\psi_s)_{s\geq 0}$ and $(\tilde{\psi}_s)_{s\geq 0}$ the two families of functions as defined in Definition 1. Assume moreover the following:*

1. *The semigroups $(P_{s,t})_{s\leq t}$ and $(Q_{s,t})_{s\leq t}$ satisfy Assumption 1, with $(\psi_s)_{s\geq 0}$ and $(\tilde{\psi}_s)_{s\geq 0}$ respectively as Lyapunov functions.*
2. *For any $s \in [0, \gamma)$, $(\psi_{s+n\gamma})_{n\in\mathbb{Z}_+}$ converges pointwise to $\tilde{\psi}_s$.*

Then, for any $\mu \in \mathcal{M}_1(E_0)$ such that $\mu(\psi_0) < +\infty$,

$$\left\|\frac{1}{t}\int_0^t \mu P_{0,s}[\psi_s \times \cdot]ds - \frac{1}{\gamma}\int_0^\gamma \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot]ds\right\|_{TV} \xrightarrow[t\to\infty]{} 0, \tag{9}$$

where $\beta_\gamma \in \mathcal{M}_1(F_0)$ is the unique invariant probability measure of the skeleton semigroup $(Q_{0,n\gamma})_{n\in\mathbb{Z}_+}$ satisfying $\beta_\gamma(\tilde{\psi}_0) < +\infty$. Moreover, for any $f \in \mathcal{B}(E)$ we have the following:

1. *For any $\mu \in \mathcal{M}_1(E_0)$,*

$$\mathbb{E}_{0,\mu}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds - \frac{1}{\gamma}\int_0^\gamma \beta_\gamma Q_{0,s}f ds\right|^2\right] \xrightarrow[t\to\infty]{} 0. \tag{10}$$

2. *If moreover $\mu(\psi_0) < +\infty$, then*

$$\frac{1}{t}\int_0^t f(X_s)ds \xrightarrow[t\to\infty]{} \frac{1}{\gamma}\int_0^\gamma \beta_\gamma Q_{0,s}f ds, \quad \mathbb{P}_{0,\mu}\text{-almost surely.} \tag{11}$$

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Remark 1. When Assumption 1 holds for $K_s = E_s$ for any s, the condition (i) in Assumption 1 implies the *Doeblin condition.*

Doeblin condition. There exist $t_0 \geq 0$, $c > 0$, and a family of probability measures $(\nu_t)_{t\geq 0}$ on $(E_t)_{t\geq 0}$ such that, for any $s \geq 0$ and $x \in E_s$,

$$\delta_x P_{s,s+t_0} \geq c\nu_{s+t_0}. \tag{12}$$

In fact, if we assume that Assumption 1(i) holds for $K_s = E_s$, the Doeblin condition holds if we set $t_0 := n_0 t_1$. Conversely, the Doeblin condition implies the conditions (i), (ii), and (iii) with $K_s = E_s$ and $\psi_s = \mathbb{1}_{E_s}$ for all $s \geq 0$, so that these conditions are equivalent. In fact, (ii) and (iii) straightforwardly hold true for $(K_s)_{s\geq 0} = (E_s)_{s\geq 0}$, $(\psi_s)_{s\geq 0} = (\mathbb{1}_{E_s})_{s\geq 0}$, $C = 1$, any $\theta \in (0, 1)$, and any $t_1 \geq 0$. If we set $t_1 = t_0$ and $n_0 = 1$, the Doeblin condition implies that, for any $s \in [0, t_1)$,

$$\delta_x P_{s,s+t_1} \geq c\nu_{s+t_1}, \quad \forall x \in E_s.$$

Integrating this inequality over $\mu \in \mathcal{M}_1(E_s)$, one obtains

$$\mu P_{s,s+t_1} \geq c\nu_{s+t_1}, \quad \forall s \in [0, t_1), \ \forall \mu \in \mathcal{M}_1(E_s).$$

Then, by the Markov property, for all $s \in [0, t_1)$, $x \in E_s$, and $n \in \mathbb{N}$, we have

$$\delta_x P_{s,s+nt_1} = (\delta_x P_{s,s+(n-1)t_1})P_{s+(n-1)t_1,s+nt_1} \geq c\nu_{s+nt_1},$$

which is (i).

Theorem 1 then implies the following corollary.

Corollary 1. *Let $(X_t)_{t\geq 0}$ be asymptotically γ-periodic in total variation distance. If $(X_t)_{t\geq 0}$ and its auxiliary semigroup satisfy a Doeblin condition, then the convergence (10) is improved to*

$$\sup_{\mu\in\mathcal{M}_1(E_0)} \sup_{f\in\mathcal{B}_1(E)} \mathbb{E}_{0,\mu}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds - \frac{1}{\gamma}\int_0^\gamma \beta_\gamma Q_{0,s}f ds\right|^2\right] \xrightarrow[t\to\infty]{} 0.$$

Moreover, the almost sure convergence (11) holds for any initial measure μ.

Remark 2. We also note that, if the convergence (6) holds for all

$$x \in \bigcup_{k=0}^{\infty} \bigcap_{l\geq k} E_{s+l\gamma} \cap F_s,$$

then this implies (6) and therefore the pointwise convergence of $(\psi_{s+n\gamma})_{n\in\mathbb{Z}_+}$ to $\tilde{\psi}_s$ (by taking $n = 0$ in (6)).

Proof of Theorem 1. The proof is divided into five steps.

First step. Since the auxiliary semigroup $(Q_{s,t})_{s\leq t}$ satisfies Assumption 1 with $(\tilde{\psi}_s)_{s\geq 0}$ as Lyapunov functions, the time-homogeneous semigroup $(Q_{0,n\gamma})_{n\in\mathbb{Z}_+}$ satisfies Assumptions 1 and 2 of [15], which we now recall (using our notation).

Assumption 2. ([15, Assumption 1].) *There exist $V : F_0 \to [0, +\infty)$, $n_1 \in \mathbb{N}$, and constants $K \geq 0$ and $\kappa \in (0, 1)$ such that*

$$Q_{0,n_1\gamma} V \leq \kappa V + K.$$

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Assumption 3. ([15, Assumption 2].) *There exist a constant* $\alpha \in (0, 1)$ *and a probability measure* ν *such that*

$$\inf_{x\in\mathcal{C}_R} \delta_x Q_{0,n_1\gamma} \geq \alpha\nu(\cdot),$$

with $\mathcal{C}_R := \{x \in F_0 : V(x) \leq R\}$ *for some* $R > 2K/(1-\kappa)$*, where* n_1*,* K*, and* κ *are the constants from Assumption 2.*

In fact, since $(Q_{s,t})_{s\leq t}$ satisfies (ii) and (iii) of Assumption 1, there exist $C > 0$, $\theta \in (0, 1)$, $t_1 \geq 0$, and $(K_s)_{s\geq 0}$ such that

$$Q_{s,s+t_1}\tilde{\psi}_{s+t_1} \leq \theta\tilde{\psi}_s + C\mathbb{1}_{K_s}, \quad \forall s \geq 0, \tag{13}$$

and

$$Q_{s,s+t}\tilde{\psi}_{s+t} \leq C\tilde{\psi}_s, \quad \forall s \geq 0, \forall t \in [0, t_1).$$

We let $n_2 \in \mathbb{N}$ be such that $\theta^{n_2}C\big(1 + \frac{C}{1-\theta}\big) < 1$. By (13) and recalling that $\tilde{\psi}_t = \tilde{\psi}_{t+\gamma}$ for all $t \geq 0$, one has for any $s \geq 0$ and $n \in \mathbb{N}$,

$$Q_{s,s+nt_1}\tilde{\psi}_{s+nt_1} \leq \theta^n\tilde{\psi}_s + \frac{C}{1-\theta}. \tag{14}$$

Thus, for all $n_1 \geq \lceil \frac{n_2t_1}{\gamma} \rceil$,

$$\begin{aligned}
Q_{0,n_1\gamma}\tilde{\psi}_0 &= Q_{0,n_1\gamma-n_2t_1}Q_{n_1\gamma-n_2t_1,n_1\gamma}\tilde{\psi}_{n_1\gamma} \\
&\leq \theta^{n_2}Q_{0,n_1\gamma-n_2t_1}\tilde{\psi}_{n_1\gamma-n_2t_1} + \frac{C}{1-\theta} \\
&\leq \theta^{n_2}C\left(1 + \frac{C}{1-\theta}\right)\tilde{\psi}_0 + \frac{C}{1-\theta},
\end{aligned}$$

where we successively used the semigroup property of $(Q_{s,t})_{s\leq t}$, (14), and (7) applied to $(Q_{s,t})_{s\leq t}$. Hence one has Assumption 2 by setting $V = \tilde{\psi}_0$, $\kappa := \theta^{n_2}C\big(1 + \frac{C}{1-\theta}\big)$, and $K := \frac{C}{1-\theta}$.

We now prove Assumption 3. To this end, we introduce a Markov process $(Y_t)_{t\geq 0}$ and a family of probability measures $(\hat{\mathbb{P}}_{s,x})_{s\geq 0, x\in F_s}$ such that

$$\hat{\mathbb{P}}_{s,x}(Y_t \in A) = Q_{s,t}\mathbb{1}_A(x), \quad \forall s \leq t,\ x \in F_s,\ A \in \mathcal{F}_t.$$

In what follows, for all $s \geq 0$ and $x \in F_s$, we will use the notation $\hat{\mathbb{E}}_{s,x}$ for the expectation associated to $\hat{\mathbb{P}}_{s,x}$. Moreover, we define

$$T_K := \inf\big\{n \in \mathbb{Z}_+ : Y_{nt_1} \in K_{nt_1}\big\}.$$

Then, using (13) recursively, for all $k \in \mathbb{N}$, $R > 0$, and $x \in \mathcal{C}_R$ (recalling that $\mathcal{C}_R$ is defined in the statement of Assumption 3), we have

$$\begin{aligned}
\hat{\mathbb{E}}_{0,x}\big[\tilde{\psi}_{kt_1}(Y_{kt_1})\mathbb{1}_{T_K>k}\big] &= \hat{\mathbb{E}}_{0,x}\big[\mathbb{1}_{T_K>k-1}\hat{\mathbb{E}}_{(k-1)t_1,Y_{(k-1)t_1}}\big(\tilde{\psi}_{kt_1}(Y_{kt_1})\mathbb{1}_{T_K>k}\big)\big] \\
&\leq \theta\hat{\mathbb{E}}_{0,x}\big[\tilde{\psi}_{(k-1)t_1}(Y_{(k-1)t_1})\mathbb{1}_{T_K>k-1}\big] \leq \theta^k\tilde{\psi}_0(x) \leq R\theta^k.
\end{aligned}$$

Since $\tilde{\psi}_{kt_1} \geq 1$ for all $k \in \mathbb{Z}_+$, we have that for all $x \in \mathcal{C}_R$, for all $k \in \mathbb{Z}_+$,

$$\hat{\mathbb{P}}_{0,x}(T_K > k) \leq R\theta^k.$$

In particular, there exists $k_0 \geq n_0$ such that, for all $k \geq k_0 - n_0$,

$$\hat{\mathbb{P}}_{0,x}(T_K > k) \leq \frac{1}{2}.$$

Hence, for all $x \in \mathcal{C}_R$,

$$\begin{aligned}\delta_x Q_{0,k_0t_1} = \hat{\mathbb{P}}_{0,x}\big(Y_{k_0t_1} \in \cdot\big) &\geq \sum_{i=0}^{k_0-n_0} \hat{\mathbb{E}}_{0,x}\Big(\mathbb{1}_{T_K=i}\hat{\mathbb{P}}_{it_1,Y_{it_1}}\big(Y_{k_0t_1} \in \cdot\big)\Big)\\ &\geq c\sum_{i=0}^{k_0-n_0} \hat{\mathbb{E}}_{0,x}\big(\mathbb{1}_{T_K=i}\big) \times \nu_{k_0t_1}\\ &= c\hat{\mathbb{P}}_{0,x}\big(T_K \leq k_0 - n_0\big)\nu_{k_0t_1}\\ &\geq \frac{c}{2}\nu_{k_0t_1}.\end{aligned}$$

Hence, for all $n_1 \geq \lceil \frac{k_0t_1}{\gamma} \rceil$, for all $x \in \mathcal{C}_R$,

$$\delta_x Q_{0,k_0t_1} Q_{k_0t_1,n_1\gamma} \geq \frac{c}{2}\nu_{k_0t_1} Q_{k_0t_1,n_1\gamma}.$$

Thus, Assumption 3 is satisfied if we take $n_1 := \lceil \frac{n_2t_1}{\gamma} \rceil \vee \lceil \frac{k_0t_1}{\gamma} \rceil$, $\alpha := \frac{c}{2}$, and $\nu(\cdot) := \nu_{k_0t_1} Q_{k_0t_1,n_1\gamma}$.

Then, by [15, Theorem 1.2], Assumptions 2 and 3 imply that $Q_{0,n_1\gamma}$ admits a unique invariant probability measure β_γ. Furthermore, there exist constants $C > 0$ and $\delta \in (0, 1)$ such that, for all $\mu \in \mathcal{M}_1(F_0)$,

$$\|\mu Q_{0,nn_1\gamma} - \beta_\gamma\|_{\tilde{\psi}_0} \leq C\mu(\tilde{\psi}_0)\delta^n. \tag{15}$$

Since β_γ is the unique invariant probability measure of $Q_{0,n_1\gamma}$, and noting that $\beta_\gamma Q_{0,\gamma}$ is invariant for $Q_{0,n_1\gamma}$, we deduce that β_γ is the unique invariant probability measure for $Q_{0,\gamma}$, and by (15), for all μ such that $\mu(\tilde{\psi}_0) < +\infty$,

$$\|\mu Q_{0,n\gamma} - \beta_\gamma\|_{\tilde{\psi}_0} \underset{n\to\infty}{\longrightarrow} 0.$$

Now, for any $s \geq 0$, note that $\delta_x Q_{s,\lceil \frac{s}{\gamma}\rceil\gamma}\tilde{\psi}_0 < +\infty$ for all $x \in F_s$ (this is a consequence of (7) applied to the semigroup $(Q_{s,t})_{s\leq t}$), and therefore, taking $\mu = \delta_x Q_{s,\lceil \frac{s}{\gamma}\rceil\gamma}$ in the above convergence,

$$\|\delta_x Q_{s,n\gamma} - \beta_\gamma\|_{\tilde{\psi}_0} \underset{n\to\infty}{\longrightarrow} 0$$

for all $x \in F_s$. Hence, since $Q_{n\gamma,n\gamma+s}\tilde{\psi}_s \leq C\big(1 + \frac{C}{1-\theta}\big)\tilde{\psi}_{n\gamma}$ by (7), we conclude from the above convergence that

$$\|\delta_x Q_{s,s+n\gamma} - \beta_\gamma Q_{0,s}\|_{\tilde{\psi}_s} \leq C\left(1 + \frac{C}{1-\theta}\right)\|\delta_x Q_{s,n\gamma} - \beta_\gamma\|_{\tilde{\psi}_0} \underset{n\to\infty}{\longrightarrow} 0. \tag{16}$$

Moreover, $\beta_\gamma(\tilde{\psi}_0) < +\infty$.

Second step. The first part of this step (up to the equality (20)) is inspired by the proof of [1, Theorem 3.11].

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We fix $s\in[0,\gamma]$. Without loss of generality, we assume that $\bigcap_{l\geq 0} E_{s+l\gamma}\cap F_s\neq\emptyset$. Then, by Definition 1, there exists $x_s\in\bigcap_{l\geq 0} E_{s+l\gamma}\cap F_s$ such that for any $n\geq 0$,

$$\left\|\delta_{x_s}P_{s+k\gamma,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma}\times\cdot]-\delta_{x_s}Q_{s,s+n\gamma}\left[\tilde{\psi}_s\times\cdot\right]\right\|_{TV}\underset{k\to\infty}{\longrightarrow}0,$$

which implies by (16) that

$$\lim_{n\to\infty}\lim_{k\to\infty}\left\|\delta_{x_s}P_{s+k\gamma,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma}\times\cdot]-\beta_\gamma Q_{0,s}\left[\tilde{\psi}_s\times\cdot\right]\right\|_{TV}=0. \tag{17}$$

Then, by the Markov property, (8), and (7), one obtains that, for any $k,n\in\mathbb{N}$ and $x\in\bigcap_{l\geq 0}E_{s+l\gamma}$,

$$\begin{aligned}
&\|\delta_x P_{s,s+(k+n)\gamma}-\delta_x P_{s+k\gamma,s+(k+n)\gamma}\|_{\psi_{s+(k+n)\gamma}}\\
&\qquad=\|\left(\delta_x P_{s,s+k\gamma}\right)P_{s+k\gamma,s+(k+n)\gamma}-\delta_x P_{s+k\gamma,s+(k+n)\gamma}\|_{\psi_{s+(k+n)\gamma}}\\
&\qquad\leq C'[P_{s,s+k\gamma}\psi_{s+k\gamma}(x)+\psi_{s+k\gamma}(x)]e^{-\kappa\gamma n}\\
&\qquad\leq C''[\psi_s(x)+\psi_{s+k\gamma}(x)]e^{-\kappa\gamma n},
\end{aligned}\tag{18}$$

where $C'':=C'\left(C\left(1+\frac{C}{1-\theta}\right)\vee 1\right)$. Then, for any $k,n\in\mathbb{N}$,

$$\|\delta_{x_s}P_{s,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma}\times\cdot]-\beta_\gamma Q_{0,s}[\tilde{\psi}_s\times\cdot]\|_{TV} \tag{19}$$

$$\leq C''[\psi_s(x)+\psi_{s+k\gamma}(x)]e^{-\kappa\gamma n}+\left\|\delta_{x_s}P_{s+k\gamma,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma}\times\cdot]-\beta_\gamma Q_{0,s}\left[\tilde{\psi}_s\times\cdot\right]\right\|_{TV},$$

which by (17) and the pointwise convergence of $(\psi_{s+k\gamma})_{k\in\mathbb{Z}_+}$ implies that

$$\begin{aligned}
&\lim_{n\to\infty}\left\|\delta_{x_s}P_{s,s+n\gamma}[\psi_{s+n\gamma}\times\cdot]-\beta_\gamma Q_{0,s}\left[\tilde{\psi}_s\times\cdot\right]\right\|_{TV}\\
&\quad=\lim_{n\to\infty}\limsup_{k\to\infty}\|\delta_{x_s}P_{s,s+(k+n)\gamma}[\psi_{s+(k+n)\gamma}\times\cdot]-\beta_\gamma Q_{0,s}\left[\tilde{\psi}_s\times\cdot\right]\|_{TV}\\
&\quad=0.
\end{aligned}\tag{20}$$

The weak ergodicity (8) implies therefore that the previous convergence actually holds for any initial distribution $\mu\in\mathcal{M}_1(E_0)$ satisfying $\mu(\psi_0)<+\infty$, so that

$$\left\|\mu P_{0,s+n\gamma}[\psi_{s+n\gamma}\times\cdot]-\beta_\gamma Q_{0,s}\left[\tilde{\psi}_s\times\cdot\right]\right\|_{TV}\underset{n\to\infty}{\longrightarrow}0. \tag{21}$$

Since

$$\left\|\mu P_{0,s+n\gamma}[\psi_{s+n\gamma}\times\cdot]-\beta_\gamma Q_{0,s}\left[\tilde{\psi}_s\times\cdot\right]\right\|_{TV}\leq 2$$

for all $\mu\in\mathcal{M}_1(E_0)$, $s\geq 0$, and $n\in\mathbb{Z}_+$, (21) and Lebesgue's dominated convergence theorem imply that

$$\frac{1}{\gamma}\int_0^\gamma\left\|\mu P_{0,s+n\gamma}[\psi_{s+n\gamma}\times\cdot]-\beta_\gamma Q_{0,s}\left[\tilde{\psi}_s\times\cdot\right]\right\|_{TV}ds\underset{n\to\infty}{\longrightarrow}0,$$

which implies that

$$\left\|\frac{1}{\gamma}\int_0^\gamma\mu P_{0,s+n\gamma}[\psi_{s+n\gamma}\times\cdot]ds-\frac{1}{\gamma}\int_0^\gamma\beta_\gamma Q_{0,s}\left[\tilde{\psi}_s\times\cdot\right]ds\right\|_{TV}\underset{n\to\infty}{\longrightarrow}0.$$

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By Cesaro's lemma, this allows us to conclude that, for any $\mu \in \mathcal{M}_1(E_0)$ such that $\mu(\psi_0) < +\infty$,

$$\left\| \frac{1}{t} \int_0^t \mu P_{0,s}[\psi_s \times \cdot] ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot] ds \right\|_{TV}$$

$$\leq \frac{1}{\lfloor \frac{t}{\gamma} \rfloor} \sum_{k=0}^{\lfloor \frac{t}{\gamma} \rfloor} \left\| \frac{1}{\gamma} \int_0^\gamma \mu P_{0,s+k\gamma}[\psi_{s+k\gamma} \times \cdot] ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot] ds \right\|_{TV}$$

$$+ \left\| \frac{1}{t} \int_{\lfloor \frac{t}{\gamma} \rfloor \gamma}^t \mu P_{0,s}[\psi_s \times \cdot] ds \right\|_{TV} \underset{t \to \infty}{\longrightarrow} 0,$$

which concludes the proof of (9) .

Third step. In the same manner, we now prove that, for any $\mu \in \mathcal{M}_1(E_0)$ such that $\mu(\psi_0) < +\infty$,

$$\left\| \frac{1}{t} \int_0^t \mu P_{0,s} ds - \frac{1}{\gamma} \int_0^\gamma \beta_\gamma Q_{0,s} ds \right\|_{TV} \underset{t \to \infty}{\longrightarrow} 0. \tag{22}$$

In fact, for any function f bounded by 1 and $\mu \in \mathcal{M}_1(E_0)$ such that $\mu(\psi_0) < +\infty$,

$$\left| \mu P_{0,s+n\gamma} \left[\psi_{s+n\gamma} \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\tilde{\psi}_s} \right] \right|$$

$$\leq \left| \mu P_{0,s+n\gamma} \left[\psi_{s+n\gamma} \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\psi_{s+n\gamma}} \right] \right| + \left| \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\psi_{s+n\gamma}} \right] \right.$$

$$\left. - \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\tilde{\psi}_s} \right] \right|$$

$$\leq \left\| \mu P_{0,s+n\gamma}[\psi_{s+n\gamma} \times \cdot] - \beta_\gamma Q_{0,s}[\tilde{\psi}_s \times \cdot] \right\|_{TV} + \left| \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\psi_{s+n\gamma}} \right] \right.$$

$$\left. - \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\tilde{\psi}_s} \right] \right|.$$

We now remark that, since $\psi_{s+n\gamma} \geq 1$ for any s and $n \in \mathbb{Z}_+$, one has that

$$\left| \frac{\tilde{\psi}_s}{\psi_{s+n\gamma}} - 1 \right| \leq 1 + \tilde{\psi}_s.$$

Since $(\psi_{s+n\gamma})_{n \in \mathbb{Z}_+}$ converges pointwise towards $\tilde{\psi}_s$ and $\beta_\gamma Q_{0,s} \tilde{\psi}_s < +\infty$, Lebesgue's dominated convergence theorem implies

$$\sup_{f \in \mathcal{B}_1(E)} \left| \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\psi_{s+n\gamma}} \right] - \beta_\gamma Q_{0,s} \left[\tilde{\psi}_s \times \frac{f}{\tilde{\psi}_s} \right] \right| \underset{n \to \infty}{\longrightarrow} 0.$$

Then, using (21), one has

$$\| \mu P_{0,s+n\gamma} - \beta_\gamma Q_{0,s} \|_{TV} \underset{n \to \infty}{\longrightarrow} 0,$$

which allows us to conclude (22), using the same argument as in the first step.

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Fourth step. In order to show the $\mathbb{L}^2$-ergodic theorem, we let $f \in \mathcal{B}(E)$. For any $x \in E_0$ and $t \geq 0$,

$$\begin{aligned}
&\mathbb{E}_{0,x}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds - \mathbb{E}_{0,x}\left[\frac{1}{t}\int_0^t f(X_s)ds\right]\right|^2\right] \\
&= \frac{2}{t^2}\int_0^t\int_s^t (\mathbb{E}_{0,x}[f(X_s)f(X_u)] - \mathbb{E}_{0,x}[f(X_s)]\mathbb{E}_{0,x}[f(X_u)])du\, ds \\
&= \frac{2}{t^2}\int_0^t\int_s^t \mathbb{E}_{0,x}[f(X_s)(f(X_u) - \mathbb{E}_{0,x}[f(X_u)])]du\, ds \\
&= \frac{2}{t^2}\int_0^t\int_s^t \mathbb{E}_{0,x}[f(X_s)(\mathbb{E}_{s,X_s}[f(X_u)] - \mathbb{E}_{s,\delta_x P_{0,s}}[f(X_u)])]du\, ds,
\end{aligned}$$

where the Markov property was used in the last line. By (8) (weak ergodicity) and (7), one obtains for any $s \leq t$

$$\left|\mathbb{E}_{s,X_s}[f(X_t)] - \mathbb{E}_{s,\delta_x P_{0,s}}[f(X_t)]\right| \leq C''\|f\|_\infty[\psi_s(X_s) + \psi_0(x)]e^{-\kappa(t-s)}, \quad \mathbb{P}_{0,x}\text{-almost surely,} \tag{23}$$

where C' was defined in the first part. As a result, for any $x \in E_0$ and $t \geq 0$,

$$\begin{aligned}
&\mathbb{E}_{0,x}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds - \mathbb{E}_{0,x}\left[\frac{1}{t}\int_0^t f(X_s)ds\right]\right|^2\right] \\
&\leq \frac{2C''\|f\|_\infty}{t^2}\int_0^t\int_s^t \mathbb{E}_{0,x}[|f(X_s)|(\psi_s(X_s) + \psi_0(x))]e^{-\kappa(u-s)}du\, ds \\
&= \frac{2C''\|f\|_\infty}{t^2}\int_0^t \mathbb{E}_{0,x}[|f(X_s)|(\psi_s(X_s) + \psi_0(x))]e^{\kappa s}\frac{e^{-\kappa s} - e^{-\kappa t}}{\kappa}ds \\
&= \frac{2C''\|f\|_\infty}{\kappa t} \times \mathbb{E}_{0,x}\left[\frac{1}{t}\int_0^t |f(X_s)|(\psi_s(X_s) + \psi_0(x))ds\right] \\
&\quad - \frac{2C''\|f\|_\infty e^{-\kappa t}}{\kappa t^2}\int_0^t e^{\kappa s}\mathbb{E}_{0,x}[|f(X_s)|(\psi_s(X_s) + \psi_0(x))]ds.
\end{aligned}$$

Then, by (9), there exists a constant $\tilde{C} > 0$ such that, for any $x \in E_0$, when $t \to \infty$,

$$\begin{aligned}
\mathbb{E}_{0,x}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds - \mathbb{E}_{0,x}\left[\frac{1}{t}\int_0^t f(X_s)ds\right]\right|^2\right] &\leq \frac{\tilde{C}\|f\|_\infty\psi_0(x)}{t} \\
&\times \frac{1}{\gamma}\int_0^\gamma \beta_\gamma Q_{0,s}[|f|\tilde{\psi}_s]ds + o\left(\frac{1}{t}\right).
\end{aligned} \tag{24}$$

Since $f \in \mathcal{B}(E)$ and by definition of the total variation distance, (22) implies that, for all $x \in E_0$,

$$\left|\frac{1}{t}\int_0^t P_{0,s}f(x) - \frac{1}{\gamma}\int_0^\gamma \beta_\gamma Q_{0,s}f ds\right| \leq \|f\|_\infty \left\|\frac{1}{t}\int_0^t \delta_x P_{0,s}ds - \frac{1}{\gamma}\int_0^\gamma \beta_\gamma Q_{0,s}ds\right\|_{TV} \underset{t\to\infty}{\longrightarrow} 0.$$

Then, using (22), one deduces that for any $x \in E_0$ and bounded function f,

$$\mathbb{E}_{0,x}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds - \frac{1}{\gamma}\int_0^\gamma \beta_\gamma Q_{0,s}fds\right|^2\right]$$

$$\leq 2\left(\mathbb{E}_{0,x}\left[\left(\frac{1}{t}\int_0^t f(X_s)ds - \frac{1}{t}\int_0^t P_{0,s}f(x)\right)^2\right] + \left|\frac{1}{t}\int_0^t P_{0,s}f(x) - \frac{1}{\gamma}\int_0^\gamma \beta_\gamma Q_{0,s}fds\right|^2\right)\underset{t\to\infty}{\longrightarrow} 0.$$

The convergence for any probability measure $\mu \in \mathcal{M}_1(E_0)$ comes from Lebesgue's dominated convergence theorem.

Fifth step. We now fix nonnegative $f \in \mathcal{B}(E)$, and $\mu \in \mathcal{M}_1(E_0)$ satisfying $\mu(\psi_0) < +\infty$. The following proof is inspired by the proof of [26, Theorem 12].

Since $\mu(\psi_0) < +\infty$, the inequality (24) implies that there exists a finite constant $C_{f,\mu} \in (0, \infty)$ such that, for t large enough,

$$\mathbb{E}_{0,\mu}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds - \mathbb{E}_{0,\mu}\left[\frac{1}{t}\int_0^t f(X_s)ds\right]\right|^2\right] \leq \frac{C_{f,\mu}}{t}.$$

Then, for n large enough,

$$\mathbb{E}_{0,\mu}\left[\left|\frac{1}{n^2}\int_0^{n^2} f(X_s)ds - \mathbb{E}_{0,\mu}\left[\frac{1}{n^2}\int_0^{n^2} f(X_s)ds\right]\right|^2\right] \leq \frac{C_{f,\mu}}{n^2}.$$

Then, by Chebyshev's inequality and the Borel–Cantelli lemma, this last inequality implies that

$$\left|\frac{1}{n^2}\int_0^{n^2} f(X_s)ds - \mathbb{E}_{0,\mu}\left[\frac{1}{n^2}\int_0^{n^2} f(X_s)ds\right]\right|\underset{n\to\infty}{\longrightarrow} 0, \quad \mathbb{P}_{0,\mu}\text{-almost surely.}$$

One thereby obtains by the convergence (22) that

$$\frac{1}{n^2}\int_0^{n^2} f(X_s)ds \underset{n\to\infty}{\longrightarrow} \frac{1}{\gamma}\int_0^\gamma \beta_\gamma Q_{0,s}fds, \quad \mathbb{P}_{0,\mu}\text{-almost surely.} \tag{25}$$

Since the nonnegativity of f is assumed, this implies that for any $t > 0$ we have

$$\int_0^{\lfloor\sqrt{t}\rfloor^2} f(X_s)ds \leq \int_0^t f(X_s)ds \leq \int_0^{\lceil\sqrt{t}\rceil^2} f(X_s)ds.$$

These inequalities and (25) then give that

$$\frac{1}{t}\int_0^t f(X_s)ds \underset{t\to\infty}{\longrightarrow} \frac{1}{\gamma}\int_0^\gamma \beta_\gamma Q_{0,s}fds, \quad \mathbb{P}_{0,\mu}\text{-almost surely.}$$

In order to conclude that the result holds for any bounded measurable function f, it is enough to decompose $f = f_+ - f_-$ with $f_+ := f \vee 0$ and $f_- = (-f) \vee 0$ and apply the above convergence to f_+ and f_-. This concludes the proof of Theorem 1. □

Proof of corollary 1. We remark as in the previous proof that, if $\|f\|_\infty \leq 1$ and $\psi_s = \mathbb{1}$, an upper bound for the inequality (24) can be obtained, which does not depend on f and x. Likewise, the convergence (21) holds uniformly in the initial measure thanks to (23). □

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Remark 3. The proof of Theorem 1, as written above, does not allow us to deal with semi-groups satisfying a Doeblin condition with time-dependent constant c_s, that is, such that there exist $t_0 \geq 0$ and a family of probability measure $(\nu_t)_{t\geq 0}$ on $(E_t)_{t\geq 0}$ such that, for all $s \geq 0$ and $x \in E_s$,

$$\delta_x P_{s,s+t_0} \geq c_{s+t_0} \nu_{s+t_0}.$$

In fact, under the condition written above, we can show (see for example the proof of the formula (2.7) of [9, Theorem 2.1]) that, for all $s \leq t$ and μ_1, $\mu_2 \in \mathcal{M}_1(E_s)$,

$$\|\mu_1 P_{s,t} - \mu_2 P_{s,t}\|_{TV} \leq 2 \prod_{k=0}^{\left\lfloor \frac{t-s}{t_0} \right\rfloor - 1} (1 - c_{t-kt_0}).$$

Hence, by this last inequality with $\mu_1 = \delta_x P_{s,s+k\gamma}$, $\mu_2 = \delta_x$, replacing s by $s + k\gamma$ and t by $s + (k+n)\gamma$, one obtains

$$\|\delta_x P_{s,s+(k+n)\gamma} - \delta_x P_{s+k\gamma,s+(k+n)\gamma}\|_{TV} \leq 2 \prod_{l=0}^{\left\lfloor \frac{n\gamma}{t_0} \right\rfloor - 1} (1 - c_{s+(k+n)\gamma - lt_0}),$$

which replaces the inequality (18) in the proof of Theorem 1. Plugging this last inequality into the formula (19), one obtains

$$\|\delta_x P_{s,s+(k+n)\gamma} - \beta_\gamma Q_{0,s}\|_{TV} \leq 2 \prod_{l=0}^{\left\lfloor \frac{n\gamma}{t_0} \right\rfloor - 1} (1 - c_{s+(k+n)\gamma - lt_0}) + \|\delta_x P_{s+k\gamma,s+(k+n)\gamma} - \beta_\gamma Q_{0,s}\|_{TV}.$$

Hence, we see that we cannot conclude a similar result when $c_s \longrightarrow 0$ as $s \to +\infty$, since, for n fixed,

$$\limsup_{k\to\infty} \prod_{l=0}^{\left\lfloor \frac{n\gamma}{t_0} \right\rfloor - 1} (1 - c_{s+(k+n)\gamma - lt_0}) = 1.$$

4. Application to quasi-stationarity with moving boundaries

In this section, $(X_t)_{t\geq 0}$ is assumed to be a time-homogeneous Markov process. We consider a family of measurable subsets $(A_t)_{t\geq 0}$ of E, and define the hitting time

$$\tau_A := \inf\{t \geq 0 : X_t \in A_t\}.$$

For all $s \leq t$, denote by $\mathcal{F}_{s,t}$ the σ-field generated by the family $(X_u)_{s\leq u\leq t}$, with $\mathcal{F}_t := \mathcal{F}_{0,t}$. Assume that τ_A is a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Assume also that for any $x \notin A_0$,

$$\mathbb{P}_{0,x}[\tau_A < +\infty] = 1 \quad \text{and} \quad \mathbb{P}_{0,x}[\tau_A > t] > 0, \ \forall t \geq 0.$$

We will be interested in a notion of *quasi-stationarity with moving boundaries*, which studies the asymptotic behavior of the Markov process $(X_t)_{t\geq 0}$ conditioned not to hit $(A_t)_{t\geq 0}$ up to the time t. For non-moving boundaries ($A_t = A_0$ for any $t \geq 0$), the *quasi-limiting distribution* is defined as a probability measure α such that, for at least one initial measure μ and for all measurable subsets $\mathcal{A} \subset E$,

$$\mathbb{P}_{0,\mu}[X_t \in \mathcal{A} | \tau_A > t] \underset{t\to\infty}{\longrightarrow} \alpha(\mathcal{A}).$$

Such a definition is equivalent (still in the non-moving framework) to the notion of *quasi-stationary distribution*, defined as a probability measure α such that, for any $t \geq 0$,

$$\mathbb{P}_{0,\alpha}[X_t \in \cdot | \tau_A > t] = \alpha. \tag{26}$$

If quasi-limiting and quasi-stationary distributions are in general well-defined for time-homogeneous Markov processes and non-moving boundaries (see [11, 23] for a general overview of the theory of quasi-stationarity), these notions are nevertheless not well-defined for time-inhomogeneous Markov processes or moving boundaries, for which they are no longer equivalent. In particular, under reasonable assumptions on irreducibility, it was shown in [24] that the notion of quasi-stationary distribution as defined by (26) is not well-defined for time-homogeneous Markov processes absorbed by moving boundaries.

Another asymptotic notion to study is the *quasi-ergodic distribution*, related to a conditional version of the ergodic theorem and usually defined as follows.

Definition 2. A probability measure β is a *quasi-ergodic distribution* if, for some initial measure $\mu \in \mathcal{M}_1(E \setminus A_0)$ and for any bounded continuous function f,

$$\mathbb{E}_{0,\mu}\left[\frac{1}{t}\int_0^t f(X_s)ds \middle| \tau_A > t\right] \underset{t\to\infty}{\longrightarrow} \beta(f).$$

In the time-homogeneous setting (in particular for non-moving boundaries), this notion has been extensively studied (see for example [2, 8, 10, 12, 13, 16–18, 24]). In the 'moving boundaries' framework, the existence of quasi-ergodic distributions has been dealt with in [24] for Markov chains on finite state spaces absorbed by periodic boundaries, and in [25] for processes satisfying a Champagnat–Villemonais condition (see Assumption (A′) below) absorbed by converging or periodic boundaries. In this last paper, the existence of the quasi-ergodic distribution is dealt with through the following inequality (see [25, Theorem 1]), which holds for any initial state x, $s \leq t$, and for some constants C, $\gamma > 0$ independent of x, s, and t:

$$\|\mathbb{P}_{0,x}(X_s \in \cdot | \tau_A > t) - \mathbb{Q}_{0,x}(X_s \in \cdot)\|_{TV} \leq Ce^{-\gamma(t-s)},$$

where the family of probability measures $(\mathbb{Q}_{s,x})_{s\geq 0, x\in E_s}$ is defined by

$$\mathbb{Q}_{s,x}[\Gamma] := \lim_{T\to\infty} \mathbb{P}_{s,x}[\Gamma | \tau_A > T], \quad \forall s \leq t, \ x \in E \setminus A_s, \ \Gamma \in \mathcal{F}_{s,t}.$$

Moreover, by [9, Proposition 3.1], there exists a family of positive bounded functions $(\eta_t)_{t\geq 0}$ defined in such a way that, for all $s \leq t$ and $x \in E_s$,

$$\mathbb{E}_{s,x}(\eta_t(X_t)\mathbb{1}_{\tau_A > t}) = \eta_s(x).$$

Then we can show (this is actually shown in [9]) that

$$\mathbb{Q}_{s,x}(\Gamma) = \mathbb{E}_{s,x}\left(\mathbb{1}_{\Gamma, \tau_A > t}\frac{\eta_t(X_t)}{\eta_s(x)}\right)$$

and that, for all $\mu \in \mathcal{M}_1(E_0)$,

$$\|\mathbb{P}_{0,\mu}(X_s \in \cdot | \tau_A > t) - \mathbb{Q}_{0,\eta_0 * \mu}(X_s \in \cdot)\|_{TV} \leq Ce^{-\gamma(t-s)},$$

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where

$$\eta_0 * \mu(dx) := \frac{\eta_0(x)\mu(dx)}{\mu(\eta_0)}.$$

By the triangle inequality, one has

$$\left\| \frac{1}{t}\int_0^t \mathbb{P}_{0,\mu}[X_s \in \cdot | \tau_A > t]ds - \frac{1}{t}\int_0^t \mathbb{Q}_{0,\eta_0 * \mu}[X_s \in \cdot]ds \right\|_{TV} \le \frac{C}{\gamma t}, \quad \forall t > 0. \tag{27}$$

In particular, the inequality (27) implies that there exists a quasi-ergodic distribution β for the process $(X_t)_{t\ge 0}$ absorbed by $(A_t)_{t\ge 0}$ if and only if there exist some probability measures $\mu \in \mathcal{M}_1(E_0)$ such that $\frac{1}{t}\int_0^t \mathbb{Q}_{0,\eta_0*\mu}[X_s \in \cdot]ds$ converges weakly to β, when t goes to infinity. In other words, under Assumption (A′), the existence of a quasi-ergodic distribution for the absorbed process is equivalent to the law of large numbers for its Q-process.

We now state Assumption (A′).

Assumption 4. *There exists a family of probability measures $(\nu_t)_{t\ge 0}$, defined on $E \setminus A_t$ for each t, such that the following hold:*

(A′1) There exist $t_0 \ge 0$ and $c_1 > 0$ such that

$$\mathbb{P}_{s,x}[X_{s+t_0} \in \cdot | \tau_A > s + t_0] \ge c_1 \nu_{s+t_0}, \quad \forall s \ge 0, \ \forall x \in E \setminus A_s.$$

(A′2) There exists $c_2 > 0$ such that

$$\mathbb{P}_{s,\nu_s}[\tau_A > t] \ge c_2 \mathbb{P}_{s,x}[\tau_A > t], \quad \forall s \le t, \ \forall x \in E \setminus A_s.$$

In what follows, we say that the pair $\{(X_t)_{t\ge 0}, (A_t)_{t\ge 0}\}$ satisfies Assumption (A′) when the assumption holds for the Markov process $(X_t)_{t\ge 0}$ considered as absorbed by the moving boundary $(A_t)_{t\ge 0}$.

The condition (A′1) is a conditional version of the Doeblin condition (12), and (A′2) is a Harnack-like inequality on the probabilities of surviving, necessary to deal with the conditioning. They are equivalent to the set of conditions presented in [1, Definition 2.2], when the non-conservative semigroup is sub-Markovian. In the time-homogeneous framework, we obtain the Champagnat–Villemonais condition defined in [5] (see Assumption (A)), shown as being equivalent to the exponential uniform convergence to quasi-stationarity in total variation.

In [25], the existence of a unique quasi-ergodic distribution is proved only for converging or periodic boundaries. However, we can expect such a result on existence (and uniqueness) for other kinds of movement for the boundary. Hence, the aim of this section is to extend the results on the existence of quasi-ergodic distributions obtained in [25] to Markov processes absorbed by asymptotically periodic moving boundaries.

Now let us state the following theorem.

Theorem 2. *Assume that there exists a γ-periodic sequence of subsets $(B_t)_{t\ge 0}$ such that, for any $s \in [0, \gamma)$,*

$$E'_s := E \setminus \bigcap_{k\in\mathbb{Z}_+} \bigcup_{l\ge k} A_{s+l\gamma} \cup B_s \ne \emptyset,$$

and there exists $x_s \in E_s$ such that, for any $n \le N$,

$$\|\mathbb{P}_{s+k\gamma,x_s}[X_{s+(k+n)\gamma} \in \cdot, \tau_A > s + (k+N)\gamma] - \mathbb{P}_{s,x_s}[X_{s+n\gamma} \in \cdot, \tau_B > s + N\gamma]\|_{TV} \underset{k\to\infty}{\longrightarrow} 0. \tag{28}$$

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Assume also that Assumption (A′) is satisfied by the pairs $\{(X_t)_{t\geq 0}, (A_t)_{t\geq 0}\}$ *and* $\{(X_t)_{t\geq 0}, (B_t)_{t\geq 0}\}$.

Then there exists a probability measure $\beta \in \mathcal{M}_1(E)$ *such that*

$$\sup_{\mu\in\mathcal{M}_1(E\setminus A_0)} \sup_{f\in\mathcal{B}_1(E)} \mathbb{E}_{0,\mu}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds - \beta(f)\right|^2 \middle| \tau_A > t\right] \underset{t\to\infty}{\longrightarrow} 0. \tag{29}$$

Remark 4. Observe that the condition (28) implies that, for any $n \in \mathbb{Z}_+$,

$$\mathbb{P}_{s+k\gamma,x_s}[\tau_A > s + (k+n)\gamma] \underset{k\to\infty}{\longrightarrow} \mathbb{P}_{s,x_s}[\tau_B > s + n\gamma].$$

Under the additional condition $B_t \subset A_t$ for all $t \geq 0$, these two conditions are equivalent, since for all $n \leq N$,

$$\begin{aligned}
&\|\mathbb{P}_{s+k\gamma,x_s}[X_{s+(k+n)\gamma} \in \cdot, \tau_A > s + (k+N)\gamma] - \mathbb{P}_{s,x_s}[X_{s+n\gamma} \in \cdot, \tau_B > s + N\gamma]\|_{TV} \\
&= \|\mathbb{P}_{s+k\gamma,x_s}[X_{s+(k+n)\gamma} \in \cdot, \tau_B \leq s + (k+N)\gamma < \tau_A]\|_{TV} \\
&\leq \mathbb{P}_{s+k\gamma,x_s}[\tau_B \leq s + (k+N)\gamma < \tau_A] \\
&= |\mathbb{P}_{s+k\gamma,x_s}[\tau_A > s + (k+N)\gamma] - \mathbb{P}_{s,x_s}[\tau_B > s + N\gamma]|,
\end{aligned}$$

where we used the periodicity of $(B_t)_{t\geq 0}$, writing

$$\mathbb{P}_{s,x_s}[X_{s+n\gamma} \in \cdot, \tau_B > s + N\gamma] = \mathbb{P}_{s+k\gamma,x_s}[X_{s+(k+n)\gamma} \in \cdot, \tau_B > s + (k+N)\gamma]$$

for all $k \in \mathbb{Z}_+$. This implies the following corollary.

Corollary 2. *Assume that there exists a* γ*-periodic sequence of subsets* $(B_t)_{t\geq 0}$*, with* $B_t \subset A_t$ *for all* $t \geq 0$*, such that, for any* $s \in [0, \gamma)$*, there exists* $x_s \in E'_s$ *such that, for any* $n \leq N$*,*

$$\mathbb{P}_{s+k\gamma,x_s}[\tau_A > s + (k+n)\gamma] \underset{k\to\infty}{\longrightarrow} \mathbb{P}_{s,x_s}[\tau_B > s + n\gamma].$$

Assume also that Assumption (A′) is satisfied by $\{(X_t)_{t\geq 0}, (A_t)_{t\geq 0}\}$ *and* $\{(X_t)_{t\geq 0}, (B_t)_{t\geq 0}\}$.

Then there exists $\beta \in \mathcal{M}_1(E)$ *such that (29) holds.*

Proof of theorem 2. Since $\{(X_t)_{t\geq 0}, (B_t)_{t\geq 0}\}$ satisfies Assumption (A′) and $(B_t)_{t\geq 0}$ is a periodic boundary, we already know by [25, Theorem 2] that, for any initial distribution μ, $t \mapsto \frac{1}{t}\int_0^t \mathbb{P}_{0,\mu}[X_s \in \cdot | \tau_B > t]ds$ converges weakly to a quasi-ergodic distribution β.

The main idea of this proof is to apply Corollary 1. Since $\{(X_t)_{t\geq 0}, (A_t)_{t\geq 0}\}$ and $\{(X_t)_{t\geq 0}, (B_t)_{t\geq 0}\}$ satisfy Assumption (A′), [25, Theorem 1] implies that there exist two families of probability measures $\left(\mathbb{Q}^A_{s,x}\right)_{s\geq 0, x\in E\setminus A_s}$ and $\left(\mathbb{Q}^B_{s,x}\right)_{s\geq 0, x\in E\setminus B_s}$ such that, for any $s \leq t$, $x \in E \setminus A_s$, $y \in E \setminus B_s$, and $\Gamma \in \mathcal{F}_{s,t}$,

$$\mathbb{Q}^A_{s,x}[\Gamma] = \lim_{T\to\infty} \mathbb{P}_{s,x}[\Gamma | \tau_A > T] \quad \text{and} \quad \mathbb{Q}^B_{s,y}[\Gamma] = \lim_{T\to\infty} \mathbb{P}_{s,y}[\Gamma | \tau_B > T].$$

In particular, the quasi-ergodic distribution β is the limit of $t \mapsto \frac{1}{t}\int_0^t \mathbb{Q}^B_{0,\mu}[X_s \in \cdot]ds$, when t goes to infinity (see [25, Theorem 5]). Also, by [25, Theorem 1], there exist constants $C > 0$ and $\kappa > 0$ such that, for any $s \leq t \leq T$, for any $x \in E \setminus A_s$,

$$\left\|\mathbb{Q}^A_{s,x}[X_t \in \cdot] - \mathbb{P}_{s,x}[X_t \in \cdot | \tau_A > T]\right\|_{TV} \leq Ce^{-\kappa(T-t)},$$

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and for any $x \in E \setminus B_s$,

$$\left\| \mathbb{Q}^B_{s,x}[X_t \in \cdot] - \mathbb{P}_{s,x}[X_t \in \cdot | \tau_B > T] \right\|_{TV} \leq Ce^{-\kappa(T-t)}.$$

Moreover, for any $s \leq t \leq T$ and $x \in E'_s$,

$$\begin{aligned}
&\|\mathbb{P}_{s,x}[X_t \in \cdot | \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot | \tau_B > T]\|_{TV} \\
&\quad = \left\| \frac{\mathbb{P}_{s,x}[X_t \in \cdot,\, \tau_A > T]}{\mathbb{P}_{s,x}[\tau_A > T]} - \frac{\mathbb{P}_{s,x}[X_t \in \cdot,\, \tau_B > T]}{\mathbb{P}_{s,x}[\tau_B > T]} \right\|_{TV} \\
&\quad = \left\| \frac{\mathbb{P}_{s,x}(\tau_B > T)}{\mathbb{P}_{s,x}(\tau_A > T)} \frac{\mathbb{P}_{s,x}[X_t \in \cdot,\, \tau_A > T]}{\mathbb{P}_{s,x}[\tau_B > T]} - \frac{\mathbb{P}_{s,x}[X_t \in \cdot,\, \tau_B > T]}{\mathbb{P}_{s,x}[\tau_B > T]} \right\|_{TV} \\
&\quad \leq \left\| \frac{\mathbb{P}_{s,x}(\tau_B > T)}{\mathbb{P}_{s,x}(\tau_A > T)} \frac{\mathbb{P}_{s,x}[X_t \in \cdot,\, \tau_A > T]}{\mathbb{P}_{s,x}[\tau_B > T]} - \frac{\mathbb{P}_{s,x}[X_t \in \cdot,\, \tau_A > T]}{\mathbb{P}_{s,x}[\tau_B > T]} \right\|_{TV} \\
&\qquad + \left\| \frac{\mathbb{P}_{s,x}[X_t \in \cdot,\, \tau_A > T]}{\mathbb{P}_{s,x}[\tau_B > T]} - \frac{\mathbb{P}_{s,x}[X_t \in \cdot,\, \tau_B > T]}{\mathbb{P}_{s,x}[\tau_B > T]} \right\|_{TV} \\
&\quad \leq \frac{|\mathbb{P}_{s,x}(\tau_B > T) - \mathbb{P}_{s,x}(\tau_A > T)|}{\mathbb{P}_{s,x}(\tau_B > T)} + \frac{\|\mathbb{P}_{s,x}[X_t \in \cdot,\, \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot,\, \tau_B > T]\|_{TV}}{\mathbb{P}_{s,x}[\tau_B > T]} \\
&\quad \leq 2\frac{\|\mathbb{P}_{s,x}[X_t \in \cdot,\, \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot,\, \tau_B > T]\|_{TV}}{\mathbb{P}_{s,x}[\tau_B > T]},
\end{aligned} \tag{30}$$

since

$$|\mathbb{P}_{s,x}(\tau_B > T) - \mathbb{P}_{s,x}(\tau_A > T)| \leq \|\mathbb{P}_{s,x}[X_t \in \cdot,\, \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot,\, \tau_B > T]\|_{TV}.$$

Then we obtain, for any $s \leq t \leq T$ and $x \in E'_s$,

$$\begin{aligned}
&\left\| \mathbb{Q}^A_{s,x}[X_t \in \cdot] - \mathbb{Q}^B_{s,x}[X_t \in \cdot] \right\|_{TV} \\
&\qquad \leq 2Ce^{-\kappa(T-t)} + 2\frac{\|\mathbb{P}_{s,x}[X_t \in \cdot,\, \tau_A > T] - \mathbb{P}_{s,x}[X_t \in \cdot,\, \tau_B > T]\|_{TV}}{\mathbb{P}_{s,x}[\tau_B > T]}.
\end{aligned} \tag{31}$$

The condition (28) implies the existence of $x_s \in E_s$ such that, for any $n \leq N$, for all $k \in \mathbb{Z}_+$,

$$\lim_{k\to\infty} \|\mathbb{P}_{s+k\gamma,x_s}[X_{s+(k+n)\gamma} \in \cdot,\, \tau_A > s + (k+N)\gamma] - \mathbb{P}_{s,x_s}[X_{s+n\gamma} \in \cdot,\, \tau_B > s + N\gamma]\|_{TV} = 0,$$

which implies by (31) that, for any $n \leq N$,

$$\limsup_{k\to\infty} \left\| \mathbb{Q}^A_{s+k\gamma,x_s}[X_{s+(k+n)\gamma} \in \cdot] - \mathbb{Q}^B_{s+k\gamma,x_s}[X_{s+(k+n)\gamma} \in \cdot] \right\|_{TV} \leq 2Ce^{-\kappa\gamma(N-n)}.$$

Now, letting $N \to \infty$, for any $n \in \mathbb{Z}_+$ we have

$$\begin{aligned}
&\lim_{k\to\infty} \left\| \mathbb{Q}^A_{s+k\gamma,x_s}[X_{s+(k+n)\gamma} \in \cdot] - \mathbb{Q}^B_{s+k\gamma,x_s}[X_{s+(k+n)\gamma} \in \cdot] \right\|_{TV} \\
&\qquad = \lim_{k\to\infty} \left\| \mathbb{Q}^A_{s+k\gamma,x_s}(X_{s+(k+n)\gamma} \in \cdot) - \mathbb{Q}^B_{s,x_s}(X_{s+n\gamma} \in \cdot) \right\|_{TV} \\
&\qquad = 0.
\end{aligned}$$

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In other words, the semigroup $\left(Q^A_{s,t}\right)_{s\le t}$ defined by

$$Q^A_{s,t}f(x) := \mathbb{E}^{\mathbb{Q}^A}_{s,x}(f(X_t)), \quad \forall s\le t,\ \forall f\in\mathcal{B}(E\setminus A_t),\ \forall x\in E\setminus A_s,$$

is asymptotically periodic (according to Definition 1, with $\psi_s = \tilde{\psi}_s = 1$ for all $s\ge 0$), associated to the auxiliary semigroup $\left(Q^B_{s,t}\right)_{s\le t}$ defined by

$$Q^B_{s,t}f(x) := \mathbb{E}^{\mathbb{Q}^B}_{s,x}(f(X_t)), \quad \forall s\le t,\ \forall f\in\mathcal{B}(E\setminus B_t),\ \forall x\in E\setminus B_s.$$

Moreover, since Assumption (A′) is satisfied for $\{(X_t)_{t\ge0}, (A_t)_{t\ge0}\}$ and $\{(X_t)_{t\ge0}, (B_t)_{t\ge0}\}$, the Doeblin condition holds for these two Q-processes. As a matter of fact, by the Markov property, for all $s\le t\le T$ and $x\in E\setminus A_s$,

$$\begin{aligned}\mathbb{P}_{s,x}(X_t\in\cdot|\tau_A>T) &= \mathbb{E}_{s,x}\left[\mathbb{1}_{X_t\in\cdot,\tau_A>t}\frac{\mathbb{P}_{t,X_t}(\tau_A>T)}{\mathbb{P}_{s,x}(\tau_A>T)}\right]\\ &= \mathbb{E}_{s,x}\left[\frac{\mathbb{1}_{X_t\in\cdot,\tau_A>t}}{\mathbb{P}_{s,x}(\tau_A>t)}\frac{\mathbb{P}_{t,X_t}(\tau_A>T)}{\mathbb{P}_{t,\phi_{t,s}(\delta_x)}(\tau_A>T)}\right]\\ &= \mathbb{E}_{s,x}\left[\mathbb{1}_{X_t\in\cdot}\frac{\mathbb{P}_{t,X_t}(\tau_A>T)}{\mathbb{P}_{t,\phi_{t,s}(\delta_x)}(\tau_A>T)}\middle|\tau_A>t\right],\end{aligned} \tag{32}$$

where, for all $s\le t$ and $\mu\in\mathcal{M}_1(E_s)$, $\phi_{t,s}(\mu) := \mathbb{P}_{s,\mu}(X_t\in\cdot|\tau_A>t)$. By (A′1), for any $s\ge0$, $T\ge s+t_0$, $x\in E\setminus A_s$, and measurable set $\mathcal{A}$,

$$\mathbb{E}_{s,x}\left[\mathbb{1}_{X_{s+t_0}\in\mathcal{A}}\frac{\mathbb{P}_{s+t_0,X_{s+t_0}}(\tau_A>T)}{\mathbb{P}_{s+t_0,\phi_{s+t_0,s}(\delta_x)}(\tau_A>T)}\middle|\tau_A>s+t_0\right]\ge c_1\int_{\mathcal{A}}\nu_{s+t_0}(dy)\frac{\mathbb{P}_{s+t_0,y}(\tau_A>T)}{\mathbb{P}_{s+t_0,\phi_{s+t_0,s}(\delta_x)}(\tau_A>T)};$$

that is, by (32),

$$\mathbb{P}_{s,x}(X_{s+t_0}\in\mathcal{A}|\tau_A>T)\ge c_1\int_{\mathcal{A}}\nu_{s+t_0}(dy)\frac{\mathbb{P}_{s+t_0,y}(\tau_A>T)}{\mathbb{P}_{s+t_0,\phi_{s+t_0,s}(\delta_x)}(\tau_A>T)}.$$

Letting $T\to\infty$ in this last inequality and using [9, Proposition 3.1], for any $s\ge0$, $x\in E\setminus A_s$, and measurable set $\mathcal{A}$,

$$\mathbb{Q}^A_{s,x}(X_{s+t_0}\in\mathcal{A})\ge c_1\int_{\mathcal{A}}\nu_{s+t_0}(dy)\frac{\eta_{s+t_0}(y)}{\phi_{s+t_0,s}(\delta_x)(\eta_{s+t_0})}.$$

The measure

$$\mathcal{A}\mapsto\int_{\mathcal{A}}\nu_{s+t_0}(dy)\frac{\eta_{s+t_0}(y)}{\phi_{s+t_0,s}(\delta_x)(\eta_{s+t_0})}$$

is then a positive measure whose mass is bounded below by c_2, by (A′2), since for all $s\ge0$ and $T\ge s+t_0$,

$$\int_{E\setminus A_{s+t_0}}\nu_{s+t_0}(dy)\frac{\mathbb{P}_{t,y}(\tau_A>T)}{\mathbb{P}_{t,\phi_{t,s}(\delta_x)}(\tau_A>T)}\ge c_2.$$

This proves a Doeblin condition for the semigroup $\left(Q^A_{s,t}\right)_{s\le t}$. The same reasoning also applies to prove a Doeblin condition for the semigroup $\left(Q^B_{s,t}\right)_{s\le t}$. Then, using (27) followed by

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Corollary 1, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{P}_{0,\mu}[X_s\in\cdot|\tau_A>t]ds=\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{Q}^A_{0,\eta_0*\mu}(X_s\in\cdot)ds$$
$$=\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{Q}^B_{0,\eta_0*\mu}[X_s\in\cdot]ds=\beta,$$

where the limits refer to convergence in total variation and hold uniformly in the initial measure.

For any $\mu\in\mathcal{M}_1(E\setminus A_0), f\in\mathcal{B}_1(E)$, and $t\geq 0$,

$$\mathbb{E}_{0,\mu}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds\right|^2\middle|\tau_A>t\right]=\frac{2}{t^2}\int_0^t\int_s^t \mathbb{E}_{0,\mu}[f(X_s)f(X_u)|\tau_A>t]du\,ds.$$

Then, by [25, Theorem 1], for any $s\leq u\leq t$, for any $\mu\in\mathcal{M}_1(E\setminus A_0)$ and $f\in\mathcal{B}(E)$,

$$\left|\mathbb{E}_{0,\mu}[f(X_s)f(X_u)|\tau_A>t]-\mathbb{E}^{\mathbb{Q}^A}_{0,\eta_0*\mu}[f(X_s)f(X_u)]\right|\leq C\|f\|_\infty e^{-\kappa(t-u)},$$

where the expectation $\mathbb{E}^{\mathbb{Q}^A}_{0,\eta_0*\mu}$ is associated to the probability measure $\mathbb{Q}^A_{0,\eta_0*\mu}$. Hence, for any $\mu\in\mathcal{M}_1(E\setminus A_0), f\in\mathcal{B}_1(E)$, and $t>0$,

$$\left|\mathbb{E}_{0,\mu}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds-\beta(f)\right|^2\middle|\tau_A>t\right]-\mathbb{E}^{\mathbb{Q}^A}_{0,\eta_0*\mu}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds-\beta(f)\right|^2\right]\right|$$
$$\leq\frac{4C}{t^2}\int_0^t\int_s^t e^{-\kappa(t-u)}du\,ds$$
$$\leq\frac{4C}{\kappa t}-\frac{4C(1-e^{-\kappa t})}{\kappa^2t^2}.$$

Moreover, since $\left(Q^A_{s,t}\right)_{s\leq t}$ is asymptotically periodic in total variation and satisfies the Doeblin condition, like $\left(Q^B_{s,t}\right)_{s\leq t}$, Corollary 1 implies that

$$\sup_{\mu\in\mathcal{M}_1(E\setminus A_0)}\sup_{f\in\mathcal{B}_1(E)}\mathbb{E}^{\mathbb{Q}^A}_{0,\eta_0*\mu}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds-\beta(f)\right|^2\right]\underset{t\to\infty}{\longrightarrow}0.$$

Then

$$\sup_{\mu\in\mathcal{M}_1(E\setminus A_0)}\sup_{f\in\mathcal{B}_1(E)}\mathbb{E}_{0,\mu}\left[\left|\frac{1}{t}\int_0^t f(X_s)ds-\beta(f)\right|^2\middle|\tau_A>t\right]\underset{t\to\infty}{\longrightarrow}0.$$

□

Remark 5. It seems that Assumption (A′) can be weakened by a conditional version of Assumption 1. In particular, such conditions can be derived from Assumption (F) in [6], as will be shown later in the paper [4], currently in preparation.

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5. Examples

5.1. Asymptotically periodic Ornstein–Uhlenbeck processes

Let $(X_t)_{t\geq 0}$ be a time-inhomogeneous diffusion process on $\mathbb{R}$ satisfying the stochastic differential equation

$$dX_t = dW_t - \lambda(t)X_t dt,$$

where $(W_t)_{t\geq 0}$ is a one-dimensional Brownian motion and $\lambda : [0, \infty) \to [0, \infty)$ is a function such that

$$\sup_{t\geq 0} |\lambda(t)| < +\infty$$

and such that there exists $\gamma > 0$ such that

$$\inf_{s\geq 0} \int_s^{s+\gamma} \lambda(u)du > 0.$$

By Itô's lemma, for any $s \leq t$,

$$X_t = e^{-\int_s^t \lambda(u)du}\left[X_s + \int_s^t e^{\int_s^u \lambda(v)dv} dW_u\right].$$

In particular, denoting by $(P_{s,t})_{s\leq t}$ the semigroup associated to $(X_t)_{t\geq 0}$, for any $f \in \mathcal{B}(\mathbb{R})$, $t \geq 0$, and $x \in \mathbb{R}$,

$$P_{s,t}f(x) = \mathbb{E}\left[f\left(e^{-\int_s^t \lambda(u)du}x + e^{-\int_s^t \lambda(u)du}\sqrt{\int_s^t e^{2\int_s^u \lambda(v)dv}du} \times \mathcal{N}(0, 1)\right)\right],$$

where $\mathcal{N}(0, 1)$ denotes a standard Gaussian variable.

Theorem 3. *Assume that there exists a γ-periodic function g, bounded on $\mathbb{R}$, such that $\lambda \sim_{t\to\infty} g$. Then the assumptions of Theorem 1 hold.*

Proof. In our case, the auxiliary semigroup $(Q_{s,t})_{s\leq t}$ of Definition 1 will be defined as follows: for any $f \in \mathcal{B}(\mathbb{R})$, $t \geq 0$, and $x \in \mathbb{R}$,

$$Q_{s,t}f(x) = \mathbb{E}\left[f\left(e^{-\int_s^t g(u)du}x + e^{-\int_s^t g(u)du}\sqrt{\int_s^t e^{2\int_s^u g(v)dv}du} \times \mathcal{N}(0, 1)\right)\right].$$

In particular, the semigroup $(Q_{s,t})_{s\leq t}$ is associated to the process $(Y_t)_{t\geq 0}$ following

$$dY_t = dW_t - g(t)Y_t dt.$$

We first remark that the function $\psi : x \mapsto 1 + x^2$ is a Lyapunov function for $(P_{s,t})_{s\leq t}$ and $(Q_{s,t})_{s\leq t}$. In fact, for any $s \geq 0$ and $x \in \mathbb{R}$,

$$\begin{aligned}P_{s,s+\gamma}\psi(x) &= 1 + e^{-2\int_s^{s+\gamma} \lambda(u)du}x^2 + e^{-2\int_s^{s+\gamma} \lambda(u)du}\int_s^{s+\gamma} e^{2\int_s^u \lambda(v)dv}du \\ &= e^{-2\int_s^{s+\gamma} \lambda(u)du}\psi(x) + 1 - e^{-2\int_s^{s+\gamma} \lambda(u)du} + e^{-2\int_s^{s+\gamma} \lambda(u)du}\int_s^{s+\gamma} e^{2\int_s^u \lambda(v)dv}du \\ &\leq e^{-2\gamma c_{\inf}}\psi(x) + C,\end{aligned}$$

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where $C \in (0, +\infty)$ and $c_{\inf} := \inf_{t\geq 0} \frac{1}{\gamma} \int_t^{t+\gamma} \lambda(u)du > 0$. Taking $\theta \in (e^{-2\gamma c_{\inf}}, 1)$, there exists a compact set K such that, for any $s \geq 0$,

$$P_{s,s+\gamma}\psi(x) \leq \theta\psi(x) + C\mathbb{1}_K(x).$$

Moreover, for any $s \geq 0$ and $t \in [0, \gamma)$, the function $P_{s,s+t}\psi/\psi$ is upper-bounded uniformly in s and t. It remains therefore to prove Assumption 1(i) for $(P_{s,t})_{s\leq t}$, which is a consequence of the following lemma.

Lemma 1. *For any $a, b_-, b_+ > 0$, define the subset $\mathcal{C}(a, b_-, b_+) \subset \mathcal{M}_1(\mathbb{R})$ as*

$$\mathcal{C}(a, b_-, b_+) := \{\mathcal{N}(m, \sigma) : m \in [-a, a], \sigma \in [b_-, b_+]\}.$$

Then, for any $a, b_-, b_+ > 0$, there exist a probability measure ν and a constant $c > 0$ such that, for any $\mu \in \mathcal{C}(a, b_-, b_+)$,

$$\mu \geq c\nu.$$

The proof of this lemma is postponed until after the end of this proof.

Since $\lambda \sim_{t\to\infty} g$ and these two functions are bounded on $\mathbb{R}_+$, Lebesgue's dominated convergence theorem implies that, for all $s \leq t$,

$$\left| \int_{s+k\gamma}^{t+k\gamma} \lambda(u)du - \int_s^t g(u)du \right| \underset{k\to\infty}{\longrightarrow} 0.$$

In the same way, for all $s \leq t$,

$$\int_{s+k\gamma}^{t+k\gamma} e^{2\int_{s+k\gamma}^u \lambda(v)dv} du \underset{k\to\infty}{\longrightarrow} \int_s^t e^{2\int_s^u g(v)dv} du.$$

Hence, for any $s \leq t$,

$$e^{-\int_{s+k\gamma}^{t+k\gamma} \lambda(u)du} \underset{k\to\infty}{\longrightarrow} e^{-\int_s^t g(u)du},$$

and

$$e^{-\int_{s+k\gamma}^{t+k\gamma} \lambda(u)du} \sqrt{\int_{s+k\gamma}^{t+k\gamma} e^{2\int_{s+k\gamma}^u \lambda(v)dv} du} \underset{k\to\infty}{\longrightarrow} e^{-\int_s^t g(u)du} \sqrt{\int_s^t e^{2\int_s^u g(v)dv} du}.$$

Using [14, Theorem 1.3], for any $x \in \mathbb{R}$,

$$\|\delta_x P_{s+k\gamma,t+k\gamma} - \delta_x Q_{s+k\gamma,t+k\gamma}\|_{TV} \underset{k\to\infty}{\longrightarrow} 0. \tag{33}$$

To deduce the convergence in ψ-distance, we will draw inspiration from the proof of [19, Lemma 3.1]. Since the variances are uniformly bounded in k (for $s \leq t$ fixed), there exists $H > 0$ such that, for any $k \in \mathbb{N}$ and $s \leq t$,

$$\delta_x P_{s+k\gamma,t+k\gamma}\left[\psi^2\right] \leq H \quad \text{and} \quad \delta_x Q_{s,t}\left[\psi^2\right] \leq H. \tag{34}$$

Since $\lim_{|x|\to\infty} \frac{\psi(x)}{\psi^2(x)} = 0$, for any $\epsilon > 0$ there exists $l_\epsilon > 0$ such that, for any function f such that $|f| \leq \psi$ and for any $|x| \geq l_\epsilon$,

$$|f(x)| \leq \frac{\epsilon\psi(x)^2}{H}.$$

Combining this with (34), and letting $K_\epsilon := [-l_\epsilon, l_\epsilon]$, we find that for any $k \in \mathbb{Z}_+$, f such that $|f| \leq \psi$, and $x \in \mathbb{R}$,

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$$\delta_x P_{s+k\gamma,t+k\gamma}[f\mathbb{1}_{K_\epsilon^c}] \le \epsilon \quad \text{and} \quad \delta_x Q_{s,t}[f\mathbb{1}_{K_\epsilon^c}] \le \epsilon.$$

Then, for any $k \in \mathbb{Z}_+$ and f such that $|f| \le \psi$,

$$|\delta_x P_{s+k\gamma,t+k\gamma} f - \delta_x Q_{s,t} f| \le 2\epsilon + |\delta_x P_{s+k\gamma,t+k\gamma}[f\mathbb{1}_{K_\epsilon}] - \delta_x Q_{s,t}[f\mathbb{1}_{K_\epsilon}]| \tag{35}$$

$$\le 2\epsilon + (1 + l_\epsilon^2)\|\delta_x P_{s+k\gamma,t+k\gamma} - \delta_x Q_{s,t}\|_{TV}. \tag{36}$$

Hence, (33) implies that, for k large enough, for any f bounded by ψ,

$$|\delta_x P_{s+k\gamma,t+k\gamma} f - \delta_x Q_{s,t} f| \le 3\epsilon, \tag{37}$$

implying that

$$\|\delta_x P_{s+k\gamma,t+k\gamma} - \delta_x Q_{s,t}\|_\psi \underset{k\to\infty}{\longrightarrow} 0.$$

We now prove Lemma 1. □

Proof of Lemma 1. Defining

$$f_\nu(x) := e^{-\frac{(x-a)^2}{2b_-^2}} \wedge e^{-\frac{(x+a)^2}{2b_-^2}},$$

we conclude easily that, for any $m \in [-a, a]$ and $\sigma \ge b_-$, for any $x \in \mathbb{R}$,

$$e^{-\frac{(x-m)^2}{2\sigma^2}} \ge f_\nu(x).$$

Imposing moreover that $\sigma \le b_+$, one has

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} \ge \frac{1}{\sqrt{2\pi} b_+} f_\nu(x),$$

which concludes the proof. □

5.2. Quasi-ergodic distribution for Brownian motion absorbed by an asymptotically periodic moving boundary

Let $(W_t)_{t\ge0}$ be a one-dimensional Brownian motion, and let h be a $\mathcal{C}^1$-function such that

$$h_{\min} := \inf_{t\ge0} h(t) > 0 \quad \text{and} \quad h_{\max} := \sup_{t\ge0} h(t) < +\infty.$$

We assume also that

$$-\infty < \inf_{t\ge0} h'(t) \le \sup_{t\ge0} h'(t) < +\infty.$$

Define

$$\tau_h := \inf\{t \ge 0 : |W_t| \ge h(t)\}.$$

Since h is continuous, the hitting time τ_h is a stopping time with respect to the natural filtration of $(W_t)_{t\ge0}$. Moreover, since $\sup_{t\ge0} h(t) < +\infty$ and $\inf_{t\ge0} h(t) > 0$,

$$\mathbb{P}_{s,x}[\tau_h < +\infty] = 1 \quad \text{and} \quad \mathbb{P}_{s,x}[\tau_h > t] > 0, \quad \forall s \le t,\ \forall x \in [-h(s), h(s)].$$

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The main assumption on the function h is the existence of a γ-periodic function g such that $h(t) \leq g(t)$, for any $t \geq 0$, and such that

$$h \sim_{t\to\infty} g \text{ and } h' \sim_{t\to\infty} g'.$$

Similarly to τ_h, define

$$\tau_g := \inf\{t \geq 0 : |W_t| = g(t)\}.$$

Finally, let us assume that there exists $n_0 \in \mathbb{N}$ such that, for any $s \geq 0$,

$$\inf\{u \geq s : h(u) = \inf_{t \geq s} h(t)\} - s \leq n_0\gamma. \tag{38}$$

This condition says that there exists $n_0 \in \mathbb{N}$ such that, for any time $s \geq 0$, the infimum of the function h on the domain $[s, +\infty)$ is reached on the subset $[s, s + n_0\gamma]$.

We first prove the following proposition.

Proposition 1. *The Markov process $(W_t)_{t\geq 0}$, considered as absorbed by h or by g, satisfies Assumption (A′).*

Proof. In what follows, we will prove Assumption (A′) with respect to the absorbing function h. The proof can easily be adapted for the function g.

- *Proof of (A′1).* Define $\mathcal{T} := \{s \geq 0 : h(s) = \inf_{t\geq s} h(t)\}$. The condition (38) implies that this set contains an infinity of times.

In what follows, the following notation is needed: for any $z \in \mathbb{R}$, define τ_z as

$$\tau_z := \inf\{t \geq 0 : |W_t| = z\}.$$

Also, let us state that, since the Brownian motion absorbed at $\{-1, 1\}$ satisfies Assumption (A) of [5] at any time (see [7]), it follows that, for a given $t_0 > 0$, there exist $c > 0$ and $\nu \in \mathcal{M}_1((-1, 1))$ such that, for any $x \in (-1, 1)$,

$$\mathbb{P}_{0,x}\left[W_{\frac{t_0}{h_{\max}^2}\wedge t_0} \in \cdot \,\middle|\, \tau_1 > \frac{t_0}{h_{\max}^2} \wedge t_0\right] \geq c\nu. \tag{39}$$

Moreover, in relation to the proof of [7, Section 5.1], the probability measure ν can be expressed as

$$\nu = \frac{1}{2}\left(\mathbb{P}_{0,1-\epsilon}[W_{t_2} \in \cdot|\tau_1 > t_2] + \mathbb{P}_{0,-1+\epsilon}[W_{t_2} \in \cdot|\tau_1 > t_2]\right), \tag{40}$$

for some $0 < t_2 < \frac{t_0}{h_{\max}^2} \wedge t_0$ and $\epsilon \in (0, 1)$.

The following lemma is very important for the next part of the argument.

Lemma 2. *For all $z \in [h_{\min}, h_{max}]$,*

$$\mathbb{P}_{0,x}[W_u \in \cdot|\tau_z > u] \geq c\nu_z, \quad \forall x \in (-z, z),\ \forall u \geq t_0,$$

where t_0 is as previously mentioned, $c > 0$ is the same constant as in (39), and

$$\nu_z(f) = \int_{(-1,1)} f(zx)\nu(dx),$$

with $\nu \in \mathcal{M}_1((-1, 1))$ defined in (40).

The proof of this lemma is postponed until after the current proof.

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Let $s \in \mathcal{T}$. Then, for all $x \in (-h(s), h(s))$ and $t \geq 0$,

$$\mathbb{P}_{s,x}[W_{s+t} \in \cdot | \tau_h > s+t] \geq \frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s+t]}{\mathbb{P}_{s,x}[\tau_h > s+t]} \mathbb{P}_{s,x}[W_{s+t} \in \cdot | \tau_{h(s)} > s+t].$$

By Lemma 2, for all $x \in (-h(s), h(s))$ and $t \geq t_0$,

$$\mathbb{P}_{s,x}[W_{s+t} \in \cdot | \tau_{h(s)} > s+t] \geq c\nu_{h(s)},$$

which implies that, for any $t \in [t_0, t_0 + n_0\gamma]$,

$$\begin{aligned} \mathbb{P}_{s,x}[W_{s+t} \in \cdot | \tau_h > s+t] &\geq \frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s+t]}{\mathbb{P}_{s,x}[\tau_h > s+t]} c\nu_{h(s)} \\ &\geq \frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s+t_0+n_0\gamma]}{\mathbb{P}_{s,x}[\tau_h > s+t_0]} c\nu_{h(s)}. \end{aligned} \tag{41}$$

Let us introduce the process X^h defined by, for all $t \geq 0$,

$$X_t^h := \frac{W_t}{h(t)}.$$

By Itô's formula, for any $t \geq 0$,

$$X_t^h = X_0^h + \int_0^t \frac{dW_s}{h(s)} - \int_0^t \frac{h'(s)}{h(s)} X_s^h ds.$$

Define

$$\left(M_t^h\right)_{t\geq 0} := \left(\int_0^t \frac{1}{h(s)} dW_s\right)_{t\geq 0}.$$

By the Dubins–Schwarz theorem, it is well known that the process M^h has the same law as

$$\left(W_{\int_0^t \frac{1}{h^2(s)} ds}\right)_{t\geq 0}.$$

Then, defining

$$I^h(s) := \int_0^s \frac{1}{h^2(u)} du$$

and, for any $s \leq t$ and for any trajectory w,

$$\mathcal{E}_{s,t}^h(w) := \sqrt{\frac{h(t)}{h(s)}} \exp\Bigg(-\frac{1}{2}\Bigg[h'(t)h(t)w_{I^h(t)}^2 - h'(s)h(s)w_{I^h(s)}^2 \tag{42}$$

$$+ \int_s^t w_{I^h(u)}^2 [(h'(u))^2 - [h(u)h'(u)]'] du \Bigg]\Bigg), \tag{43}$$

Girsanov's theorem implies that, for all $x \in (-h(s), h(s))$,

$$\mathbb{P}_{s,x}[\tau_h > s+t_0] = \mathbb{E}_{I^h(s), \frac{x}{h(s)}}\left[\mathcal{E}_{s,s+t_0}^h(W) \mathbb{1}_{\tau_1 > \int_0^{s+t_0} \frac{1}{h^2(u)} du}\right]. \tag{44}$$

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On the event

$$\left\{\tau_1 > \int_0^{s+t_0} \frac{1}{h^2(u)} du\right\},$$

and since h and h' are bounded on $\mathbb{R}_+$, the random variable $\mathcal{E}^h_{s,s+t_0}(W)$ is almost surely bounded by a constant $C > 0$, uniformly in s, such that for all $x \in (-h(s), h(s))$,

$$\mathbb{E}_{I^h(s), \frac{x}{h(s)}}\left[\mathcal{E}^h_{s,s+t_0}(W)\mathbb{1}_{\tau_1 > \int_0^{s+t_0} \frac{1}{h^2(u)} du}\right] \le C\mathbb{P}_{0, \frac{x}{h(s)}}\left[\tau_1 > \int_s^{s+t_0} \frac{1}{h^2(u)} du\right]. \tag{45}$$

Since $h(t) \ge h(s)$ for all $t \ge s$ (since $s \in \mathcal{T}$),

$$I^h(s+t_0) - I^h(s) \le \frac{t_0}{h(s)^2}.$$

By the scaling property of the Brownian motion and by the Markov property, one has for all $x \in (-h(s), h(s))$

$$\begin{aligned}
\mathbb{P}_{s,x}[\tau_{h(s)} > s + t_0] &= \mathbb{P}_{0,x}[\tau_{h(s)} > t_0] \\
&= \mathbb{P}_{0, \frac{x}{h(s)}}\left[\tau_1 > \frac{t_0}{h^2(s)}\right] \\
&= \mathbb{E}_{0, \frac{x}{h(s)}}\left[\mathbb{1}_{\tau_1 > \int_s^{s+t_0} \frac{1}{h^2(u)} du}\mathbb{P}_{0, W_{\int_s^{s+t_0} \frac{1}{h^2(u)} du}}\left[\tau_1 > \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(s)} ds\right]\right] \\
&= \mathbb{P}_{0, \frac{x}{h(s)}}\left[\tau_1 > \int_s^{s+t_0} \frac{1}{h^2(u)} du\right] \\
&\quad \mathbb{P}_{0, \phi_{I^h(s+t_0) - I^h(s)}(\delta_x)}\left[\tau_1 > \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)} du\right],
\end{aligned}$$

where, for any initial distribution μ and any $t \ge 0$,

$$\phi_t(\mu) := \mathbb{P}_{0,\mu}[W_t \in \cdot | \tau_1 > t].$$

The family $(\phi_t)_{t \ge 0}$ satisfies the equality $\phi_t \circ \phi_s = \phi_{t+s}$ for all $s, t \ge 0$. By this property, and using that

$$I^h(s+t_0) - I^h(s) \ge \frac{t_0}{h^2_{\max}}$$

for any $s \ge 0$, the minorization (39) implies that, for all $s \ge 0$ and $x \in (-1, 1)$,

$$\phi_{I^h(s+t_0) - I^h(s)}(\delta_x) \ge c\nu.$$

Hence, by this minorization, and using that h is upper-bounded and lower-bounded positively on $\mathbb{R}_+$, one has for all $x \in (-1, 1)$

$$\begin{aligned}
&\mathbb{P}_{0, \phi_{I^h(s+t_0) - I^h(s)}(\delta_x)}\left[\tau_1 > \frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)} du\right] \\
&\qquad \ge c\mathbb{P}_{0,\nu}\left[\tau_1 > \inf_{s \ge 0}\left\{\frac{t_0}{h^2(s)} - \int_s^{s+t_0} \frac{1}{h^2(u)} du\right\}\right];
\end{aligned}$$

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that is to say,

$$\frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s+t_0]}{\mathbb{P}_{0,\frac{x}{h(s)}}\left[\tau_1 > \int_s^{s+t_0}\frac{1}{h^2(u)}du\right]} \geq c\mathbb{P}_{0,\nu}\left[\tau_1 > \inf_{s\geq 0}\left\{\frac{\gamma}{h^2(s)} - \int_s^{s+t_0}\frac{1}{h^2(u)}du\right\}\right].$$

In other words, we have just shown that, for all $x \in (-h(s), h(s))$,

$$\frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s+t_0]}{\mathbb{P}_{s,x}[\tau_h > s+t_0]} \geq \frac{c}{C}\mathbb{P}_{0,\nu}\left[\tau_1 > \inf_{s\geq 0}\left\{\frac{t_0}{h^2(s)} - \int_s^{s+t_0}\frac{1}{h^2(u)}du\right\}\right] > 0. \tag{46}$$

Moreover, by Lemma 2 and the scaling property of the Brownian motion, for all $x \in (-h(s), h(s))$,

$$\begin{aligned}\frac{\mathbb{P}_{s,x}[\tau_{h(s)} > s+t_0+n_0\gamma]}{\mathbb{P}_{s,x}[\tau_{h(s)} > s+t_0]} &= \mathbb{P}_{0,\mathbb{P}_{0,x}[W_{t_0}\in\cdot|\tau_{h(s)}>t_0]}[\tau_{h(s)} > n_0\gamma] \\ &\geq c\mathbb{P}_{0,\nu_{h(s)}}[\tau_{h(s)} > n_0\gamma] \\ &= c\int_{(-1,1)}\nu(dy)\mathbb{P}_{h(s)y}[\tau_{h(s)} > n_0\gamma] \\ &\geq c\mathbb{P}_{0,\nu}\left[\tau_1 > \frac{n_0\gamma}{h^2_{\min}}\right] > 0. \end{aligned}\tag{47}$$

Thus, combining (41), (46), and (47), for any $x \in (-h(s), h(s))$ and any $t \in [t_0, t_0+n_0\gamma]$,

$$\mathbb{P}_{s,x}[W_{s+t} \in \cdot|\tau_h > s+t] \geq c_1\nu_{h(s)}, \tag{48}$$

where

$$c_1 := c\mathbb{P}_{0,\nu}\left[\tau_1 > \frac{n_0\gamma}{h^2_{\max}}\right] \times \frac{c}{C}\mathbb{P}_{0,\nu}\left[\tau_1 > \inf_{s\geq 0}\left\{\frac{\gamma}{h^2(s)} - \int_s^{s+\gamma}\frac{1}{h^2(u)}du\right\}\right]c.$$

We recall that the Doeblin condition (48) has, for now, been obtained only for $s \in \mathcal{T}$. Consider now $s \notin \mathcal{T}$. Then, by the condition (38), there exists $s_1 \in \mathcal{T}$ such that $s < s_1 \leq s + n_0\gamma$. The Markov property and (48) therefore imply that, for any $x \in (-h(s), h(s))$,

$$\mathbb{P}_{s,x}[W_{s+t_0+n_0\gamma} \in \cdot|\tau_h > s+t_0+n_0\gamma] = \mathbb{P}_{s_1,\phi_{s_1,s}}[W_{s+t_0+n_0\gamma} \in \cdot|\tau_h > s+t_0+n_0\gamma] \geq c_1\nu_{h(s_1)},$$

where, for all $s \leq t$ and $\mu \in \mathcal{M}_1((-h(s), h(s)))$,

$$\phi_{t,s}(\mu) := \mathbb{P}_{s,\mu}[W_t \in \cdot|\tau_h > t].$$

This concludes the proof of (A′1).

- *Proof of (A′2).* Since $(W_t)_{t\geq 0}$ is a Brownian motion, note that for any $s \leq t$,

$$\sup_{x\in(-1,1)}\mathbb{P}_{s,x}[\tau_h > t] = \mathbb{P}_{s,0}[\tau_h > t].$$

Also, for any $a \in (0, h(s))$,

$$\inf_{[-a,a]}\mathbb{P}_{s,x}[\tau_h > t] = \mathbb{P}_{s,a}[\tau_h > t].$$

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Thus, by the Markov property, and using that the function $s \mapsto \mathbb{P}_{s,0}[\tau_g > t]$ is non-decreasing on $[0, t]$ (for all $t \geq 0$), one has, for any $s \leq t$,

$$\mathbb{P}_{s,a}[\tau_h > t] \geq \mathbb{E}_{s,a}[\mathbb{1}_{\tau_0 < s+\gamma < \tau_h} \mathbb{P}_{\tau_0,0}[\tau_h > t]] \geq \mathbb{P}_{s,a}[\tau_0 < s + \gamma < \tau_h] \mathbb{P}_{s,0}[\tau_h > t]. \quad (49)$$

Defining $a := \frac{h_{\min}}{h_{\max}}$, by Lemma 2 and taking $s_1 := \inf\{u \geq s : u \in \mathcal{T}\}$, one obtains that, for all $s \leq t$,

$$\begin{aligned}\mathbb{P}_{s,\nu_{h(s_1)}}[\tau_h > t] &= \int_{(-1,1)} \nu(dx) \mathbb{P}_{s,h(s_1)x}[\tau_h > t] \\ &\geq \nu([-a, a]) \mathbb{P}_{s,h(s_1)a}[\tau_h > t] \\ &\geq \nu([-a, a]) \mathbb{P}_{0,h_{\min}}[\tau_0 < \gamma < \tau_h] \sup_{x \in (-h(s), h(s))} \mathbb{P}_{s,x}[\tau_h > t].\end{aligned}$$

This concludes the proof, since, using (40), one has $\nu([-a, a]) > 0$. □

We now prove Lemma 2.

Proof of Lemma 2. This result comes from the scaling property of a Brownian motion. In fact, for any $z \in [h_{\min}, h_{\max}]$, $x \in (-z, z)$, and $t \geq 0$, and for any measurable bounded function f,

$$\begin{aligned}\mathbb{E}_{0,x}[f(W_t) | \tau_z > t] &= \mathbb{E}_{0,x}\left[f\left(z \times \frac{1}{z} W_{z^2 \frac{t}{z^2}} \right) \middle| \tau_z > t \right] \\ &= \mathbb{E}_{0,\frac{x}{z}}\left[f\left(z \times W_{\frac{t}{z^2}} \right) \middle| \tau_1 > \frac{t}{z^2} \right].\end{aligned}$$

Then the minorization (39) implies that for any $x \in (-1, 1)$,

$$\mathbb{P}_{0,x}\left[W_{\frac{t_0}{h_{\max}^2}} \in \cdot \middle| \tau_1 > \frac{t_0}{h_{\max}^2} \right] \geq c\nu.$$

This inequality holds for any time greater than $\frac{t_0}{h_{\max}^2}$. In particular, for any $z \in [h_{\min}, h_{\max}]$ and $x \in (-1, 1)$,

$$\mathbb{P}_{0,x}\left[W_{\frac{t_0}{z^2}} \in \cdot \middle| \tau_1 > \frac{t_0}{z^2} \right] \geq c\nu.$$

Then, for any $z \in [a, b]$, f positive and measurable, and $x \in (-z, z)$,

$$\mathbb{E}_{0,x}[f(W_{t_0}) | \tau_z > t_0] \geq c\nu_z(f),$$

where $\nu_z(f) := \int_E f(z \times x)\nu(dx)$. This completes the proof of Lemma 2. □

We now conclude the section by stating and proving the following result.

Theorem 4. *For any $s \leq t$, $n \in \mathbb{N}$, and any $x \in \mathbb{R}$,*

$$\mathbb{P}_{s+k\gamma,x}[\tau_h \leq t + k\gamma < \tau_g] \underset{k\to\infty}{\longrightarrow} 0.$$

In particular, Corollary 2 holds for $(W_t)_{t\geq 0}$ absorbed by h.

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Proof. Recalling (43), by the Markov property for the Brownian motion, one has, for any $k, n \in \mathbb{N}$ and any $x \in \mathbb{R}$,

$$\mathbb{P}_{s+k\gamma,x}[\tau_h > t+k\gamma] = \sqrt{\frac{h(t+k\gamma)}{h(s+k\gamma)}}\mathbb{E}_{0,x}\left[\exp\left(-\frac{1}{2}\mathcal{A}^h_{s,t,k}(W)\right)\mathbb{1}_{\tau_1 > I^h(t+k\gamma)-I^h(s+k\gamma)}\right],$$

where, for any trajectory $w = (w_u)_{u\geq 0}$,

$$\mathcal{A}^h_{s,t,k}(w) = h'(t+k\gamma)h(t+k\gamma)w^2_{I^h(t+k\gamma)-I^h(s+k\gamma)} - h'(s+k\gamma)h(s+k\gamma)w_0^2$$
$$+\int_0^{t-s} w^2_{I^h(u+s+k\gamma)-I^h(s+k\gamma)}[(h'(u+s+k\gamma))^2 - [h(u+s+k\gamma)h'(u+s+k\gamma)]']du.$$

Since $h \sim_{t\to\infty} g$, one has for any $s, t \in [0, \gamma]$

$$\sqrt{\frac{h(t+k\gamma)}{h(s+k\gamma)}} \underset{k\to\infty}{\longrightarrow} \sqrt{\frac{g(t)}{g(s)}}.$$

For the same reasons, and using that the function h is bounded on $[s+k\gamma, t+k\gamma]$ for all $s \leq t$, Lebesgue's dominated convergence theorem implies that

$$I^h(t+k\gamma) - I^h(s+k\gamma) \underset{k\to\infty}{\longrightarrow} I^g(t) - I^g(s)$$

for all $s \leq t \in [0, \gamma]$. Moreover, since $h \sim_{t\to\infty} g$ and $h' \sim_{t\to\infty} g'$, one has for all trajectories $w = (w_u)_{u\geq 0}$ and $s \leq t \in [0, \gamma]$

$$\mathcal{A}^h_{s,t,k}(w) \underset{k\to\infty}{\longrightarrow} g'(t)g(t)w^2_{I^g(t)-I^g(s)} - g'(s)g(s)w_0^2 + \int_s^t w^2_{I^g(u)}[(g'(u))^2 - [g(u)g'(u)]']du.$$

Since the random variable

$$\exp\left(-\frac{1}{2}\mathcal{A}^h_{s,t,k}(W)\right)\mathbb{1}_{\tau_1 > I^h(t+k\gamma)-I^h(s+k\gamma)}$$

is bounded almost surely, Lebesgue's dominated convergence theorem implies that

$$\mathbb{P}_{s+k\gamma,x}[\tau_h > t+k\gamma] \underset{k\to\infty}{\longrightarrow} \mathbb{P}_{s,x}[\tau_g > t],$$

which concludes the proof. □

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